Original Paper *Forma*, **17**, 1–17, 2002

A Large Class of Random Tessellations with the Classic Poisson Polygon Distributions

Roger E. MILES[1]* and Margaret S. MACKISACK[2]

[1] *1476 Sutton Road, Sutton, NSW 2620, Australia*
[2] *Department of Mathematical Sciences, University of Technology Sydney, P.O. Box 123 Broadway, NSW 2007, Australia*
**E-mail address: fgh@netspeed.com.au*

(Received March 18, 2002; Accepted June 17, 2002)

Keywords: Random Tessellation of Convex Polygons, Poisson Anisotropic Lines in the Plane, Ergodic Polygon Distributions, Rectangular Tessellation, Superposition and Nesting

Abstract. One special case of Arak, Clifford and Surgailis' 1993 point-based polygon models for random graphs yields an isotropic random tessellation of convex polygons, with all vertices T-vertices. It is shown that its polygon distributions coincide with those of the random tessellation determined by Poisson isotropic random lines in the plane, for which all vertices are X-vertices (cf. Fig. 4). This surprising property extends to general orientation distributions, e.g. to rectangular tessellations stemming from a two atom distribution. Applying this property, it is shown that a wide variety of distinct random tessellations obtained from these two by superposition, nesting, etc. possess those very same polygon distributions.

1. Introduction and Summary

To begin (Sec. 2), basic properties of homogeneous Poisson lines **P** in the plane R^2 with orientation distribution Θ, and the random tessellation $\mathscr{P}$ of convex polygons they determine, are developed. Stemming from **P**, next (Sec. 3) the stochastic process $\mathbf{A}(C)$ of interconnected line segments—"I-segments"—with orientation distribution Θ within an arbitrary convex domain C of R^2 is specified, employing an advancing fixed orientation line (AFOL). The probability element of its realisation within C is determined, thus incidentally showing that the orientation of the AFOL is immaterial for this stochastic construction. Moreover $\mathbf{A}(C)$ is shown to be consistent in the sense that, for any $C' \supset C$, the restriction of $\mathbf{A}(C')$ to C is stochastically equivalent to $\mathbf{A}(C)$. A complete stochastic analysis of the I-segments of $\mathbf{A}(C)$ is carried out.

Ignoring edge effects, $\mathbf{A}(C)$ has the effect of partitioning C into a random tessellation $\mathscr{A}(C)$ of convex polygons, with all vertices being T-vertices, the intersections of pairs of I-segments. It is shown that the probability element for any given convex polygon coincides with the corresponding element for $\mathscr{P}$, so demonstrating that their polygon distributions are identical.

To conclude (Sec. 4), a wide variety of distinct new random tessellations, generated from these two ($\mathscr{P}$, $\mathscr{A}$) by superposition, nesting and/or partition and restitution of Θ, are shown to possess the same "Poisson" polygon distributions.

Certain of our theory duplicates that of ARAK *et al*. (1993), but since the tessellations of interest are but a special case of their rather complex general model, our specific and hence simpler self-contained treatment may be valuable as a simpler speedy access to this area for the general reader.

History. This paper has an interesting history. In MACKISACK and MILES (1996) we investigated several models for homogeneous rectangular tessellations in R^2 with exclusively T-vertices, the main one of which—Gilbert's—proved to be rather intractable. Subsequent to that, we arrived at a fully tractable such tessellation with rather nice properties. This was presented at meetings in 1995 (MACKISACK and MILES , 1996) and 1997 (MILES, 1998), at the first of which its similarity to one of the models of ARAK *et al*. (1993) was pointed out to us. The only difference transpired to be our two atom orientation distribution versus their isotropic one. In fact our theory was found to extend to the case of *arbitrary* orientation distributions, as presented here. Thus we are happy to acknowledge the original presentation of this splendid model in ARAK *et al*. (1993). Our most important contribution, in this paper, is an analysis of its specific properties, with identification of its distributions as those of the classic random tessellation determined by Poisson lines in R^2.

2. Anisotropic Poisson Line Process in the Plane

First we specify (infinite unoriented) lines in the plane R^2 by

$$G = G(h,\theta) \quad (-\infty < h < \infty,\ \ 0 \le \theta < \pi),$$

where (h, θ) are polar coordinates of the perpendicular from the origin O to the line. The relevant invariant integral geometric density (SANTALÓ, 1976) is $dG = dhd\theta$. We write $\mathbf{P}_\lambda(0, d)$ (d = 1, 2, ...) for a Poisson point process of intensity λ in d-dimensional euclidean space R^d (notationally this conforms to " $\mathbf{P}(s, d)$" for Poisson s-flats in R^d, cf. MILES (1971)). Then we define the *anisotropic Poisson line process*

$$\mathbf{P} \equiv \mathbf{P}_\tau^\Theta(1,2)$$

in R^2 of intensity τ and orientation distribution Θ by $\{G_i\}$, i.e. $\{h_i, \theta_i\}$, where

(i) $\{h_i\}$ is $\mathbf{P}_\tau(0, 1)$ on R^1; and, independently,

(ii) $\{\theta_i\}$ are independent identically distributed from Θ.

Thus $\{h_i, \theta_i\}$ is, in general, an inhomogeneous Poisson point process in the strip $-\infty < h < \infty$, $0 \le \theta < \pi$. The isotropic case corresponds to Θ uniform on $[0, \pi)$, in which case $\{h_i, \theta_i\}$ is (homogeneous) $\mathbf{P}_{\tau/\pi}(0, 2)$ in the same strip. The distribution Θ is quite general, but we exclude the degenerate case in which its probability is concentrated on a single point (in which case no line/line intersections may occur).

The basic "prob el" (probability element) applicable to $\mathbf{P}$ is

$$\begin{aligned} dp &\equiv dp(h,\theta) \\ &= \Pr\{\text{there is a line of } \mathbf{P} \text{ within } (dh, d\theta)\} \\ &= \tau dh \Theta(d\theta) \end{aligned}$$

(=$(\tau/\pi)dhd\theta$ in the isotropic case). In what follows we shall make much use of the "complete independence" of Poisson processes. It is of interest that the (isotropic) **P** occurs as a limiting example of ARAK *et al*. (1993)'s general model (their Case 1).

We next develop some of the basic properties of **P**.

P *is homogeneous*. That is, **P** is stochastically invariant under arbitrary translations in R^2. This stems from its construction, specifically the homogeneity of $\{h_i\}$, and $\{\theta_i\}$ being independent identically distributed.

Hitting distributions for **P**. Let C be a bounded convex domain in R^2 and

$$V(C) = \{(h,\theta): G(h,\theta) \cap C \neq \varnothing\},$$

so that the number of lines of **P** hitting C equals the number of points $\{h_i, \theta_i\}$ within $V(C)$, which is Poisson distributed, with mean value

$$\begin{aligned} \iint_{V(C)} dp &= \iint_{V(C)} \tau dh \Theta(d\theta) \\ &= \tau \int_0^\pi w_C(\theta)\Theta(d\theta) \end{aligned}$$

where $w_C(\theta)$ is the width of C at orientation θ

$$\equiv \tau\overline{C}, \quad \text{say;} \tag{1}$$

i.e. $\overline{C}$ is the Θ-weighted mean width of C.

Examples. (i) In the isotropic case $\overline{C} = S(C)/\pi$, where S denotes perimeter; (ii) For a disc $Q(q)$ of radius q, $\overline{Q(q)} = 2q$ for all Θ.

For a line segment L of length l and orientation ϕ,

$$\overline{L} = l\chi(\phi) \tag{2}$$

where

$$\chi(\phi) \equiv \int_0^\pi \langle\theta,\phi\rangle\Theta(d\theta)$$

and

$$\langle \theta, \phi \rangle \equiv |\sin(\theta - \phi)|.$$

Note $\chi(\phi) = 2/\pi$ in the isotropic case.

Line sections of **P**. Consider the point process section of **P** by an arbitrary line $G(b, \phi)$ with orientation ϕ. The prob el dp may alternatively be expressed in terms of (u, θ), where u measures length along $G(b, \phi)$ from some arbitrary origin, by

$$\begin{aligned} dp &= \tau dh \Theta(d\theta) \\ &= \tau du \langle \theta, \phi \rangle \Theta(d\theta), \end{aligned} \tag{3}$$

from which we conclude that the section is a **P**(0, 1) with intensity $\tau\chi(\phi)$; the orientation distribution of the line through each point of this process being given by

$$\Pr\{d\theta | b, \phi\} \propto \langle \theta, \phi \rangle \Theta(d\theta). \tag{4}$$

These two properties permit a stochastic construction of **P** with respect to any given $G(b, \phi)$. In fact, a given line of **P** itself also intersects the rest of **P** in exactly the same way, i.e. same point process and (independent) orientations.

The planar point process **P**|**P** *of X-vertices of* **P**. That is, the aggregate of line/line intersection points for **P**. Appealing to Poisson independence, the joint **P** prob el for two lines in R^2 is

$$\begin{aligned} dp_1 dp_2 &= \tau dh_1 \Theta(d\theta_1) \cdot \tau dh_2 \Theta(d\theta_2) \\ &= \tau^2 da \langle \theta_1, \theta_2 \rangle \Theta(d\theta_1) \Theta(d\theta_2), \end{aligned} \tag{5}$$

where da is an area element of R^2. It follows that the intensity of **P**|**P** is

$$\rho_{\mathbf{P}} = \frac{1}{2} \cdot \tau^2 \int_0^\pi \int_0^\pi \langle \theta_1, \theta_2 \rangle \Theta(d\theta_1) \Theta(d\theta_2), \tag{6}$$

the factor 1/2 stemming from the double representation pertaining; moreover, the prob el of the joint orientation distribution of the two lines through a uniform random point of **P**|**P** is

$$\propto \langle \theta_1, \theta_2 \rangle \Theta(d\theta_1) \Theta(d\theta_2).$$

Note $\rho_{\mathbf{P}} = \tau^2$ in the isotropic case.

The random tessellation $\mathscr{P}$. **P** has the effect of partitioning R^2 into an aggregate $\mathscr{P}$ of random convex polygons—a random tessellation—the characteristics of which conform to (almost sure ergodic) distributions (MILES, 1973), e.g.

(i) the distribution of in-radii is exponential (2τ);

(ii) the conditional distribution of $\overline{D}$ for a uniform random member D of $\mathcal{P}$, given the number N of its sides or vertices, is $\Gamma(N-2,\ \tau)$, i.e. the distribution of the sum of $N-2$ independent exponential (τ) random variables.

Many (ergodic) moments are also known, especially in the isotropic case (MILES, 1986). Our main interest in this paper is in this and various other such random tessellations of R^2.

The probability element for polygonal cells of $\mathcal{P}$. Consider an arbitrary convex polygon D in R^2, which suppose has N sides or vertices —i.e. a convex N-gon. We may represent D by its line segment sides L_i ($i = 1, \ldots, N$) or, more simply, by the lines containing them:

$$D = D(G_1, \ldots, G_N).$$

The corresponding **P** prob el, by complete Poisson independence, is

$$\begin{aligned}\Pr\{dD\} &= \Pr\{\text{there is a cell of } \mathcal{P} \text{ within } (dG_1,\ \ldots,\ dG_N)\}\\ &= \Pr\{\text{there are lines of } \mathbf{P} \text{ within each of}\\ &\qquad\qquad dG_1,\ \ldots,\ dG_N \textit{ and } \text{ there are no (other) lines of } \mathbf{P} \text{ hitting } D\}\\ &= \left(\prod_1^N dp_i\right)\prod_{V(D)}(1-dp).\end{aligned}$$

Again, by Poisson independence,

$$\begin{aligned}\prod_{V(D)}(1-dp) &= \exp\left\{-\tau\iint_{V(D)} dh\Theta(d\theta)\right\}\\ &= \exp\left(-\tau\overline{D}\right).\end{aligned}$$

Thus we have the *basic prob el*

$$\Pr\{dD\} = \exp\left(-\tau\overline{D}\right)\prod_1^N dp_i \tag{7}$$

for $\mathcal{P}$.

3. ACS (Arak-Clifford-Surgailis) Random Tessellations

Specification of the line segment process **A**(C) *within an arbitrary bounded convex domain C*. This specification is best explained by way of a typical realisation (Fig. 1).

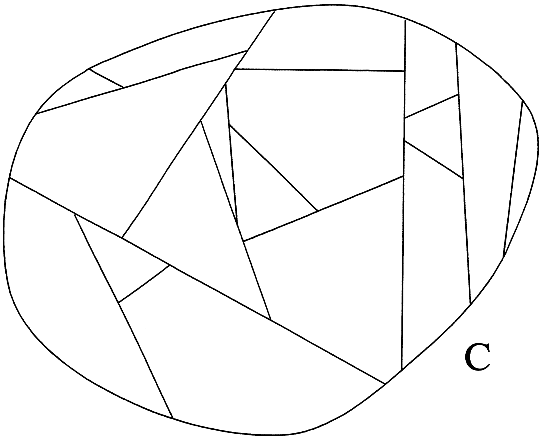

Fig. 1. Illustrative realisation of $\mathbf{A}(C)$.

Thus the realisation comprises a number of interconnecting line segments within C. Each end of each such segment is either a T-junction with another segment or an intersection with the boundary ∂C of C. When terminated by T-junctions at both ends, we refer to these segments as I-segments (MACKISACK and MILES, 1996).

$$\mathbf{A}(C) \equiv \mathbf{A}_\tau^\Theta(C)$$

is now specified in terms of the Poisson line process $\mathbf{P} = \mathbf{P}_\tau^\Theta$ discussed above. Consider a fixed orientation line $G(t, \phi)$. Allowing t to increase from $-\infty$ to ∞, we have an advancing fixed orientation line (AFOL); it may be useful to think of t as a time variable. Let $t_0 < t_1$ specify the tangent positions of $G(t, \phi)$ with C. Let $\mathbf{P}(C)$ denote those lines of $\mathbf{P}$ which intersect C, and $\mathbf{R}_\phi^-(C)$ denote the ray parts of them extending from $t = -\infty$ to their first contact with C. $\mathbf{A}_\phi(C)$ stems from the extension of the members of $\mathbf{R}_\phi^-(C)$ into C, subject to the following unfolding random mechanism within C, as t increases from t_0 to t_1.

(A_1) Within $(t, t + dt)$ any advancing (i.e. relative to $G(t, \phi)$) segment $L(\psi)$ with orientation ψ may "give birth" to a new advancing segment, having prob el

$$\frac{1}{2}dp \quad \left(t_0 \le t \le t_1\right),$$

i.e. with a prob el relative to $L(\psi)$ of $(1/2)[\tau du\langle\psi, \theta\rangle\Theta(d\theta)]$, where u measures length along $L(\psi)$ (cf. Equations (3) and (4) above).

(A_2) If within dt any two advancing segments meet, then a fair coin toss (probabilities 1/2, 1/2) determines which of them advances and which is blocked ("dies").

One minor restriction is necessary: the orientation ϕ of the AFOL must not coincide with a positive atom of Θ's probability. This completes the specification of $\mathbf{A}_\phi(C)$. In the isotropic case it is the model presented in Case 3 and figure 6 of ARAK *et al*. (1993).

It is also useful to define the combined line/line segment process $\mathbf{PA}(C)$ over R^2 as follows. Suppose advancing line segments within C reaching ∂C extend outside C as rays to $t = \infty$, the union of which denote by $\mathbf{R}_\phi^{+}(C)$. Then, letting $\mathbf{P}^c(C)$ denote the lines of $\mathbf{P}$ *not* intersecting C, define

$$\mathbf{PA}_\phi(C) = \mathbf{A}_\phi(C) \cup \mathbf{R}_\phi^{-}(C) \cup \mathbf{R}_\phi^{+}(C) \cup \mathbf{P}^c(C).$$

Thus, as t increases from $-\infty$, $\mathbf{PA}_\phi(C)$ coincides with $\mathbf{P}$ up to t_0, but thereafter is modified by the presence of C. It is a segment process within, and a line process outside, C. The reader should be satisfied as to how, for example, Fig. 1 is sequentially generated by the AFOL; and that it could well have been thus generated with respect to (almost) any ϕ-value in $[0, \pi)$. Ignoring edge effects near ∂C, $\mathbf{A}_\phi(C)$ has the effect of partitioning C into an aggregate $\mathscr{A}_\phi(C)$ of convex polygons. Likewise $\mathbf{PA}_\phi(C)$ generates the random tessellation $\mathscr{PA}_\phi(C)$ of R^2 (with no such edge effects!).

Probability element for a given configuration of $\mathbf{A}_\phi(C)$. Suppose the given configuration, e.g. that of Fig. 1, comprises n line segments:

$$B = (L_1, \ldots, L_n)$$

where L_i ($\subset$line G_i) has length l_i and orientation ψ_i. We now derive its prob el $\Pr\{dB\}$, making abundant use of the complete Poisson independence prevailing. Thus $\Pr\{dB\}$ stems from the product of the following individual independent prob els:

(a) Πdp_i over all advancing $\mathbf{P}$ lines entering C ($\mathbf{R}_\phi^{-}(C)$);

(b) $\Pi(1 - dp)$ over all advancing rays entering C which *don't* "carry" a line of $\mathbf{P}$;

(c) $\Pi \frac{1}{2}$ over all blocks occurring within C;

(d) $\Pi\left(\frac{1}{2} dp_j\right)$ over all births occurring within C; and

(e) $\Pi\left(1 - \frac{1}{2} dp\right)$ over all possible births which *don't* take place within C.

In other words

$$\Pr\{dB\} = \left(\frac{1}{2}\right)^m \exp\left(-\iint_{V(C)} dp\right) \prod_1^n \left\{\exp\left(-\frac{1}{2}\iint_{V(L_i)} dp\right)\right\} \prod_1^n dp_i, \tag{8}$$

where m is the total number of T-junctions in B. Applying Eq. (1), this reduces to

$$\Pr\{dB\} = \left(\frac{1}{2}\right)^m \exp\left(-\tau\overline{C}\right) \exp\left(-\frac{\tau}{2}\sum_1^n \overline{L_i}\right) \prod_1^n dp_i. \tag{9}$$

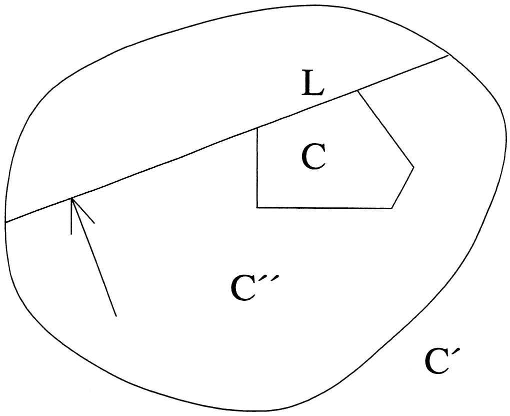

Fig. 2. Illustrating the argument leading to the Consistency Theorem.

Since this is independent of the fixed angle ϕ of the AFOL, we have the *Optional Orientation Theorem*. $\mathbf{A}_\phi(C)$ does not depend upon ϕ, and hence may be written $\mathbf{A}(C)$. Similarly we may write $\mathscr{A}(C)$, $\mathbf{PA}(C)$, etc.

Corollary. Suppose T is any given tangent to C and $H(T)$ is the half-plane bounded by T which does not contain C. Then $\mathbf{PA}(C)$ within $H(T)$ is stochastically equivalent to $\mathbf{P}$ within $H(T)$.

Consistency. Suppose C, C' are bounded convex domains with $C \subset C'$, and $\mathbf{A}(C|C')$ denotes the restriction of $\mathbf{A}(C')$ to C.

Consistency Theorem. $\mathbf{A}(C|C')$ is stochastically equivalent to $\mathbf{A}(C)$, for all $C' \supset C$.

Proof. Suppose C is a convex N-gon, let L be one of its sides, $H(L)$ be the half-plane bounded by L containing C, and $C'' = H(L) \cap C'$ (Fig. 2). Now, generating $\mathbf{A}(C)$ in the arrowed direction orthogonal to L, as is allowable by the Optional Orientation Theorem, we see that $\mathbf{A}(C''|C')$ is s.e. (stochastically equivalent) to $\mathbf{A}(C''|C'')$ and hence $\mathbf{A}(C|C')$ is s.e. to $\mathbf{A}(C|C'')$. This argument may be repeated another $N-1$ times, to show that $\mathbf{A}(C|C')$ is s.e. to $\mathbf{A}(C, C) = \mathbf{A}(C)$. Clearly this result extends from convex N-gons to general bounded convex C.

Corollary. $\mathbf{A}(C)$ is homogeneous in the sense that, if $C_1 \subset C$, $C_2 \subset C$ with C_1, C_2 translates of each other, then $\mathbf{A}(C_1|C)$ is stochastically equivalent to $\mathbf{A}(C_2|C)$, i.e. after translation.

Arbitrary line section of $\mathbf{A}(C)$. Suppose the line G (of orientation ϕ) partitions C into the two convex subdomains C_1, C_2. Write a_1, a_2 for the opposite advance directions orthogonal to G, a_i being through C_i towards G ($i = 1, 2$). Now, by the stochastic construction, the section by G of $\mathbf{A}(C)$ is s.e. to the section by G of $\mathbf{A}(C_2)$ via the advance a_2. But, by the Optional Orientation Theorem, this is s.e. to the section by G of $\mathbf{A}(C_2)$ via the advance a_1 which, again by the stochastic construction, coincides with the corresponding $\mathbf{P}$ result (Eqs. (3) and (4)), viz. $\mathbf{P}_{\tau\chi(\phi)}$ (0, 1).

Probability element for the T-junction vertices of $\mathbf{A}(C)$. By the Consistency Theorem and homogeneity, we may take as our configuration B a T-junction within C taken as a disc $Q(\varepsilon)$

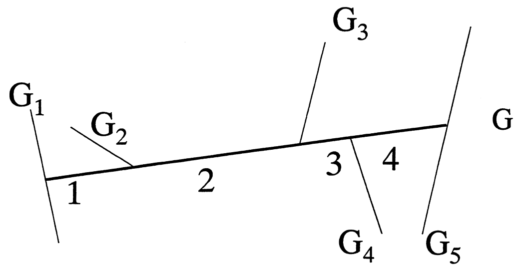

Fig. 3. An I-segment, with its associated J- and K-segments.

of radius ε centred at the T-junction. Let the "top" (horizontal) and "bottom" (vertical) bars of the "T" lie within lines G_1, G_2 respectively. Then, by Eq. (9) with $m = 1$, $n = 2$, the corresponding prob el is

$$\frac{1}{2}\left\{-\tau\overline{Q(\varepsilon)}\right\}\exp\left(-\frac{\tau}{2}\sum_1^2\overline{L_i}\right)\prod_1^2 dp_i.$$

Going to the limit $\varepsilon \to 0$, the desired prob el is

$$dp_1 \cdot \frac{1}{2} dp_2 \tag{10}$$

which may be re-expressed by means of Eq. (5). Allowing the top/bottom to be G_2/G_1, Eq. (10) is doubled, yielding identity between the **P** and **A** values. Thus, writing **A**|**A** for the point process of such T-vertices, geometrical considerations imply that its intensity $\rho_{\mathbf{A}} = 2\rho_{\mathbf{P}}$, $\rho_{\mathbf{P}}$ being given by Eq. (6).

Geometry of I-segment structures. Besides I-segments there are also J- and K-segments, best illustrated by an example (Fig. 3). See also MACKISACK and MILES (1996, Sec. 2). J-segments are the sides of the polygons of $\mathscr{A}(C)$, while K-segments are the common boundaries of adjacent polygons of $\mathscr{A}(C)$. Thus, in Fig. 3, the single I-segment contains five distinct J-segments (1, 2, 34; 123, 4) and four K-segments (1, 2, 3, 4).

Such a homogeneous random structure engenders certain geometric expectation identities (MACKISACK and MILES, 1996, Sec. 3). Thus, since each T-junction serves both as an "intermediate" point and an "end" point,

$$E\{\text{number of T-junctions along a uniform random I-segment}\} = 2.$$

Moreover, for their lengths (subject to uniform weighting),

$$E\{l_I\} = 2E\{l_J\} = 3E\{l_K\}. \tag{11}$$

Probability element for the I-segments of **A(**C). To find the prob el for a given I-segment occurring within **A(**C), we proceed as for the T-junction above, but for simplicity omit the shrinking of a containing convex domain (C) down to the I-segment itself (C_0). The geometry and notation are illustrated in Fig. 3, where the I-segment joins $G \cap G_1$ and $G \cap G_5$ ($n = 5$) and each of G_2, G_3 and G_4 support a line segment, each of which may lie on either side of G. Thus the associated configuration

$$B_n = \left(G; G_1, G_2^{\pm}, \ \ldots, \ G_{n-1}^{\pm}, G_n\right)$$

where ± relates to the side of G. Let $G = G(h, \phi)$, the orientations of $G_1, \ldots, G_n$ be $\theta_1, \ldots, \theta_n$ and $u_1, \ldots, u_n$ locate their intersections with G (from some arbitrary origin on G), so the I-segment length $I = u_n - u_1$. Then, by Eq. (9),

$$\Pr\left\{dB_n\right\} = \left(\frac{1}{2}\right)^n dp\left(\prod_1^n dp_i\right)\exp\left(-\tau\overline{C_0}\right)\exp\left(-\frac{\tau}{2}\overline{C_0}\right)$$

which, by Eqs. (2)–(4),

$$= \left(\frac{\tau}{2}\right)^n dp\left[\prod_1^n \langle\phi, \theta_i\rangle du_i \Theta\left(d\theta_i\right)\right]\exp\left\{-\frac{3\tau}{2}\chi(\phi) I\right\}$$

(2^{n-2} relations, corresponding to the ±'s). The θ_i dependences enter in an independent way, so may be integrated out:

$$\begin{aligned}
&\Pr\left\{dG; du_1, \ \ldots, \ du_n\right\} \\
&= \left\{\tau\chi(\phi)/2\right\}^n \exp\left\{-\frac{3\tau}{2}\chi(\phi)\left(u_n - u_1\right)\right\} dp \prod_1^n du_i \\
&= \left(\kappa(\phi)/3\right)^n \exp\left\{-\kappa(\phi)\left(u_n - u_1\right)\right\} dp \prod_1^n du_i
\end{aligned}$$

where

$$\kappa(\phi) \equiv 3\tau\chi(\phi)/2$$

(=$3\tau/\pi$ in the isotropic case). Thus

$$\begin{aligned}\Pr\left\{du_1,\ \ldots,\ du_n|G\right\} &= \left(\kappa(\phi)/3\right)^n \exp\left\{-\kappa(\phi)(u_n-u_1)\right\}\prod_1^n du_i\\ &= \frac{1}{3}\kappa(\phi)du_1 \quad \left[\frac{1}{3}\cdot\kappa(\phi)\exp\left\{-\kappa(\phi)(u_2-u_1)\right\}du_2\right]\\ &\qquad \left[\frac{1}{3}\cdot\kappa(\phi)\exp\left\{-\kappa(\phi)(u_3-u_2)\right\}du_3\right]\\ &\qquad \vdots\\ &\qquad \left[\frac{1}{3}\cdot\kappa(\phi)\exp\left\{-\kappa(\phi)(u_n-u_{n-1})\right\}du_n\right].\end{aligned}$$

This expresses the prob el of such an I-segment in sequential form; save for the angles θ_i, which are independent, with the usual prob els $\propto\langle\phi,\ \theta_i\rangle\Theta(d\theta_i)$ (Eq. (4)).

It is natural to couch these prob els in terms of probability *distributions*. Thus, starting from $G \cap G_1$ the successive intersections $G \cap G_i$ conform to a Poisson process with intensity $\kappa(\phi)$, the factors 1/3 reflecting the three choices at each intersection: birth, block and terminate. In other words, the K-segments have exponential $\{\kappa(\phi)\}$ distributions, and n has a geometric (1/3) distribution. From these two facts it follows that I has an exponential $\{\kappa(\phi)/3\}$ distribution. The condition for a J-segment is that successive probability 1/3 events, birth or block, be the same at all its intersections, from which it follows that the J-segments have an exponential $\{2\kappa(\phi)/3\}$ distribution.

As for mean lengths, $\mu_I = 2\mu_J = 3\mu_K$, agreeing with the more general result equation (24) in MACKISACK and MILES (1996)—Eq. (11) above. One may also show that, associated with a uniform random I-segment,

$$E\{\text{number of J-segments (summed over both sides)}\} = 4,$$

$$E\{\text{number of K-segments}\} = 3.$$

To summarize, for an I-segment with given orientation ϕ,

$$\left.\begin{array}{l}\text{I is exponential }(\lambda)\\ \text{J - values are exponential }(2\lambda)\\ \text{K - values are exponential }(3\lambda)\end{array}\right\} \qquad (12)$$

($\lambda \equiv \kappa(\phi)/3$ or $\tau\chi(\phi)/2$).

To combine these results for all I-segments in a large domain C, one must weight accordingly. For example, if $f_\phi(X)$, $f(X)$ are the densities of $X = I,\ J,\ K$ then

$$f(X)=\frac{\int_0^\pi f_\phi(X)\kappa(\phi)\Theta(d\phi)}{\int_0^\pi \kappa(\phi)\Theta(d\phi)}$$
$$=\frac{\int_0^\pi\int_0^\pi f_\phi(X)\langle\theta,\phi\rangle\Theta(d\theta)\Theta(d\phi)}{\int_0^\pi\int_0^\pi\langle\theta,\phi\rangle\Theta(d\theta)\Theta(d\phi)},$$

the denominator being $2\rho_{\mathbf{P}}/\tau^2 = \rho_{\mathbf{A}}/\tau^2$.

Isotropic case. Such complexity disappears in the isotropic case, where $\kappa(\phi) = 3\tau/\pi$. Thus, for the *totality* of such segments, I, J and K have exponential distributions with parameters τ/π, $2\tau/\pi$ and $3\tau/\pi$, respectively.

Probability element for a given N-gon polygonal cell of $\mathscr{A}(C)$. The derivation closely parallels that for I-segments, with the domain C being shrunk down to the N-gon D itself. To render D as a valid configuration B within C (=D!), at each of its N vertices extend one (arbitrary) edge outside C. Now $\Pr\{dD\}$ is the product of two factors, obtained as above (Eq. (8)):

(a) The prob el for the N sides of D is

$$\left(\frac{1}{2}\right)^N\prod_1^N dp_i.$$

(b) To take account, in an element of $\partial C = \partial D$, of:

(i) external **P** lines not penetrating D: a factor $1 - dp\cdot(1/2)$;

(ii) no internal births: a factor $1 - (1/2)dp$;

the combined effect of which is $1 - dp$. The corresponding prob el is the product of this over all (line segment) secants of D, viz

$$\exp\left\{-\tau\iint_{V(D)} dh\Theta(d\theta)\right\}=\exp\left(-\tau\overline{D}\right).$$

Combining (a) and (b), and summing over all 2^N choices of edge extensions, the required prob el

$$\Pr\{dD\}=2^N\left(\frac{1}{2}\right)^N\left(\prod_1^N dp_i\right)\exp\left(-\tau\overline{D}\right)$$
$$=\exp\left(-\tau\overline{D}\right)\prod_1^N dp_i. \tag{13}$$

Note that no restriction whatever has been placed here on the **P** lines blocked by D, and that *it coincides with the corresponding prob el* $\Pr\{dD\}$ *for* $\mathscr{P}$ (Eq. (7)). Note also that, as is *now* expected, the prob el of J-segments of $\mathbf{A}(C)$ with orientation ϕ (Eq. (12)) coincides with that of the sides $\mathscr{P}$ with orientation ϕ (Eqs. (3) and (4)).

Rectangular tessellations. In MACKISACK and MILES (1996) we investigated various stochastic models of homogeneous rectangular tessellations, in which each cell was an (aligned) rectangle and, desirably, all vertices were T-vertices. Only after its publication did we discover what turned out to be $\mathbf{A}_\tau^\Theta$, with Θ concentrated on two orthogonal atoms:

$$\left.\begin{array}{l}\xi \text{ at } \theta = 0\\ \eta \text{ at } \theta = \pi/2\end{array}\right\}\quad (\xi + \eta = 1).$$

Without a doubt this seems to us to be the nicest and most tractable nontrivial model possible for a rectangular tessellation. For it,

$$\kappa(\phi) = 3\tau\chi(\phi)/2 = (3\tau/2)(\xi|\sin\phi| + \eta|\cos\phi|).$$

The sides of a uniform random rectangle are independent exponential random variables, with mean values $1/\tau\xi$, $1/\tau\eta$; as they are, of course, for the corresponding, but somewhat trivial, $\mathscr{P}$ rectangular tessellation formed by two families of orthogonal random lines.

4. Further Random Tessellations Derived by Superposition, Nesting, etc.

In the previous section we have seen that the stochastic structures of $\mathbf{A}(C)$ (T-junctions, I-segments, polygons) have precisely the same prob els as those of the corresponding structures for **P** in R^2. For example, the prob el for the polygon of $\mathscr{P}$ containing O coincides with that of the polygon D of $\mathscr{A}(C)$ containing a fixed point of C (provided $D \subset$ sufficiently large C). In other words, they have the same stochastic construction: that which is implicit in the definition of **P** in Sec. 2. Since C is an arbitrary bounded convex domain of R^2, these considerations beg the question: What can be said about the empiric distributions of $\mathscr{A}(Q(q))$ as $q \to \infty$ (assuming edge effects near $\partial Q(q)$ become asymptotically negligible)? Do they coincide with the corresponding limit values for $\mathscr{P}(Q(q))$ explored elsewhere (MILES, 1964, 1973)? In fact a parallel theory for $\mathscr{A}(Q(q))$ may be set out, the successive steps being similar to those for $\mathscr{P}(Q(q))$. Because of this, and because its inclusion here would upset the balance of the paper, also rendering it considerably longer, it is omitted. The interested reader is referred to the cognate references MILES (1970, 1971, 1974). Thus the empiric distributions of $\mathscr{A}(Q(q))$ converge almost surely to the same distributions as $\mathscr{P}(Q(q))$, since these limit distributions coincide with the (identical, i.e. $\mathscr{A} \sim \mathscr{P}$) prob els above.

Ergodic bridge. The ergodic essence of these results is epitomised by an "ergodic bridge" (MILES, 1995). Write A for area and Z for a general translation invariant continuous (possibly vector) characteristic of a convex polygon. Then we may write $^\circ f(A, Z)$ for the joint p.d.f. of (A, Z) for the polygon $^\circ D$ of $\mathscr{P}$ containing O. Writing $f(A, Z)$ for the limit

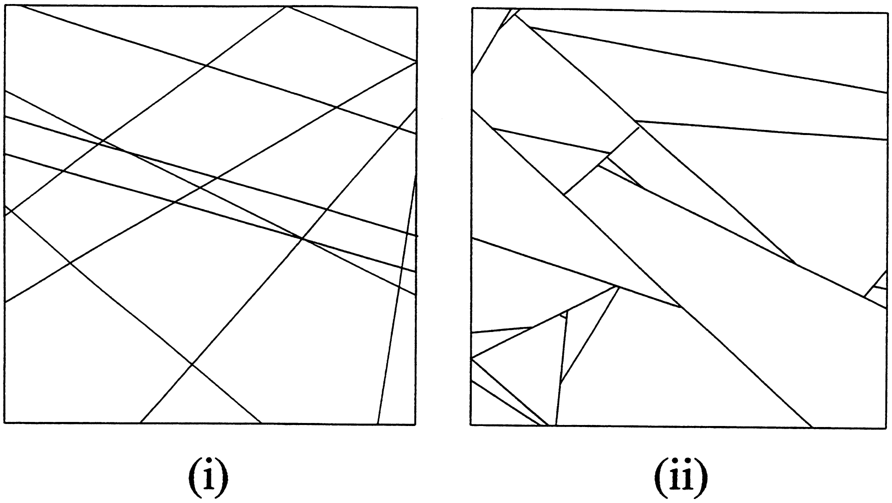

Fig. 4. Simulations of (i) **P**, $\mathscr{P}$ and (ii) **A**, $\mathscr{A}$ with common intensity τ and orientation distribution Θ (having density $4/(5\pi)$ for $\theta \in (\pi/8, \pi/2)$ and $14/(5\pi)$ for $\theta \in (3\pi/4, \pi)$). The polygons of $\mathscr{P}$ and $\mathscr{A}$ have the same distributions!

(ergodic) almost sure p.d.f of (A, Z) for the equi-weighted aggregate of polygons of $\mathscr{P}(Q(q))$ as $q \to \infty$, we have the basic *ergodic bridge* relation

$$^\circ f(A,Z) = \frac{Af(A,Z)}{E(A)},$$

where $E(A)$ is the mean value of A with respect to f. This relation stems from the fact that O is like a "uniform random point" relative to the homogeneous **P**, so that the "probability" that it falls in any polygon is $\propto$ the area of the polygon. It is an ergodic bridge in the sense that knowledge of $^\circ f$, depending upon a single polygon for all realisations, interrelates with that of f, depending (almost surely) upon all polygons for a single realisation. Knowledge of either determines, in principle, knowledge of the other. Since the prob els $\Pr\{dD\}$ for $\mathscr{P}$ and $\mathscr{A}$ coincide (Eqs. (7) and (13)), it follows that $^\circ f$ for $\mathscr{P}$ and $\mathscr{A}$ coincide; and hence also, by the ergodic bridge, that f for $\mathscr{P}$ and $\mathscr{A}$ coincide. For example, the distribution of in-radii and the conditional distribution of $\overline{D}$ given N mentioned in Sec. 2.

By way of illustration, Fig. 4 shows simulations of **P**, $\mathscr{P}$, and **A**, $\mathscr{A}$ in a square, with the same τ and Θ values. (We are indebted to Hao He for the **A**, $\mathscr{A}$ simulation.)

Now we have the following stochastic construction equivalent to the definition of **P** $= \mathbf{P}_\tau^\Theta(1, 2)$ in Sec. 2. A stochastic construction of $^\circ D$ for either $\mathbf{P}_\tau^\Theta$ or $\mathbf{A}_\tau^\Theta$ is as follows. Suppose $\{h_i\}$ are the points of a $\mathbf{P}_\tau(0, 1)$ and $\{\theta_i\}$ are independent identically distributed from Θ. Writing H_i for the half-plane bounded by $G(h_i, \theta_i)$ which contains O, we have $\cap H_i$ is stochastically equivalent to $^\circ D$. This property is now repeatedly exploited, to exhibit a wealth of further random tessellations possessing these very same distributions. To ease the

notation in the sequel, we write $\mathbf{P}[\tau, \Theta]$ for $\mathbf{P}_\tau^\Theta$, $\mathbf{A}[\tau, \Theta]$ for $\mathbf{A}_\tau^\Theta$ and $*[\tau, \Theta]$ for either.
Superposition. As an example consider the (independent) superposition

$$\mathbf{P} \wedge \mathbf{A} \equiv \mathbf{P}[\tau_1, \Theta_1] \wedge \mathbf{A}[\tau_2, \Theta_2]$$

of these two processes. Since the intersection of two convex polygons is a convex polygon (or $\varnothing$), we have that $\mathbf{P} \wedge \mathbf{A}$ determines a tessellation of random convex polygons. Consider the stochastic construction of $^\circ D$ with respect to $\mathbf{P} \wedge \mathbf{A}$. It coincides with that of $\cap H_i(h_i, \theta_i)$, where $\{h_i\}$ are $\mathbf{P}_{\tau_1+\tau_2}$ (0, 1) and each θ_i is chosen from

$$\begin{cases} \Theta_1 \text{ with probability } \tau_1 / (\tau_1 + \tau_2), \text{ and} \\ \Theta_2 \text{ with probability } \tau_2 / (\tau_1 + \tau_2); \end{cases}$$

which is equivalent to selecting each θ_i from

$$\Theta = \frac{\tau_1\Theta_1 + \tau_2\Theta_2}{\tau_1 + \tau_2}.$$

Thus, for example, $^\circ D\{\mathbf{P}[\tau_1, \Theta_1] \wedge \mathbf{A}[\tau_2, \Theta_2]\}$ is stochastically equivalent to $^\circ D\{\mathbf{P}[\tau_1 + \tau_2, (\tau_1\Theta_1 + \tau_2\Theta_2)/(\tau_1 + \tau_2)]\}$ with, by the ergodic bridge, a corresponding result for the ergodic equi-weight distributions. This result generalises immediately to the superposition of n mutually independent processes, each of which is either $\mathbf{P}$ or $\mathbf{A}$:

$$^\circ D\left\{\wedge *_i \left[\tau_i, \Theta_i\right]\right\} \text{ is stochastically equivalent to } {}^\circ D\left\{*\left[\sum \tau_i, \left(\sum \tau_i \Theta_i\right) / \sum \tau_i\right]\right\}.$$

Nesting. Associate with each cell D_j of $*_1[\tau_1, \Theta_1]$ a realisation r_j of $*_2[\tau_2, \theta_2]$, the r_j being mutually independent. Let $*_1\{*_2\}$ comprise $*_1[\tau_1, \Theta_1]$ together with r_j within each of its D_j's, i.e. $*_2$ nested within $*_1$. Clearly this nesting may be iterated, to yield

$$*_{1\ldots n} \equiv *_1\{*_2\{\ldots\{*_n\}\ldots\}\},$$

and the above argument extends, to show

$$^\circ D\left\{*_{1\ldots n}\right\} \text{ is stochastically equivalent to } {}^\circ D\left\{*\left[\sum \tau_i, \left(\sum \tau_i \Theta_i\right) / \sum \tau_i\right]\right\},$$

with a corresponding ergodic result for the equi-weighted tessellations.
Intermediate case. These results apply also to the intermediate case, in which each cell of $*_1$ is selected (independently) in category j of m categories with probability p_j ($j = 1, \ldots, m$). The nesting involves taking the *single* realisation r_j of $*_2$ within all category j cells ($j = 1, \ldots, m$). The details are left to the reader.

Partition and restitution of Θ. Nesting may be combined with partition of Θ to yield further useful results. We illustrate with a simple example. Suppose Θ_{123} comprises three equal atoms of 1/3 at 0, $\pi/3$ and $2\pi/3$; Θ_{12} comprises two equal atoms of 1/2 at 0, $\pi/3$ (etc.); and Θ_1 comprises a single atom of 1 at 0 (etc.). Then the polygons of the following tessellations have the same distributions:

$$*[\tau, \Theta_{123}]$$

$$*[2\tau/3, \Theta_{12}]\{*[\tau/3, \Theta_3]\} \quad \text{(single nest)}$$

$$*[\tau/3, \Theta_1]\{*[\tau/3, \Theta_2]\{*[\tau/3, \Theta_3]\}\} \quad \text{(double nest).}$$

The reader might care to sketch realisations of these tessellations. Clearly a wealth of differing such tessellation models possess common "Poisson" polygon distributions.

5. Closing Remarks

A *within non-convex domains.* Suppose we replace the convex C by a pretty general bounded domain X, e.g. one bounded by a simple closed continuous curve, or by a disjoint union of such curves. Again an AFOL may be advanced across X, applying the **A** mechanism (A_1), (A_2) of Sec. 3 within X and simple ray extension in X^c. This results in a line segment process within X and a line process within X^c, the net result being a partition of R^2 into a random tessellation of convex polygons. The entire above theory extends in straightforward manner, the key again being prob el expressions. For example, the prob el for a given N-gon, even straddling $\partial X = \partial X^c$, is unchanged! The manifold possible choices of X generate a very general range of tractable stochastic models conforming to the classic Poisson polygon distributions.

Dimensional extensions? **P**, $\mathcal{P}$ may be effortlessly extended to Poisson hyperplanes in R^d, $\mathbf{P}(d-1, d)$, with generated Poisson polytopes $\mathcal{P}_j$; and more generally to Poisson s-flats in R^d, $\mathbf{P}(s, d)$, $(0 \le s < d)$ (MILES, 1971, 1974). However, there seems to be no corresponding extension possible of **A** to higher dimensionalities, e.g. by advancing a fixed orientation plane in R^3, against a background of Poisson lines $\mathbf{P}(1, 3)$ or planes $\mathbf{P}(2, 3)$. There appears to be no mechanism for such lines or planes to be born and die, with a *complete balance* between the two, analogous to the balance achieved by (A_1), (A_2). The geometrical requirements for such an extension appear quite excessive.

REFERENCES

ARAK, T., CLIFFORD, P. and SURGAILIS, D. (1993) Point-based polygonal models for random graphs, *Adv. Appl. Prob.*, **25**, 348–372.

MACKISACK, M. S. and MILES, R. E. (1996) Homogeneous rectangular tessellations, *Adv. Appl. Prob.*, **28**, 993–1013.

MILES, R. E. (1964) Random polygons determined by random lines in a plane, *Proc. Nat. Acad. Sci (USA)*, **52**, 901–907, II 1157–1160.

MILES, R. E. (1970) On the homogeneous planar Poisson point process, *Math. Biosciences*, **6**, 85–127.

MILES, R. E. (1971) Poisson flats in Euclidean spaces. Part II: Homogeneous Poisson flats and the Complementary Theorem, *Adv. Appl. Prob.*, **3**, 1–43.

MILES, R. E. (1973) The various aggregates of random polygons determined by random lines in a plane, *Adv. Math*, **10**, 256–290.

MILES, R. E. (1974) A synopsis of "Poisson flats in Euclidean spaces", in *Stochastic Geometry* (eds. E. F. Harding and D. G. Kendall), pp. 202–227, Wiley, London.

MILES, R. E. (1986) Random tessellations, in *Encyclopaedia of Statistical Sciences* (eds. S. Kotz and N. L. Johnson), Vol. 7, pp. 567–572, Wiley, New York.

MILES, R. E. (1995) A heuristic proof of a long-standing conjecture of D. G. Kendall concerning the shapes of certain large random polygons, *Adv. Appl. Prob.*, **27** 397–417.

MILES, R. E. (1998) A large class of random tessellations with the classic Poisson polygon distributions (abstract), *Adv. Appl. Prob. (SGSA)*, **30**, 285.

MILES, R. E. and MACKISACK, M. S. (1996) Further random tessellations with the classic Poisson polygon distributions (abstract), *Adv. Appl. Prob. (SGSA)*, **28**, 338–339.

SANTALÓ, L. A. (1976) *Integral Geometry and Geometric Probability, Encycl. Math. Applic.*, Vol. 1. Addison-Wesley, Mass.