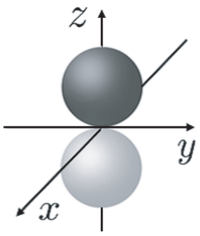

Fig. 3. Precise form of $|z>$.

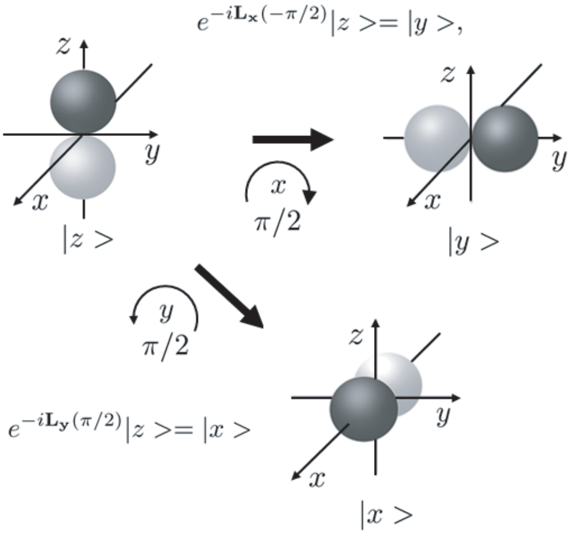

Fig. 4. Forms of the $l = 1$ members.

This comes from Eq. (70) and Fig. 1, later explicitly shown in Fig. 4. From Eqs. (72), (73) and (74), we have

$$<\theta = \alpha|z>= A\cos\alpha, \tag{75}$$

where the constant A is given by $A \equiv <\theta = 0|z>$ and we set A to be real and positive.

Then the wave function can be written as

$$Y_{1,z}(\theta,\phi) \equiv <\theta,\phi|z>= A\cos\theta. \tag{76}$$

To illustrate this, let $r = |Y_{1,z}(\theta,\phi)|$. Then

$$r = A\frac{z}{r}\ \ (z>0),\ \ r = -A\frac{z}{r}\ \ (z<0).$$

Then we have

$$x^2 + y^2 + (z \pm A/2)^2 = (A/2)^2, \quad \text{for } z \lessgtr 0. \tag{77}$$

From Eq. (76), the two spheres with centers $(0, 0, A/2)$ and $(0, 0, -A/2)$ have opposite phases. Thus, we show them with different contrast in Fig. 3.

To conclude this section, we show all the forms of the $l = 1$ members i.e., the $|x>$, $|y>$, and $|z>$ states, and their relations in Fig. 4, as obtained from Eqs. (69), (70) and (77).

5. $l = 2$ (d-state) Case

Next we consider the $l = 2$ case. The matrix elements of the angular moment can be calculated as

$$[\mathbf{L_j}]_{mn} \equiv <2, m|\hat{L}_j|2, n>, \quad (j = x, y, z) \tag{78}$$

with the notation

$$|2,2> = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\ |2,1>= \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\ |2,0>= \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},$$

$$|2,-1> = \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix},\ |2,-2>= \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}. \tag{79}$$

Then we have the explicit matrix forms

$$\mathbf{L_x} = \begin{pmatrix}0 & 1 & 0 & 0 & 0\\ 1 & 0 & \sqrt{6}/2 & 0 & 0\\ 0 & \sqrt{6}/2 & 0 & \sqrt{6}/2 & 0\\ 0 & 0 & \sqrt{6}/2 & 0 & 1\\ 0 & 0 & 0 & 1 & 0\end{pmatrix}, \tag{80}$$

$$\mathbf{L_y} = i\begin{pmatrix}0 & -1 & 0 & 0 & 0\\ 1 & 0 & -\sqrt{6}/2 & 0 & 0\\ 0 & \sqrt{6}/2 & 0 & -\sqrt{6}/2 & 0\\ 0 & 0 & \sqrt{6}/2 & 0 & -1\\ 0 & 0 & 0 & 1 & 0\end{pmatrix}, \tag{81}$$

$$\mathbf{L_z} = \begin{pmatrix}2 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -2\end{pmatrix}. \tag{82}$$

Then the rotation matrices around the x, y, and z axes are calculated by the Taylor expansion of the matrix-valued exponents, the same way as the case of $l = 1$:

$$e^{-i\mathbf{L_x}\phi} = \begin{pmatrix}A & iB & C & iD & E\\ iB & F & iG & H & iD\\ C & iG & J & iG & C\\ iD & H & iG & F & iB\\ E & iD & C & iB & A\end{pmatrix}, \tag{83}$$

$$e^{-i\mathbf{L_y}\phi} = \begin{pmatrix}A & B & -C & -D & E\\ -B & F & G & -H & -D\\ -C & -G & J & G & -C\\ D & -H & -G & F & B\\ E & D & -C & -B & A\end{pmatrix}, \tag{84}$$

$$e^{-i\mathbf{L_z}\phi} = \begin{pmatrix}e^{-2i\phi} & 0 & 0 & 0 & 0\\ 0 & e^{-i\phi} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & e^{+i\phi} & 0\\ 0 & 0 & 0 & 0 & e^{+2i\phi}\end{pmatrix}, \tag{85}$$