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Euler Number and Connectivity Indexes of a Three Dimensional Digital Picture

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Abstract. Fundamental properties of topological structure of a 3D digitized picture are presented including the concept of neighborhood and connectivity among volume cells (voxels) of 3D digitized binary pictures defined on a cubic grid, the concept of simplicial decomposition of a 3D digitized object, and two algorithms for calculating the Euler number (genus). First we define four types of connectivity. Second we present two algorithms to calculate the Euler number of a 3D figure. Thirdly we introduce new local features called the connectivity number (CN) and the connectivity index to study topological properties of a 3D object in a 3D digitized picture. The CN at a 1-voxel (=a voxel with the value 1) x is defined as the change in the Euler number of the object caused by changing the value of x into zero, that is, caused by deleting x. Finally, by using them, we prove a necessary and sufficient condition that a 1-voxel is deletable. A 1-voxel x is said to be deletable if deletion of x causes no decrease and no increase in the numbers of connected components, holes and cavities in a given 3D picture.

1. Introduction

A three dimensional (3D) digital picture is a straightforward extension of a two dimensional (2D) digital image now widely used in the world. It is obtained by digitizing the 3D space in which an object of interest is contained (Fig. 1).

A 3D picture is obtained practically by recording successive 2D cross sections of an object by appropriate methods. A typical example is the CT (computed tomography) recording cross sections of the human body, which appeared in 1972 in the medical field. By the development of helical (spiral) scan CT in the beginning of 90’s, technology to take 3D images of the human body made great progress. Nowadays it has become possible to take a 3D X-ray image of the human body with the spatial resolution of 0.1 mm if it is needed for diagnosis and treatment.

More generally a 3D picture is considered as a model for a stack of 2D pictures such

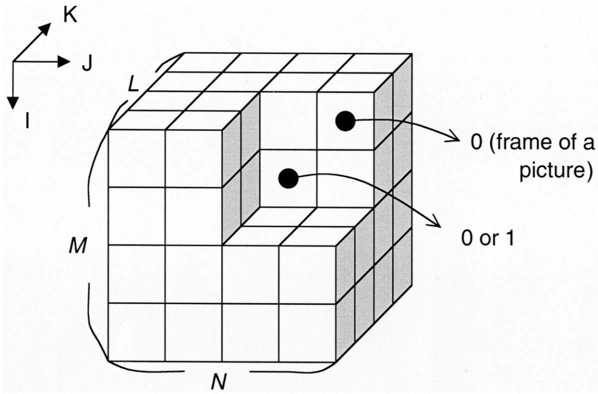


Fig. 1. Three dimensional digitized binray picture with the size $L \times M \times N$. The density value is assumed to be 0 on the frame of a picture. Other voxels can take either 0 or 1 as density values.

as a set of CT pictures and a set of microscope pictures of successive cross sections of pathological samples, or a model for a sequence of time-varying 2D pictures.

Various techniques and algorithms have been developed for 2D picture processing (ROSENFELD and KAK, 1982; TORIWAKI and YOKOI, 1985; TORIWAKI, 1988, 2002a; WATT and POLICARPO, 1998). Methods and algorithms similar to them have been shown to be applicable to the analysis of 3D pictures, such as filtering, thinning, smoothing, labeling, and distance transformation. Some of them may be straightforward extensions of 2D picture processing techniques, and others not. A typical example of the latter is digital topology. As is expected from topology of the continuous space in mathematics, topological properties of 3D digitized pictures are much more complicated than those of 2D digitized pictures.

Study of digital geometry and relating topics of 3D pictures began in 1970's. The earliest report the authors know is (PARK and ROSENFELD, 1971). Several surveys are available concerning research in 1970' and 80's including exhaustive list of references (TORIWAKI and YOKOI, 1985; KONG and ROSCOE, 1985; KONG and ROSENFELD, 1989). A few monographs and collections of survey articles have been published (HERMAN, 1998; KLETTE *et al.*, 1998; IMIYA and ECKHARDT, 1999; NIKOLARIDIS, 2001; BERTRAND *et al.*, 2001; DAVIS, 2001). However, topics they cover are limited, and work published in Japan was hardly referred except (TORIWAKI and YOKOI, 1985). A comprehensive textbook has not been published yet, except for only one Japanese book by one of the authors (TORIWAKI, 2002a).

In this article we present fundamental properties of topological structure of a 3D digitized picture, and provide a method to treat topology of digitized 3D figure. We first establish the concept of neighborhood and connectivity among volume cells (voxels) of 3D digitized binary pictures defined on a cubic grid. Basing upon four types of connectivity (6-, 18-, 18′- and 26-connectivity) introduced here, the concept of triangulation (or simplicial decomposition) of a 3D digitized object is made clear by listing all of zero-dimensional to three-dimensional simplexes in an explicit form.

Next, two algorithms for calculating the Euler number (genus) are given; one is to compute the value of a polynomial of binary variables representing voxel densities, and the other to perform local pattern matching. Both are performed using only the information of a $(2 \times 2 \times 2)$ local space.

Thirdly we present local features called the connectivity number (CN) and the connectivity index to study topological properties of a 3D object in a 3D digitized picture. The CN at a 1-voxel (=a voxel with the value 1) x is defined as the change in the Euler number of the object caused by changing the value of x into zero, that is, caused by deleting x. The connectivity index is a set of three numbers called the component index R(x), the hole index H(x) and the cavity index Y(x) defined at each voxel x and its 26-neighborhood N(x). Roughly speaking, they are described as follows: R(x) is the number of connected components connected to the voxel x, H(x) and Y(x) are the number of holes and the number of cavities created in N(x) by deleting x, respectively. Methods for calculating these properties are given. We also discuss relationship among those indexes and topological features of a 3D object.

Finally we prove a necessary and sufficient condition that a 1-voxel is deletable. A 1-voxel x is said to be deletable if deletion of x causes no decrease and no increase in the numbers of connected components, holes and cavities in a given 3D picture. It is shown that a 1-voxel x is deletable if and only if R(x) = 1, H(x) = 0, and Y(x) = 0, or equivalently the connectivity number $N_c(x) = 1$ and R(x) = 1.

Almost all of the contents of this article were derived by authors and colleagues in Nagoya University during 1978~82 and published as original papers in Japanese academic journals (YONEKURA *et al.*, 1980a, b, c). They had not been published in English except for very small portion of them in short papers in conference proceedings and short notes (YONEKURA *et al.*, 1980d, e; TORIWAKI *et al.*, 1982). The equations of binary variables to calculate the above features were first given by the authors during 1980~82 for all of four types of connectivity (YONEKURA *et al.*, 1982a, b, c).

2. Basic Definitions and Notations

A digitized three-dimensional picture (3D picture) defined on a 3D cubic grid is a straightforward extension of a two-dimensional (2D) picture to the 3D space. A 3D digitized picture is represented by a 3D array $F = \{ f_{ijk} \}$, when i, j, and k are integers such that $1 \leqq i \leqq M$, $1 \leqq j \leqq N$, $1 \leqq k \leqq L$, and f_{ijk} represents a density value at a sample point (or a cube) located on the i-th row and the j-th column on the k-th plane (Fig. 1). Each cube is called a voxel (picture element, or simply a point). A picture is called a binary picture if each voxel has a value of either 1 or 0. Voxels of a picture with values 0 and 1 are called 0-voxels and 1-voxels, respectively. The set of all 1-voxels, and its complement may be called a figure (or an object) and the background, respectively. It is assumed that the first row, the Mth row, the first column, the Nth column, the first plane and the Lth plane are all filled with 0-voxels. These rows, columns and planes are called the frame of the picture.

[Definition 1] (Neighborhood) For each voxel $x = (i, j, k)$, three kinds of neighborhood, the 6-neighborhood (6-n) $N^{(6)}(x)$, the 18-neighborhood (18-n) $N^{(18)}(x)$, and the 26-neighborhood (26-n) $N^{(26)}(x)$ are defined as follows:

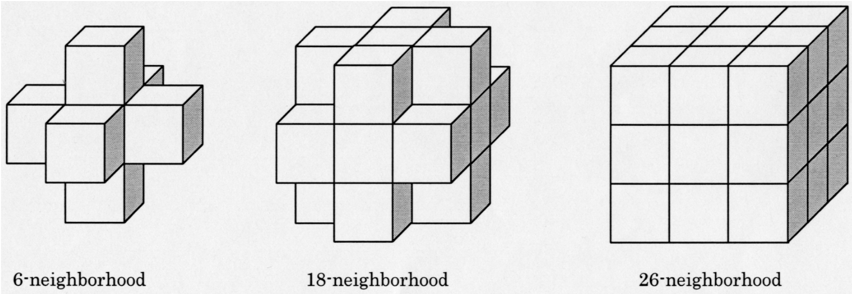


Fig. 2. Three kinds of neighborhood.

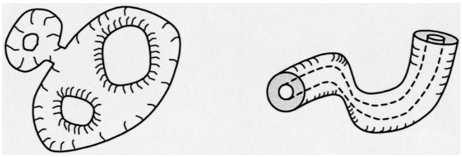


Fig. 3. Holes (handles, tunnels).

$$N^{(6)}(x) = \{(p, q, r); |p-i| + |q-j| + |r-k| = 1\}$$

$$N^{(18)}(x) = \{(p, q, r); 0 < |p-i|^2 + |q-j|^2 + |r-k|^2 \leqq 2\}$$

$$N^{(26)}(x) = \{(p, q, r); \max(|p-i|, |q-j|, |r-k|) = 1\}.$$

Any voxel P in the 6-n (18-n, 26-n) of a voxel x shares at least one face (edge, vertex) of the voxel with x (Fig. 2). If a voxel P is in the m-neighborhood of a voxel Q, the voxel P is said to be m-adjacent to the voxel Q, and vice versa (m = 6, 18, 18′, 26).

[Definition 2] (Connectivity) Two voxels P and Q with a common density value are said to be 6-connected (6-c), (18-connected (18-c), 26-connected (26-c)), if a sequence of voxels P_0 (=P), P_1, P_2, ..., P_n (=Q) exists, such that each P_i is in the 6-n (18-n, 26-n) of P_{i-1} ($l \leqq i \leqq n$) and all P_i's have the same value as P and Q.

It is clear that 6-connectendness, 18-connectendness, and 26-connectendness thus defined give a kind of equivalence relationship among voxels with value 1 (or value 0).

[Definition 3] (Connected component) Each equivalence class of voxels defined by 6-connectedness is called a 6-connected (6-c) component. An 18-c component and a 26-c component are defined in a similar way. A connected component of 0-voxels is called a 0-component, and that of 1-voxels is called 1-component.

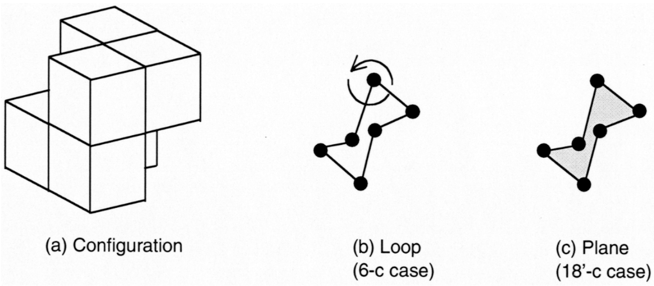


Fig. 4. Illustration of the difference between the 6-connectivity (6-c) and 18′-connectivity (18′-c).

Table 1. Permissible combinations of connectivities for 1-components and 0-components.

1-component	0-component
6-c	26-c
18-c	18′-c
18′-c	18-c
26-c	6-c

[Definition 4] (Cavity) Any 0-component which does not contain the frame of a picture is called a cavity. A cavity which connects to the frame of a picture when all 1-voxels of a 1-component *C* are changed into 0-voxels is called a cavity of the 1-component *C*.

[Definition 5] (Hole) Consider a division of 1-component *C* into a set of simplexes. The number of holes of the 1-component *C* is defined as the number of independent one-dimensional homology classes.

Concrete forms of simplexes of a digitized 3D figure will be shown in the next section. Intuitively, the number of holes of a 3D figure is interpreted as the number of handles or tunnels (Fig. 3). It is difficult to give intuitive explanation of a hole itself in a simple form. For details, see textbooks of topology.

[Definition 6] (Simply connected) A connected component of 1-voxels which has neither a hole nor a cavity is said to be simply connected, and otherwise, multiply connected. The fourth type of connectivity should be introduced for consistent discussion on topological properties of a 3D figure. In this new type called the 18′-connectivity (18′-c), definitions of the neighborhood and connectivity are exactly the same as those of the 6-c except the only voxel configuration shown in Fig. 4(a). This configuration is considered as a plane of six voxels in the 18′-c while it is regarded as a loop of six voxels in the 6-c.

Anyone of 6-, 18-, 18′- or 26-connectivity may be adopted for analysis of any particular problem. Care must be taken, however, to avoid the contradiction concerning the connectivity of 1-voxels and that of 0-voxels. Only four combinations shown in Table 1 are permissible.

3. Euler Number of 3D Digitized Object

3.1. Definition of the Euler number

[Definition 7] Let us consider a 3D digitized object and its simplicial decomposition. Denoting the numbers of k-dimensional simplexes (k-simplexes) by n_k's (k = 0, 1, 2, 3), the Euler number (Euler characteristic, genus) E of the object is defined as follows.

$$\mathrm{E} = n_0 - n_1 + n_2 - n_3. \tag{1}$$

The following relation is well known in topology as the Euler-Poincare's formula.

$$\mathrm{E} = b_0 - b_1 + b_2 \tag{2}$$

where b_0 = number of 1-components (0-dimensional Betti number); b_1 = number of holes in all of 1-components (1-dimensional Betti number); b_2 = number of cavities in all of 1-components (2-dimensional Betti number).

Alternative representation of the above property is given by considering a closed surface in the 3D space. For each single closed netted surface the following relation holds (Euler's formula):

$$n - e + f = 2 - q, \tag{3}$$

where n denotes the number of nodal points of the net, e denotes the number of edges, and f denotes the number of faces. A number q is often called the connectivity number in topology.

In the subsequent parts of the paper we will use Eq. (1) or (2) and reserve the name "connectivity number" for another characteristic number we will define in Subsec. 4.1 of the paper.

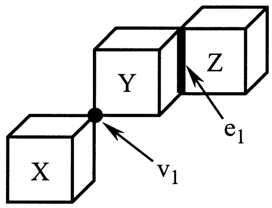
The Euler number is an important topological feature of a 3D object. We will present in this section two methods for calculating the Euler number of a 3D digitized object, which are derived from different viewpoints. The first viewpoint is to regard each voxel as a cube in the 3D continuous space. Since a 3D object (a set of 1-voxels) is decomposed into a set of 1-voxels (cube), this decomposition is considered as a triangulation or a simplicial decomposition of the object. The Euler number is obtained by applying Eq. (1) to this triangulation (*triangulation method*).

The second point of view is to count the numbers of zero-, one, two, and three-dimensional digital simplexes included in a given 3D object, and substitute them to Eq. (1) (*simplex counting method*).

In the following two sections, we will present details of the above two methods.

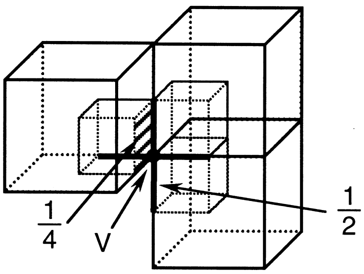
3.2. Triangulation method for calculating the Euler number

In the decomposition of a 3D object to cubes corresponding to 1-voxels, a k-dimensional simplexe (k = 0, 1, 2) corresponds to a vertex (k = 0), an edge (k = 1), and a face (k = 2), of a 1-voxel, respectively, and a three-dimensional simplex corresponds to a



	6-c	18-c	18'-c	26-c
n_0	24	22	24	21
n_1	36	35	36	35
n_2	18	18	18	18
n_3	3	3	3	3
E	3	2	3	1

Fig. 5. Example for the triangulation method. Let us consider a figure consisting of three voxels, X, Y, and Z as shown here. Then values of n_k's in Eq. (1) are given as in the table above corresponding to each type of connectivity. For example, n_0 is the number of vertexes in this figure. Since each voxel (a cube) has eight vertexes, the number of vertexes in this figure n_0 is 24 (=8 × 3) for the 6-connectivity case. In the 26-connectivity case, voxels X and Y are connected and the vertex v_1 is not a vertex of this figure. The edge e_1 is not the edge of the figure, because v_1 and e_1 are inside of the figure in the 26-c case. Other cases are considered in the same way. For details, see (GRAY, 1971; TORIWAKI, 2002a).



	6-c	18-c	18'-c	26-c
Δn_0	2	1	2	1
Δn_1	$7\times\frac{1}{2}$	$6\times\frac{1}{2}$	$7\times\frac{1}{2}$	$6\times\frac{1}{2}$
Δn_2	$6\times\frac{1}{4}$	$6\times\frac{1}{4}$	$6\times\frac{1}{4}$	$6\times\frac{1}{4}$
Δn_3	$3\times\frac{1}{8}$	$3\times\frac{1}{8}$	$3\times\frac{1}{8}$	$3\times\frac{1}{8}$

$$\Delta E(v) = \Delta n_0 - \Delta n_1 + \Delta n_2 - \Delta n_3$$

Fig. 6. Illustrative example for calculating the Euler number by the trianguration method.

voxel itself. Thus, n_k's in Eq. (1) is given as follows:

n_0 = number of vertexes of 1-voxels contained in a 3D object,

n_1 = number of edges of 1-voxels contained in a 3D object,

n_2 = number of faces of 1-voxels contained in a 3D object,

and

n_3 = number of 1-voxels contained in a 3D object.

The type of connectivity should be taken into consideration in counting those numbers. In the case of Fig. 5, for example, the vertex v_1 should be counted twice for the 6-c, 18-c, and 18′-c, while only once for the 26-c, because it belongs to separating two voxels X and Y in the first three cases and not a verttex for the 26-c case. The edge e_1 should be counted twice for the 6-c, and 18-c cases, because it belongs to both of two different voxels Y and Z. Thus the result is as shown in Fig. 5.

To obtain the total sum of n_k's over the whole of a given 3D picture, it is convenient to count them at each vertex of a 1-voxel. Let us consider a vertex V and a set of 2 × 2 × 2 voxels S(V) sharing this vertex (Fig. 6).

Let Δn_k's (k = 0, 1, 2, 3) denote the following quantities.

Δn_0 = number of vertexes of 1-voxels in S(V) which share V.

Δn_1 = (number of edges of 1-voxels in S(V) containing the vertex V) × 1/2.

Table 2. Possible 2 × 2 × 2 subpatterns and their contributions to the Euler number.

	Vertex Δn_0			Edge $-\Delta n_1 \cdot 2$			Face $\Delta n_2 \cdot 4$			Volume $-\Delta n_3 \cdot 8$			Euler differential ΔE(V)***		
Connectivity*	6	18	26	6	18	26	6	18	26	6	18	26	6	18	26
Subpattern**															
Q_{01}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Q_{11}	1	1	1	–3	–3	–3	3	3	3	–1	–1	–1	1	1	1
Q_{21}	1	1	1	–4	–4	–4	5	5	5	–2	–2	–2	0	0	0
Q_{22}	2	1	1	–6	–5	–5	6	6	6	–2	–2	–2	2	–2	–2
Q_{23}	2	2	1	–6	–6	–6	6	6	6	–2	–2	–2	2	2	–6
Q_{31}	1	1	1	–5	–5	–5	7	7	7	–3	–3	–3	–1	–1	–1
Q_{32}	2	1	1	–7	–6	–6	8	8	8	–3	–3	–3	1	–3	–3
Q_{33}	3	1	1	–9	–6	–6	9	9	9	–3	–3	–3	3	–1	–1
Q_{41}	1	1	1	–5	–5	–5	8	8	8	–4	–4	–4	0	0	0
Q_{42}	1	1	1	–6	–6	–6	9	9	9	–4	–4	–4	–2	–2	–2
Q_{43}	1	1	1	–6	–6	–6	9	9	9	–4	–4	–4	–2	–2	–2
Q_{44}	2	1	1	–8	–6	–6	10	10	10	–4	–4	–4	0	0	0
Q_{45}	2	1	1	–8	–6	–6	10	10	10	–4	–4	–4	0	0	0
Q_{46}	4	1	1	–12	–6	–6	12	12	12	–4	–4	–4	4	4	4
Q_{51}	1	1	1	–6	–6	–6	10	10	10	–5	–5	–5	–1	–1	–1
Q_{52}	1	1	1	–7	–6	–6	11	11	11	–5	–5	–5	–3	1	1
Q_{53}	2	1	1	–9	–6	–6	12	12	12	–5	–5	–5	–1	3	3
Q_{61}	1	1	1	–6	–6	–6	11	11	11	–6	–6	–6	0	0	0
Q_{62}	1	1	1	–7	–6	–6	12	12	12	–6	–6	–6	–2	2	2
Q_{63}	0	1	1	–6	–6	–6	12	12	12	–6	–6	–6	–6	2	2
Q_{71}	1	1	1	–6	–6	–6	12	12	12	–7	–7	–7	1	1	1
Q_{81}	1	1	1	–6	–6	–6	12	12	12	–8	–8	–8	0	0	0
Q_{63} (18′-c)	1	1	1	–6	–6	–6	12	12	12	–6	–6	–6	2	2	2

*For the 18′-c, all except Q_{63} is the same as the 6-c.
**Codes of subpatterns are shown in the attached figure.
***$\Delta E(V) = \Delta n_0 - \Delta n_1 + \Delta n_2 - \Delta n_3$.

Δn_2 = (number of faces of 1-voxels in S(V) containing the vertex V) × 1/4.

Δn_3 = (number of 1-vowels in S(V) containing the vertex V) × 1/8.

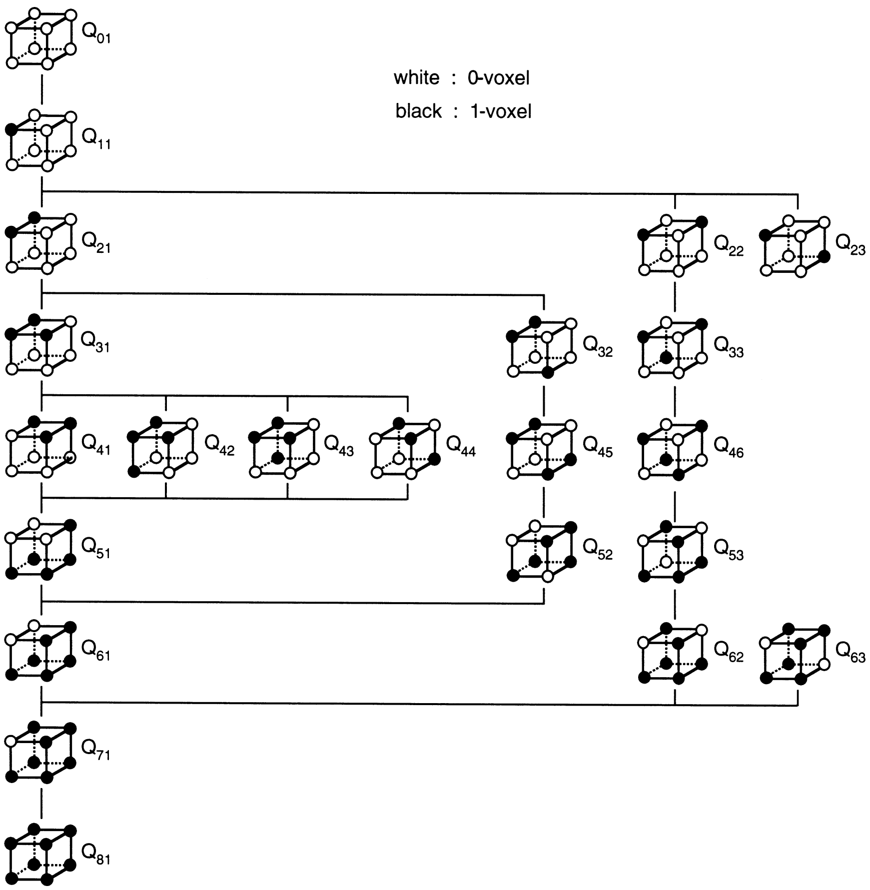
Then, the amount of the contribution E(V) to the Euler number E at the vertex V is given by,

$$\Delta E(v) = \Delta n_0 - \Delta n_1 + \Delta n_2 - \Delta n_3. \tag{4}$$

The Euler number E is obtained by adding ΔE(V) of all vertexes in a 3D object, that is,

$$E = \sum_v \Delta E(v). \tag{5}$$

Table 2. (continued).



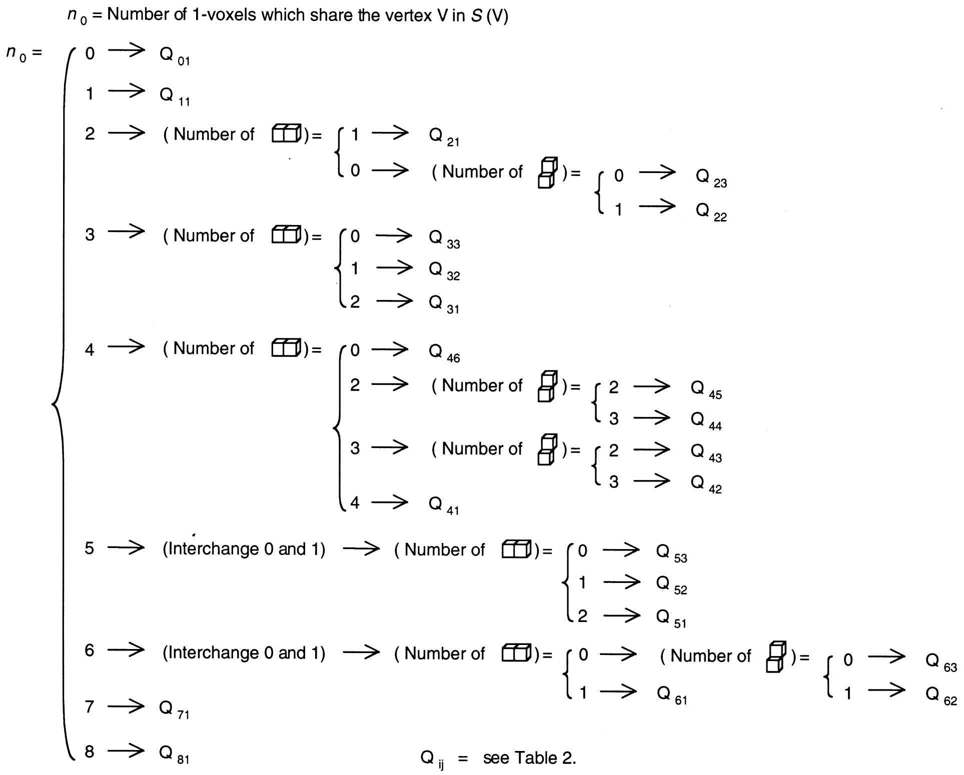


Fig. 7. Flow chart of Algorithm 1 for calculating the Euler number (pattern matching).

The type of connectivity should be taken into consideration again in the similar way as was presented concerning Fig. 5. An example is shown in Fig. 6.

The value of ΔE(V) defined above is uniquely determined by the configuration of 1-voxels in *S*(V). Since there are 256 possible configurations and it is known by considering various symmetric relations that only 22 among them are basically different patterns, the value of ΔE(V) for each configuration can be calculated beforehand and stored in the form of a table. Thus, the computation of the Euler number is reduced to the iterative table searches and additions. All of possible $2 \times 2 \times 2$ configurations are shown in Table 2 with the values of ΔE(V).

An efficient algorithm to find which pattern among these 22 cases a given $2 \times 2 \times 2$ subpattern corresponds to is given as follows.

[Algorithm 1] Denote the number of 1-voxels in *S*(V) by n_0. Then the value of the contribution to the Euler number ΔE(V) at the vertex V is determined by the flow chart in Fig. 7 and Table 2.

Note that this algorithm does not depend on the type of the connectivity. The value of ΔE(V) for the desired type of connectivity can be found from the table prepared beforehand.

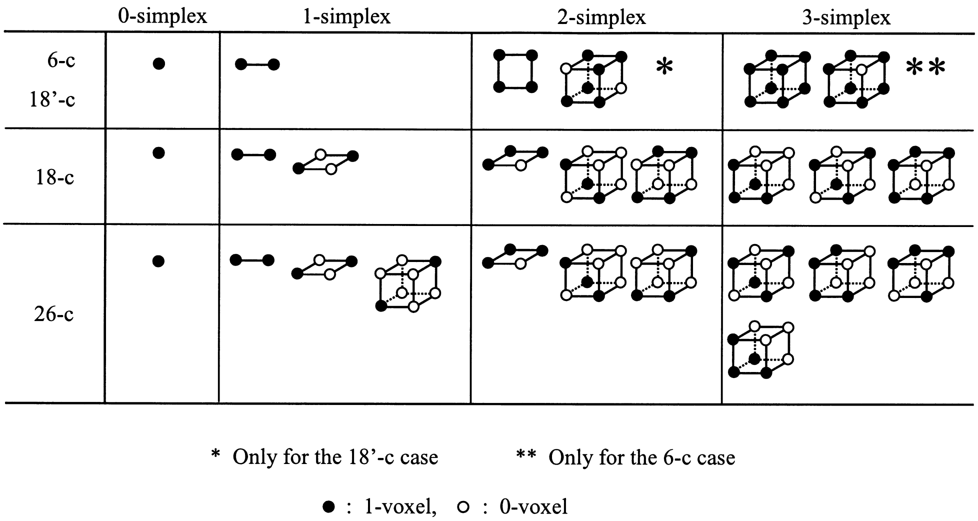


Fig. 8. All possible simplexes.

This method for obtaining the Euler number is an extension of the method for two-dimensional figures given by GRAY (1971). LOBREGT *et al.* (1980) derived a similar method for only the 6-c and 26-c cases basing upon a closed netted surface model and Eq. (3).

3.3. Simplex counting method for calculating the Euler number

Simplexes used for decomposition of a 3D digitized picture should be defined as follows:

(1) 0-simplex = each of 1-voxels.

(2) 1-simplex = pair of two 1-voxels neighboring each other.

(3) 2-simplex = a set of three 1-voxels, anyone of which is adjacent to the remaining two voxels, or, a set of four voxels which exist on the same plane, and anyone of which is adjacent to two of the remaining three voxels.

(4) 3-simplex = a set of four 1-voxels, anyone of which is adjacent to all of the remaining three voxels, or, a set of five or more 1-voxels which do not exist on the same plane and make a closed polyhedron with the minimum number of faces.

The 1-, 2-, and 3-simplexes are called the edge element, the face element and the volume element, respectively. Concrete forms of simplexes vary according to the type of the connectivity. We show in Fig. 8 all of basically different possible simplexes for four types of connectivity.

The simplicial decomposition must satisfy the following requirements (Fig. 9).

(1) Neighboring two face elements share only one common 1-voxel or share only one common edge element.

(2) Connection between a face element and an edge element happens either in the form that one of vertexes of the edge element coincides with one of vertexes of the face

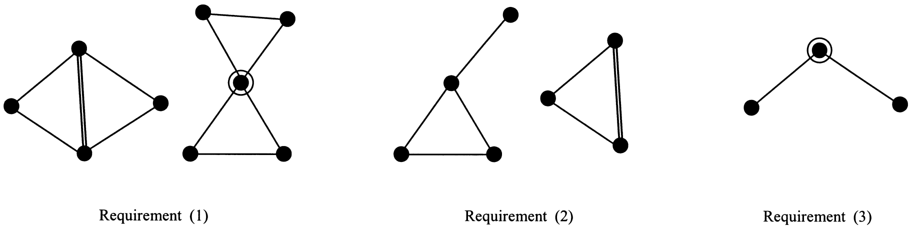


Fig. 9. Illustration of requirements to the simplecial decomposition.

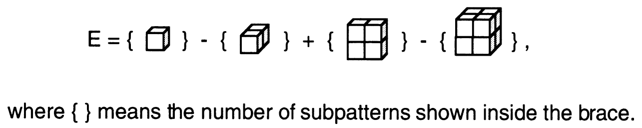


Fig. 10. Calculation of the Euler number E for the 6-connectivity case.

element, or that the edge element coincides with one of edges of the face element.

(3) Two edge elements are connected by one common vertex only.

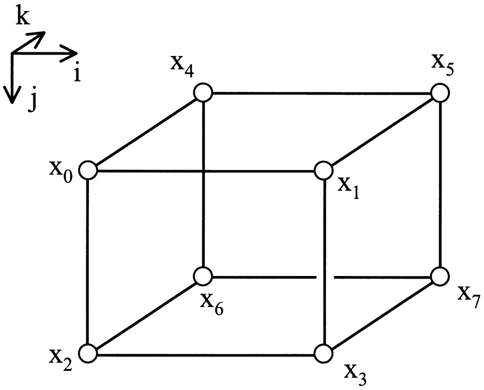
The Euler number is calculated by counting simplexes of each dimension contained in a 3D object, and substituting the results for n_r's in Eq. (1). In the 6-c case, for example, the Euler number E is calculated as shown in Fig. 10.

Counting of simplexes presented above may be performed by local template matching over a $2 \times 2 \times 2$ local area, or equivalently by calculating a value of a pseudo-Boolean expression defined over the $2 \times 2 \times 2$ local area. The latter procedure is presented by the following theorem.

[Theorem 1] Let $X_0, X_1, \ldots, X_6$, and X_7 denote density values (0 or 1) of voxels $P = (i, j, k)$, and its neighbors $(i + 1, j, k)$, and $(i + 1, j + 1, k + 1)$ in the $2 \times 2 \times 2$ local area as shown in Fig. 11 ($1 \leqq i \leqq L$, $1 \leqq j \leqq M$, $1 \leqq k \leqq N$). Then the Euler number $E^{(K)}$ where K represents the type of the connectivity (K = 6, 18, 18′, or 26) is given by the following equations.

$$E^{[6]} = \sum_{i=2}^{L-1}\sum_{j=2}^{M-1}\sum_{k=2}^{N-1}\left\{ x_0\left[1 - x_1(1 - x_4 \cdot x_5) - x_2(1 - x_1 \cdot x_3) - x_4(1 - x_2 \cdot x_6) - \prod_{i=0}^{7} x_i \right]\right\} \tag{6}$$

$$E^{(18)} = \sum_{i=2}^{L-1}\sum_{j=2}^{M-1}\sum_{k=2}^{N-1}\left\{ \bar{x}_0\left[1 - \bar{x}_1(1 - \bar{x}_4 \cdot \bar{x}_5) - \bar{x}_2(1 - \bar{x}_1 \cdot \bar{x}_3) - \bar{x}_4(1 - \bar{x}_2 \cdot \bar{x}_6) - \prod_{i=0}^{7} \bar{x}_i \right]\right\} + \sum_{m=0}^{3}\left[x_m \cdot x_{7-m} \cdot \bar{x}^{6}{}_{\mathrm{else}} \right] \tag{7}$$



$x_0 = (\, i, j, k\,)$
$x_1 = (\, i+1, j, k\,)$
$x_2 = (\, i, j+1, k\,)$
$x_3 = (\, i+1, j+1, k\,)$
$x_4 = (\, i, j, k+1\,)$
$x_5 = (\, i+1, j, k+1\,)$
$x_6 = (\, i, j+1, k+1\,)$
$x_7 = (\, i+1, j+1, k+1\,)$

Fig. 11. Variables in Theorem 1.

$$E^{(18')} = \sum_{i=2}^{L-1}\sum_{j=2}^{M-1}\sum_{k=2}^{N-1}\left\{x_0\left[1 - x_1\left(1 - x_4\cdot x_5\right) - x_2\left(1 - x_1\cdot x_3\right) - x_4\left(1 - x_2\cdot x_6\right) - \prod_{i=0}^{7} x_i\right]\right\}$$
$$+ \sum_{m=0}^{3}\left[\bar{x}_m\cdot\bar{x}_{7-m}\cdot x^6{}_{\mathrm{else}}\right] \qquad (8)$$

$$E^{[26]} = \sum_{i=2}^{L-1}\sum_{j=2}^{M-1}\sum_{k=2}^{N-1}\left\{\bar{x}_0\left[1 - \bar{x}_1\left(1 - \bar{x}_4\cdot \bar{x}_5\right) - \bar{x}_2\left(1 - \bar{x}_1\cdot \bar{x}_3\right) - \bar{x}_4\left(1 - \bar{x}_2\cdot \bar{x}_6\right) - \prod_{i=0}^{7} \bar{x}_i\right]\right\} \qquad (9)$$

where $\bar{x}_i = 1 - x_i$, and $x^6{}_{\mathrm{else}}$ ($\bar{x}^6{}_{\mathrm{else}}$) means the product of all six values x_i's ($\bar{x}_i$'s) such that $i = 0, 1, ..., 7, i \neq m, i \neq 7 - m$.

(Proof) [6-c case] Considering simplexes shown in Fig. 8,

$$\text{number of 0-simplex} = \Sigma_i\Sigma_j\Sigma_k x_0$$

$$\text{number of 1-simplex} = \Sigma_i\Sigma_j\Sigma_k x_0(x_1 + x_2 + x_4)$$

$$\text{number of 2-simplex} = \Sigma_i\Sigma_j\Sigma_k x_0(x_1x_2x_3 + x_2x_4x_6 + x_1x_4x_5)$$

$$\text{number of 3-simplex} = \Sigma_i\Sigma_j\Sigma_k\Pi_{l=0}{}^{7}x_l \qquad (10)$$

where $\Sigma_i\Sigma_j\Sigma_k$ means the summation over the whole of a given picture.

Substituting them into Eq. (1), we obtain Eq. (6).

[26-c case] Let $F = \{ f_{ijk} \}$, and $\overline{F} = \{ \bar{f}_{ijk} \}$ denote an input picture F and its inversion, that is, $\overline{F} = \{1 - f_{ijk}\}$. Denoting the Euler number of the picture for the *m*-c case (*m* = 6, 18, 18′, 26) by $E^{(m)}(F)$, following relations hold:

$$E^{(6)}(F) = E^{(26)}\left(\overline{F}\right), \quad E^{(26)}(F) = E^{(6)}\left(\overline{F}\right) \tag{11}$$

$$E^{(18)}(F) = E^{(18')}\left(\overline{F}\right), \quad E^{(18')}(F) = E^{(18)}\left(\overline{F}\right). \tag{12}$$

These are proved by calculating $\Delta E(V)$ of Eq. (4) for all $2 \times 2 \times 2$ configurations given in Table 2. Equation (7) is derived immediately from Eq. (11).

[18′-c case] From the definition of the 18′-connectivity, $E^{(18')}$ is obtained by adding to the $E^{(6)}$ (Eq. (6)) the contribution of the configuration shown in Fig. 4 which is a 2-simplex for the 18′-c case. By doing this and slightly rearranging expression, we obtain Eq. (8).

[18-c case] Equation (7) is immediately derived from Eqs. (8) and (12) (Q.E.D).

4. Connectivity Index and Deletability

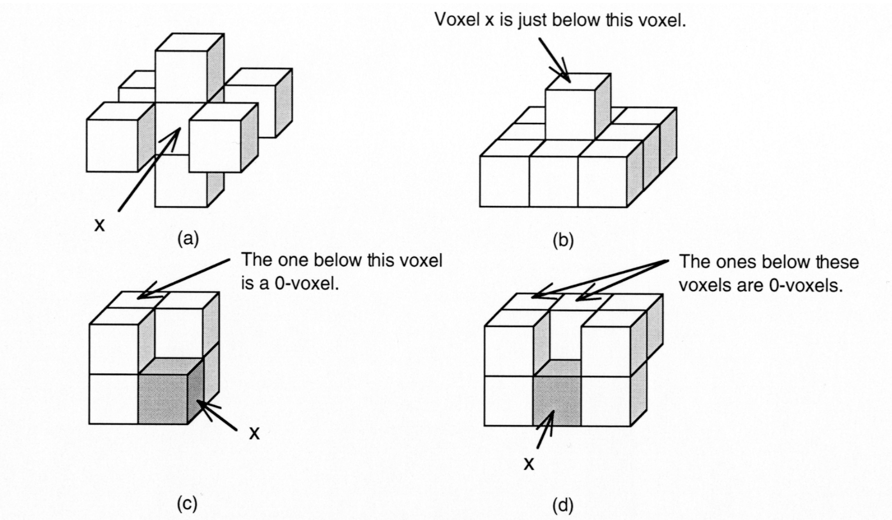
4.1. The Euler number and topological properties of 3D objects

The Euler number plays a very important role in the analysis of topological properties of a 3D object. However, its characteristic is quite different from that of a two-dimensional picture. The most remarkable difference is that preservation of the value of the Euler number in a particular picture transformation does not always mean the invariance of topological properties of a 3D object. Let us consider, for example, a pattern shown in Fig. 12(b). In the 6-c case, this pattern consists of a single connected component, without any hole and any cavity, that is, $b_0 = 1$, $b_1 = b_2 = 0$, in Eq. (2), hence $E^{(6)} = 1$. If we delete a 1-voxel x (change the 1-voxel x into a 0-voxel), the object is divided into two parts, a loop and single 1-voxel. Therefore, $b_0 = 2$, $b_1 = 1$, $b_2 = 0$ and still $E^{(6)} = 2 - 1 = 1$. It is obvious from this example that keeping the Euler number unchanged is not the necessary and sufficient condition for deletability of a 1-voxel in a 3D object. However it is still significant as a necessary condition for deletability because change in the value of the Euler number always means change in topological features of a 3D object.

4.2. Connectivity number and connectivity index

The first step to investigate the effect of picture transformation on the topological properties is to evaluate the change in the value of the Euler number. We will define several features called a connectivity number and a connectivity index to represent the effect of the deletion of a 1-voxel. In the subsequent parts of the paper we mean by "delete a 1-voxel x" that we change the 1-voxel x into a 0-voxel.

[Definition 8] (Connectivity number) The connectivity number (CN) $N_c^{(m)}(x)$ at a 1-voxel x is defined as follows.



	$N_c^{(m)}(x)$				$R^{(m)}(x)$				$H^{(m)}(x)$				$Y^{(m)}(x)$			
m	6	18	18'	26	6	18	18'	26	6	18	18'	26	6	18	18'	26
a	2	6	2	-2	2	6	2	1	0	0	0	3	0	0	0	0
b	1	1	2	1	2	1	2	1	1	0	1	0	0	0	0	0
c	2	1	1	1	2	1	1	1	0	0	0	0	0	0	0	0
d	2	1	2	1	2	1	2	1	0	0	0	0	0	0	0	0

Fig. 12. Examples of the connectivity index.

$$N_c^{(m)}(x) = E^{(m)}(x) - E^{(m)}(\bar{x}) + 1 \tag{13}$$

where $E^{(m)}(x)$ and $E^{(m)}(\bar{x})$ denote the value of the Euler number of a 3D object before and after deletion of the 1-voxel x, respectively, and *m* denotes the type of connectivity (*m* = 6, 18, 18′ and 26).

Obviously the CN takes different values at different voxels. Also it differs for the different type of the connectivity.

[Definition 9] (Connectivity index) In a $3 \times 3 \times 3$ local area consisting of a 1-voxel x and its 26-neighborhood, we define the following there indexes.

Component index $R^{(m)}(x)$ = the number of m-connected components which are connected to x and exist in the 18-neighborhood for $m = 6$ and $18'$ (in the 26-neighborhood for $m = 18$ and 26). (The subpattern Q_{63} of Table 2 is exceptionally regarded as a single connected component for $m = 18'$)

Hole index $H^{(m)}(x)$ = the number of holes which are newly generated by the deletion of x, or equivalently the decrease in the number of $\overline{m}$-connected components of 0-voxels in this local area caused by the deletion of x ($\overline{m}$ means the type of connectivity determined from m by Table l).

Cavity index $Y^{(m)}(x)$ = the number of cavities which are newly generated by the deletion of x.

The CN is a rather straightforward extension of the feature of the same name in two-dimensional picture processing. The CN of a two-dimensional picture is equal to the number of connected component in a 3×3 local area except for very few special cases (YOKOI *et al*., 1975). We cannot give a simple geometrical interpretation to the CN of a 3D object, because complicated relations exist among the above three indexes. This means that the local configuration of a 3D object has much more complicated topological characteristics than those of a two-dimensional picture.

Next we will present several properties concerning the CN and the connectivity index.

[Property 1] The following relation holds among the CN at a 1-voxel x, $N_c^{(m)}(x)$ and the Betti numbers.

$$N_c^{(m)}(x) = 1 + \Delta b_0^{(m)} - \Delta b_1^{(m)} + \Delta b_2^{(m)} \tag{14}$$

where $\Delta b_k^{(m)} = b_k^{(m)\prime} - b_k^{(m)}$, $k = 0, 1, 2$, $b_k^{(m)}$ and $b_k^{(m)\prime}$ are the Betti numbers of the order k before and after the deletion of the 1-voxel x, respectively, and m denotes the type of connectivity.

(Proof) The property is derived immediately from Eqs. (2) and (13). Considering that the Betti numbers b_k's ($k = 0, 1, 2$) represent the number of connected components ($k = 0$), that of holes ($k = 1$) and that of cavities ($k = 2$), the value of the CN $N_c(x)$ reduced by one is equal to "(the change in the number of connected components) – (the change in the number of the holes) + (the change in the number of cavities)" caused by the deletion of the 1-voxel x (Q.E.D.).

[Property 2] Assume that the component index at a 1-voxel x is $(l, 0, 0)$ and that, by the deletion of x, the number of connected components increases by α and the number of holes decreases by β over the whole of a given 3D picture. Then

$$l = \alpha + \beta + 1. \tag{15}$$

(Proof) Let N(x) denote the neighborhood of x used to define the component index R(x) in Definition 9. By the definition, l means the number of connected components connected to x in N(x). Consider any two of those components. If they are connected outside N(x), a hole is lost by the deletion of x (Fig. 13(a)), and otherwise, a connected component including

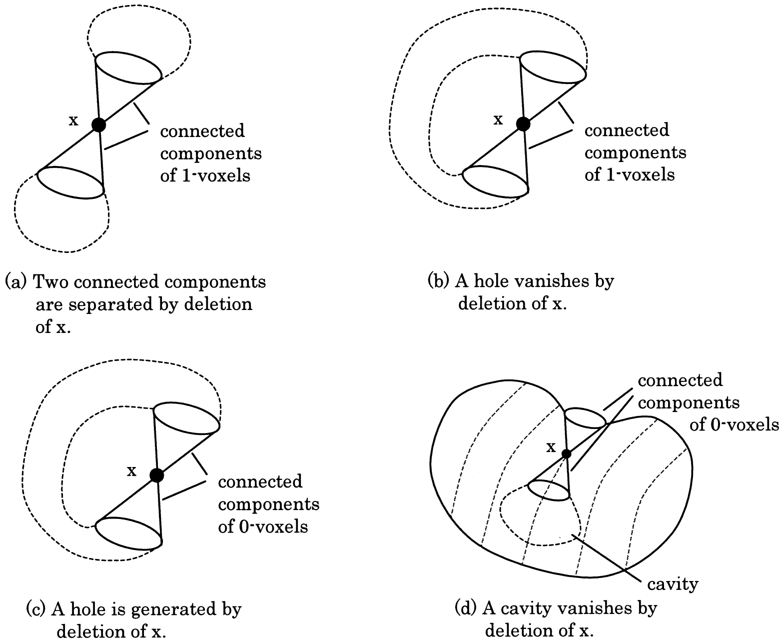


Fig. 13. Illustrations for explanation of Property 2, 3, and 4.

x is disconnected to increase the number of connected components in the picture by one (Fig. 13(b)). By the assumption, the former happens β times and the latter α times. Since there are l connected components in N(x), either of the above two cases happen $(l - 1)$ times by the deletion of x. Hence, $l - 1 = \alpha + \beta$ which means the Eq. (15) (Q.E.D).

[Property 3] Assume that the component index at a 1-voxel x is $(l, n, 0)$ and that, by the deletion of x, the number of holes increases by β' and the number of cavities decreases by γ over the whole of a given input picture. Then,

$$n = \beta' + \gamma. \tag{16}$$

(Proof) By the definition of the hole index, the value n is equal to "(the number of connected components of 0-voxels connected to x) – 1" in the neighborhood of x, N(x). Consider any two of such connected components of 0-voxels. If they are connected outside N(x), they are connected once again by the deletion of x, which means generation of a hole in the component of 1-voxels (Fig. 13(c)). If those two components of 0-voxels are not connected outside N(x), one or both of them belongs to a cavity. In this case, the deletion of x makes those two components of 0-voxels to be connected, which causes vanishment of a cavity (Fig. 13 (d)). By the assumption, the former happens β' times and the latter γ times when x is deleted. Since there are $n + 1$ connected components of 0-voxels in N(x), either of the above two cases happens n times by the deletion of x.

Thus,

$$n = \beta' + \gamma. \qquad \text{(Q.E.D.)}$$

[Property 4] Assume that the connectivity index at a 1-voxel x is (l, n, 0). If the deletion of x creates α connected components and β' cavities, and vanishes β holes, then

$$l = \alpha + \beta + 1$$

$$n = \beta' + \gamma. \tag{17}$$

(Proof) Equation (17) is derived immediately from Property 2 and 3.

The next theorem is important to discuss the deletability of a 1-voxel.

[Theorem 2] The following relation holds among the connectivity number and the connectivity index ($R^{(m)}(x)$, $H^{(m)}(x)$, $Y^{(m)}(x)$) at a 1-voxel x.

$$N_c^{(m)}(x) = R^{(m)}(x) - H^{(m)}(x) + Y^{(m)}(x) \tag{18}$$

where m (m = 6, 18, 18′, 26) denotes the type of connectivity.

(Proof) It is obvious that, if $Y^{(m)}(x) = 1$ (see Eq. (15)), $R^{(m)}(x) = 1$, $H^{(m)}(x) = 0$, and $N_c^{(m)}(x) = 2$. Thus Eq. (18) holds.

Assume that $Y^{(m)}(x) = 0$. Putting that $R^{(m)}(x) = l$ and $H^{(m)}(x) = n$, from Eq. (14),

$$N_c^{(m)}(x) = 1 + \Delta b_0^{(m)} - \Delta b_1^{(m)} + \Delta b_2^{(m)}$$

and

$$\Delta b_0^{(m)} = \alpha, \quad \Delta b_1^{(m)} = (\beta' - \beta), \quad \Delta b_2^{(m)} = -\gamma$$

where α, β, β' and γ are the same as those in Property 4.

Then, by Property 4,

$$N_c^{(m)}(x) = \alpha - (\beta' - \beta) - \gamma + 1 = \alpha + \beta + 1 - (\beta' + \gamma) = l - n. \qquad \text{(Q.E.D.)}$$

4.3. Calculation of the connectivity number (CN)

Methods for calculating the CN are obtained from its definition (Definition 8) and the methods for determining the Euler number presented in Subsec. 3.2 and 3.3 as are shown in the following properties.

[Property 5] Let $n_k^{(m)}$, and $n_k^{(m)\prime}$ denote the number of k-simplexes (k = 0, 1, 2, 3) contained in a 3D object before and after the deletion of a 1-voxel x, respectively. Denoting the CN of the 1-voxel x by $N_c^{(m)}(x)$,

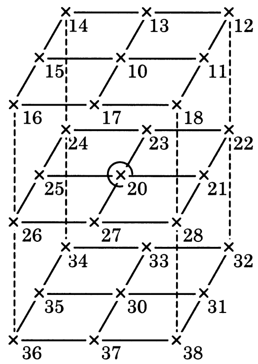


Fig. 14. Variables representing pixel values used in calculation of the connectivity number at the voxel x.

$$N_c^{(m)}(x) = \Delta n_1^{(m)} - \Delta n_2^{(m)} + \Delta n_3^{(m)} \tag{19}$$

where $\Delta n_k^{(m)} = n_k^{(m)} - n_k^{(m)\prime}$, and m represents the type of connectivity.

(Proof) Denoting the Euler number of an object before and after the deletion of the 1-voxel x by $E^{(m)}(x)$ and $E^{(m)}(\bar{x})$, respectively,

$$E^{(m)}(x) = n_0^{(m)} - n_1^{(m)} + n_2^{(m)} - n_3^{(m)}$$

$$E^{(m)}(\bar{x}) = n_0^{(m)\prime} - n_1^{(m)\prime} + n_2^{(m)\prime} - n_3^{(m)\prime}$$

and

$$n_0^{(m)\prime} - n_0^{(m)} = -1.$$

This and the definition of the CN imply Eq. (19) (Q.E.D.).

Thus, the simplex counting method for calculating the Euler number can be applied again to determine the value of the CN at each 1-voxel. By using the triangulation method, on the other hand, we can derive a set of pseudo-Boolean expressions to calculate the CN. To do so we will denote 1-voxels in a $3 \times 3 \times 3$ local area by the variables shown in Fig. 14. Here we represent by x_{ijk} both voxel itself and its density value which takes 0 or 1.

[Property 6] Let $X = \{x_{10}, x_{11}, \ldots, x_{38}\}$ (x_{20} is excluded) denote a set of 26 voxels in the 26-neighborhood of the voxel x_{20} in Fig. 14. Let us use the expression $N_c^{[m]}(X, x)$ instead of $N_c(x)$ so that we may show explicitly that the CN depends a set X as well as x itself. Then following equations hold.

$$N_c^{[26]}(X, x) = 2 - N_c^{[6]}(\bar{X}, x)$$

$$N_c^{[18]}(X, x) = 2 - N_c^{[18']}(\bar{X}, x) \tag{20}$$

where $\overline{X}$ is a set of variables which are complements of elements of X, that is $\overline{X} = \{\overline{x}_{10}, \overline{x}_{11}, \ldots, \overline{x}_{38}\}$, where $\overline{x}_{ij} = 1 - x_{ij}$. $N_c^{(6)}(\overline{X}, x)$ represents the value of the CN for the configuration consisting of $\overline{x}$ and $\overline{X}$ instead of x and X.

(Proof) Each term relating to a voxel x in the equations to calculate the Euler number is also a function of the set X and the variable x. Denote this term by $\Delta E(X, x)$. Then, by the definition of the CN,

$$\begin{aligned}
N_c^{(26)}(X,x) &= \Delta E^{(26)}(X,\overline{x}) - \Delta E^{(26)}(X,x) + 1 \\
&= \Delta E^{(6)}\left(\overline{X},\overline{x}\right) - \Delta E^{(6)}\left(\overline{X},x\right) + 1 \\
&= -\left(N_c^{(6)}\left(\overline{X},x\right) - 1\right) + 1 \\
&= 2 - N_c^{(6)}\left(\overline{X},x\right).
\end{aligned}$$

The second equation in Eq. (20) is also derived in the same way (Q.E.D).

[Property 7] The connectivity number $N_c^{(m)}(x)$ at a 1-voxel x is calculated by the following equation.

$$\begin{aligned}
N_c^{(6)}(x) = &\sum_{h=1,3} x_{h,0}\left(1 - \sum_{k\in S_1} x_{h,k}\cdot x_{2,k}\right) \\
&+ \sum_{k\in S_1} x_{2,k}\left\{1 - x_{2,k+1}\cdot x_{2,k+2}\left(1 - \sum_{h=1,3} x_{h,0}\cdot x_{h,k}\cdot x_{h,k+1}\cdot x_{h,k+2}\right)\right\}
\end{aligned} \tag{21}$$

$$\begin{aligned}
N_c^{(18)'}(x) = &\, x_{1,0} + x_{3,0} + \sum_{k\in S_1} x_{2,k}\left(1 - x_{2,k+1}\cdot x_{2,k+2}\right) \\
&- \sum_{k\in S_1}\sum_{h=1,3}\Big[x_{h,0}\cdot x_{h,k}\cdot x_{2,k}\left(1 - x_{h,k+2}\cdot x_{2,k+1}\cdot x_{2,k+2}\right) + x_{h,k+1} \\
&\qquad \cdot\Big\{x_{h,0}\cdot x_{2,k+1}\cdot\left(\overline{x}_{h,k}\cdot\overline{x}_{2,k+2}\cdot x_{2,k}\cdot x_{h,k+2} + \overline{x}_{2,k}\cdot\overline{x}_{h,k+2}\cdot x_{h,k}\cdot x_{2,k+2}\right) \\
&\qquad + \overline{x}_{h,0}\cdot\overline{x}_{2,k+1}\cdot x_{h,k}\cdot x_{2,k}\cdot x_{2,k+2}\cdot x_{h,k+2}\Big\}\Big]
\end{aligned} \tag{22}$$

$$\begin{aligned} N_c^{(18)}(x) = 2 - \bar{x}_{1,0} - \bar{x}_{3,0} - \sum_{k \in S_1} \bar{x}_{2,k}\left(1 - \bar{x}_{2,k+1} \cdot \bar{x}_{2,k+2}\right) \\ + \sum_{k \in S_1} \sum_{h=1,3} \Big[\bar{x}_{h,0} \cdot \bar{x}_{h,k} \cdot \bar{x}_{2,k}\left(1 - \bar{x}_{h,k+2} \cdot \bar{x}_{2,k+1} \cdot \bar{x}_{2,k+2}\right) + \bar{x}_{h,k+1} \\ \cdot \Big\{\bar{x}_{h,0} \cdot \bar{x}_{2,k+1} \cdot \left(x_{h,k} \cdot x_{2,k+2} \cdot \bar{x}_{2,k} \cdot \bar{x}_{h,k+2} + x_{2,k} \cdot x_{h,k+2} \cdot \bar{x}_{h,k} \cdot \bar{x}_{2,k+2}\right) \\ + x_{h,0} \cdot x_{2,k+1} \cdot \bar{x}_{h,k} \cdot \bar{x}_{2,k} \cdot \bar{x}_{2,k+2} \cdot \bar{x}_{h,k+2}\Big\}\Big] \end{aligned} \tag{23}$$

$$\begin{aligned} N_c^{(26)}(x) = 2 - \sum_{h=1,3} x_{h,0}\left(1 - \sum_{k \in S_1} \bar{x}_{h,k} \cdot \bar{x}_{2,k}\right) \\ + \sum_{k \in S_1} \bar{x}_{2,k}\left\{1 - \bar{x}_{2,k+1} \cdot \bar{x}_{2,k+2}\left(1 - \sum_{h=1,3} \bar{x}_{h,0} \cdot \bar{x}_{h,k} \cdot \bar{x}_{h,k+1} \cdot \bar{x}_{h,k+2}\right)\right\} \end{aligned} \tag{24}$$

where,

$$S_1 = \{1, 3, 5, 7\}, \quad x_{a,9} \equiv x_{a,1} \ (a = 1, 2, 3, 4)$$

and $\bar{x}_{a,b} \equiv 1 - x_{a,b}$.

(Proof) Equations (21)–(24) are derived from Theorem 1 and Definition 8.

4.4. Calculation of the connectivity index and voxel classification

The connectivity index ($R^{(m)}(x)$, $H^{(m)}(x)$, $Y^{(m)}(x)$) at each 1-voxel x is calculated by the following procedure.

(1) Component index $R^{(m)}(x)$: $R^{(m)}(x)$ is equal to the number of connected components connected to x, existing in the suitable neighborhood of x presented in Definition 9. Therefore component labeling is performed first in a local area including x and its neighborhood presented above. A labeling procedure was shown in YONEKURA *et al.* (1982a). Next, we extract all 1-voxels having the same label as that of x in the above area. Finally, after deleting the 1-voxel x, we again perform labeling on a set of the remaining 1-voxels. The resulting number of connected components equals $R^{(m)}(x)$.

Another method using a projection graph is also available. That is, if we automatically generate a projection graph, the number of connected components of the graph equals $R^{(m)}(x)$. Details are omitted here (YONEKURA *et al.*, 1980b, 1982b, 1982c).

(2) Cavity index $Y^{(m)}(x) = 0$ for all configurations except three cases in Fig. 15 where $Y^{(m)}(x) = 1$.

(3) Hole index $H^{(m)}(x)$: $H^{(m)}(x)$ is obtained by substituting into Eq. (18) the values of $R^{(m)}(x)$ and $Y^{(m)}(x)$ calculated by the above procedures and the value of $N_c^{(m)}(x)$ determined by the methods presented in Subsec. 4.3. Alternatively the value of $H^{(m)}(x)$ is calculated by counting connected components of 0-voxels in a suitably determined neighborhood of x and utilizing the following property.

[Property 8] Let $X = \{x_{10}, x_{11}, ..., x_{38}\}$ (x_{20} is excluded) denote a set of 26 voxels in the 26-neighborhood of a 1-voxel x ($=x_{20}$) in Fig. 14. Let us denote the connectivity index of the 1-voxel x_{20} by $(R^{(m)}(X, x), H^{(m)}(X, x), Y^{(m)}(X, x))$ instead of $(R^{(m)}(x), H^{(m)}(x), Y^{(m)}(x))$ so that dependency to the set X may be explicitly shown. If we define a set $\overline{X} = \{\overline{x}_{10}, \overline{x}_{11}, ..., \overline{x}_{38}\}$, where $\overline{x}_{ij} = 1 - x_{ij}$, and if we denote by $(R^{(m)}(\overline{X}, x), H^{(m)}(\overline{X}, x), Y^{(m)}(\overline{X}, x))$ the connectivity index of x with the local subpattern $\overline{X}$ in its 26-neighborhood, the following relations hold.

$$R^{(6)}(X,x) - 1 = H^{(26)}(\overline{X},x) - Y^{(26)}(\overline{X},x)$$

$$R^{(26)}(X,x) - 1 = H^{(6)}(\overline{X},x) - Y^{(6)}(\overline{X},x)$$

$$R^{(18)}(X,x) - 1 = H^{(18')}(\overline{X},x) - Y^{(18')}(\overline{X},x)$$

$$R^{(18')}(X,x) - 1 = H^{(18)}(\overline{X},x) - Y^{(18)}(\overline{X},x). \tag{25}$$

Furthermore, $Y^{(m)}(\overline{X}, x) = 1$ if and only if $R^{(m)}(\overline{X}, x) = H^{(m)}(\overline{X}, x) = 0$, and otherwise, $Y^{(m)}(\overline{X}, x) = 0$.

(Proof) Equation (25) are derived by enumerating possible projection graphs. Details are omitted here (YONEKURA *et al.*, 1982b).

All possible values of the connectivity number and the connectivity index are shown in Table 3. The state of a 1-voxel x is classified using these values. Several examples are presented below (*m* denotes the type connectivity) and Fig. 12.

(1) Interior voxel. A 1-voxel x is called an interior voxel if $R^{(m)}(x) = 1$, $H^{(m)}(x) = 0$ and $Y^{(m)}(x) = 1$, thus $N_c^{(m)}(x) = 2$. The cavity index $Y^{(m)}(x)$ is positive if and only if x is an interior voxel. All voxels in the *m*-neighborhood of an interior voxel have a value l for the *m*-c case (Fig. 15).

(2) Boundary voxel. A 1-voxel which is not the 3D interior voxel is called a 3D boundary voxel. A boundary voxel for the *m*-c case has at least one 0-voxel in its *m*-neighborhood.

(3) Connecting voxel. A 1-voxel x at which $N_c^{(m)}(x) = 1$ and $(R^{(m)}(x), H^{(m)}(x), Y^{(m)}(x)) = (2, 0, 0)$ is called a 3D connecting voxel because x has two connected components in its *m*-neighborhood.

Table 3. Possible values of the connectivity number (Nc) and the connectivity indexes (R, H, Y).

6-c*				18-c				18′-c				26-c			
Nc	R	H	Y	Nc	R	H	Y	Nc	R	H	Y	Nc	R	H	Y
–6	1	7	0												
–5	1	6	0												
–4	1	5	0	–4	1	5	0	–4	1	5	0	–4	1	5	0
–3	1	4	0	–3	1	4	0	–3	1	4	0	–3	1	4	0
–2	1	3	0	–2	1	3	0	–2	1	3	0	–2	1	3	0
	2	4	0												
–1	1	2	0	–1	1	2	0	–1	1	2	0	–1	1	2	0
	2	3	0										2	3	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	0		1	1	0		1	1	0		1	1	0
	2	2	0		2	2	0		2	2	0		2	2	0
1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
	2	1	0		2	1	0		2	1	0		2	1	0
	3	2	0										3	2	0
2	1	0	1	2	1	0	1	2	1	0	1	2	1	0	1
	2	0	0		2	0	0		2	0	0		2	0	0
	3	1	0		3	1	0		3	1	0		3	1	0
3	3	0	0	3	3	0	0	3	3	0	0	3	3	0	0
	4	1	0										4	1	0
4	4	0	0	4	4	0	0	4	4	0	0	4	4	0	0
													5	1	0
5	5	0	0	5	5	0	0	5	5	0	0	5	5	0	0
6	6	0	0	6	6	0	0	6	6	0	0	6	6	0	0
												7	7	0	0
												8	8	0	0

*6 (18, etc.) – c = 6 (18, etc.) connectivity case.

(4) Isolating voxel. A 1-voxel x at which $N_c^{(m)}(x) = 0$ and $(R^{(m)}(x), H^{(m)}(x), Y^{(m)}(x)) = (0, 0, 0)$ is called an isolating voxel. An isolating voxel has no 1-voxel in its *m*-neighborhood for the *m*-c case.

(5) Two-dimensional (2D) interior voxel. A 1-voxel x at which $N_c^{(m)}(x) = 0$ and $(R^{(m)}(x), H^{(m)}(x), Y^{(m)}(x)) = (1, 1, 0)$ is called a two-dimensional (2D) interior voxel.

[Property 9]

(1) The hole index $H^{(m)}(x)$ at a 1-voxel x is equal to the amount of increase in the one-dimensional Betti number in the 26-neighborhood of x caused by deleting x. In other words, $H^{(m)}(x)$ is equal to the number of holes in the $3 \times 3 \times 3$ local area consisting of x and its 26-neighborhood created by the deletion of x. Equivalently it equals the number of separate connected components of 0-voxels that are connected by the deletion of x.

(2) $0 \leqq H^{(m)}(x) \leqq 7$.

(3) $0 \leqq R^{(m)}(x) \leqq 8$.

(4) $-6 \leqq N_c^{(m)}(x) \leqq 8$.

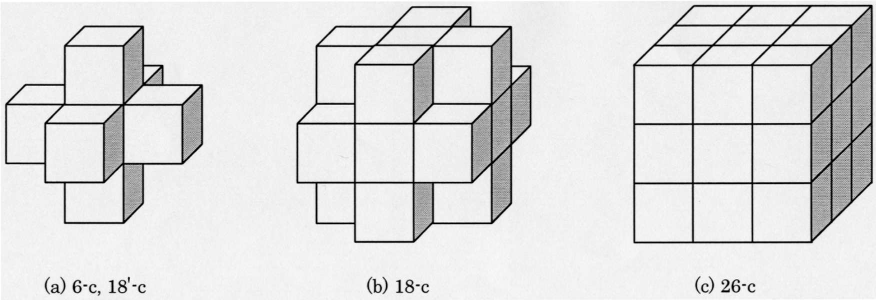


Fig. 15. Three-dimensional interior voxel (central voxel in the figures). All voxels show here are 1-voxeles. Others may be arbitrary.

5. Deletability of a 1-voxel

It is of crucial importance to delete a 1-voxel (replacing a 1-voxel by a 0-voxel) without changing connectivity and other topological properties of a 3D object for fundamental procedures of 3D picture processing such as thinning, shrinking, and feature measurements. We will derive here the necessary and sufficient condition that a 1-voxel is deletable.

[Definition 10] (deletability) A 1-voxel x is said to be deletable (strictly, m-deletable where m represents the type of connectivity. $m = 6, 18, 18', 26$), if its deletion (conversion into a 0-voxel) causes none of separation of connected components, vanishment of holes and cavities, and creation of new holes and cavities.

Obviously, an original 3D figure and the one resulted from deletion of a 1-voxel x are homeomorphic if x is deletable. It should be noted again that preservation of the Euler characteristic does not always mean the deletability in the above sense. The following property provides a necessary condition that a 1-voxel is deletable.

[Property 10] If a 1-voxel x is m-deletable,

$$N_c^{(m)}(x) = 1.$$

(Proof) Obviously, the Euler number of a 3D object is not changed by deleting a 1-voxel x, if x is deletable. This and the definition of the CN imply the property to be proved (Q.E.D.).

In order to derive the necessary and sufficient condition for the deletability we need to refer to the connectivity index. The result is shown in the following theorem.

[Theorem 3] Assume that density value of a 1-voxel x and its 26-neighborhood is given. Then the 1-voxel x is m-deletable if and only if the connectivity index $(R^{(m)}(x), H^{(m)}(x), Y^{(m)}(x)) = (1, 0, 0)$.

(Proof) It is obvious that a 1-voxel x is not deletable if it is a 3D interior voxel, that is, $Y^{(m)}(x) = 1$. Therefore $Y^{(m)}(x)$ should be equal to zero for x to be *m*-deletable.

Let us assume that, by deleting a 1-voxel x with the connectivity index $(l, n, 0)$, α connected components and β holes are created and β' holes and γ cavities vanish. Then, x is deletable if and only if

$$\alpha = \beta = \beta' = \gamma = 0. \tag{26}$$

From Property 4, Eq. (26) is equivalent to "$l = 1$ and $n = 0$" (Q.E.D.).

[Corollary 1] A 1-voxel x is *m*-deletable if and only if

$$N_c^{(m)}(x) = 1 \quad \text{and} \quad R^{(m)}(x) = 1. \tag{27}$$

To test the deletability condition we must know the values of the connectivity index, that is, $(R^{(m)}(x), H^{(m)}(x), Y^{(m)}(x))$. Since calculation of the hole index $H^{(m)}(x)$ requires the value of the CN $N_c^{(m)}$ as was presented in Subsec. 4.4, a procedure to check the deletability is reduced to the test of Corollary 1. $N_c^{(m)}(x)$ is calculated by using pseudo-Boolean expressions Eqs. (21)–(24). Calculation of $R^{(m)})(x)$ takes rather long time of computation because it includes labeling of a connected component.

A more efficient procedures is obtained by first testing whether "$N_c^{(m)}(x) = 1$" holds or not and then testing the equation $R^{(m)}(x) = 1$ if "$N_c^{(m)}(x) = 1$" is true. Details was presented in YONEKURA *et al*. (1982c).

6. Conclusion

We presented in this paper fundamental properties of topological structure of a 3D digitized picture. First we defined four types of connectivity (6-, 18-, 18′- and 26-connectivity) among picture volume cells (voxels) of 3D digitized picture defined on a cubic grid. Basing upon these four types of connectivity, we discussed simplicial decomposition of a 3D digitized picture with a list of all possible simplexes of zero to three dimensions.

Second we showed two algorithms for calculating the Euler number of a 3D digitized object, one is to compute a polynomial of binary variables representing voxel densities, and the other to perform local pattern matching.

Thirdly we introduced local features called the connectivity number and the connectivity index, and studied their properties in detail. The connectivity number at a 1-voxel x is defined as the difference between the Euler numbers of a 3D object before and after the deletion of the 1-voxel x. The connectivity index is a set of three numbers R(x), H(x), and Y(x) which are related to the number of connected components connected to x, increase or decrease in the numbers of holes and cavities caused by deleting the 1-voxel x, respectively.

Finally we proved the necessary and sufficient condition that a 1-voxel is deletable. That is, it was shown that a 1-voxel is deletable if and only if $R(x) = 1$, $H(x) = 0$ and $Y(x) = 0$, or equivalently the connectivity number $N_c(x) = 1$ and $R(x) = 1$.

Results presented here have many applications in recognition of 3D pictures. In particular they are important in the analysis of 3D pictures of the human body for computer aided diagnosis (TORIWAKI, 2002a, b). Calculation of the Euler number is effectively used for extracting shape features from a 3D figure. Test of deletability of a 1-voxel and connectivity indexes are of critical importance for deriving thinning algorithm (TORIWAKI and MORI, 2001). It was successfully applied to extract medial surfaces and medial lines of 3D objects such as vessels, liver, and tumor in 3D X-ray CT images of the human body in computer (SAITO and TORIWAKI, 1995, 1996; TORIWAKI and MORI, 2001; TORIWAKI, 2002a, b).

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