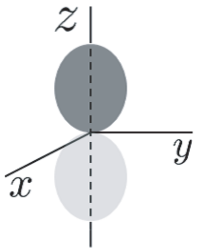

Fig. 1. Schematic diagram of $|z>$ state.

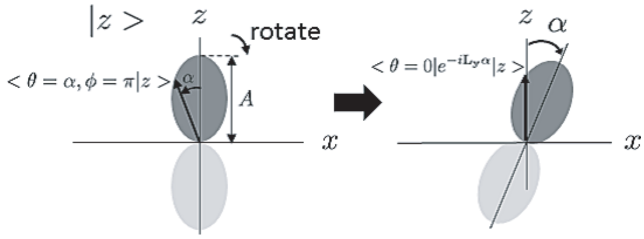

Fig. 2. Rotation of $|z>$ around y axis by angle α.

To carry out this calculation, we predict the form of the matrix $(\mathbf{L}_j)^n$ and prove it by mathematical induction. Then we calculate the sum of the series. We obtain

$$e^{-i\mathbf{L_x}\phi} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\cos\phi - \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\sin\phi, \quad (61)$$

$$e^{-i\mathbf{L_y}\phi} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\cos\phi - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\sin\phi, \quad (62)$$

$$e^{-i\mathbf{L_z}\phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cos\phi - i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\sin\phi. \quad (63)$$

We construct the real spherical harmonics (hereafter abbreviated to RSHs) to show their form graphically. The SH itself is a complex function, and the method of constructing the RSH from the SH is well known [7]. In accordance with this construction, we define the following states:

$$|x> \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|1,-1> - |1,1>), \quad (64)$$

$$|y> \equiv \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}}(|1,1> + |1,-1>), \quad (65)$$

$$|z> \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |1,0> . \quad (66)$$

Then we discuss the form of the states. We start with the $|z>$ state. First, this state has rotational symmetry around the z-axis:

$$e^{-i\mathbf{L_z}\phi}|z> = |z> . \quad (67)$$

Second, the $|z>$ state has parity odd for $\hat{P}_z$ from Eq. (50), which means that the state has xy-plane as the node plane.

$$\mathbf{P_z}|z> = -|z> . \quad (68)$$

Then the form of $|z>$ ($r = |<\theta,\phi|z>|$) is considered to be that in Fig. 1, where the different contrasts show the phase inversion. Figure 1 is a rough sketch and the precise form will be discussed later.

Next we consider the forms of the other two states.

First, from Eqs. (61), (65) and (66), we obtain

$$e^{-i\mathbf{L_x}(-\pi/2)}|z> = |y> . \quad (69)$$

Second, from Eqs. (62), (64) and (66), we obtain

$$e^{-i\mathbf{L_y}(\pi/2)}|z> = |x> . \quad (70)$$

Therefore, these three states have the same form but different orientations. We obtain the form of $|z>$ more precisely as follows. The rotation of the $|z>$ state around the y axis by angle α gives the following relation from Eqs. (62), (64) and (66):

$$e^{-i\mathbf{L_y}\alpha}|z> = \cos\alpha|z> + \sin\alpha|x> . \quad (71)$$

Then we examine the z direction. For this purpose, we multiply "bra" $<\theta = 0|$ from the left. We obtain

$$<\theta = 0|e^{-i\mathbf{L_y}\alpha}|z> = \cos\alpha <\theta = 0|z> + \sin\alpha <\theta = 0|x> . \quad (72)$$

The l.h.s. can be calculated as

$$e^{+i\mathbf{L_y}\alpha}|\theta = 0> = |\theta = \alpha, \phi = \pi> .$$

By taking the Hermitian conjugate, we obtain

$$<\theta = 0|e^{-i\mathbf{L_y}\alpha}|z> = <\theta = \alpha, \phi = \pi|z> = <\theta = \alpha|z>, \quad (73)$$

where the final equality originates from the rotational symmetry of $|z>$ about the z axis. This situation is shown in Fig. 2. $<\theta = \alpha, \phi = \pi|z>$ is shown by the arrow in the left figure. To obtain the length of this arrow, we rotate the state $|z>$ around the y axis by angle α and examine the z direction.

Furthermore, from the form of $|x>$, we have

$$<\theta = 0|x> = 0. \quad (74)$$