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Local Patterns and Connectivity Indexes in a Three Dimensional Digital Picture

Junichiro TORIWAKI[1]* and Tatsuhiro YONEKURA[2]

[1]*Department of Information Engineering, Graduate School of Engineering, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-8603, Japan*
[2]*Department of Computer and Information Sciences, School of Engineering, Ibaraki University, 4-12-1, Nakanarusawa-cho, Hitachi, Ibaraki 316-8511, Japan*
**E-mail address: toriwaki@nuie.nagoya-u.ac.jp*

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Abstract. In this paper we present the basic data of the local pattern on the area of the size of $3 \times 3 \times 3$ voxels denoted as $N_{333}(x)$ on a three dimensional digital binary picture. Topological features called the connectivity index are studied by calculating them for all possible patterns. Main results are summarized as follows.

(1) List of all possible values of the connectivity index

(2) Examples of local patterns on $N_{333}(x)$ which take specific values of the connectivity index.

(3) Numbers of different patterns on $N_{333}(x)$ for all possible values of the connectivity index.

All arrangements of 0 and 1 on $N_{333}(x)$ (local patterns) were systematically generated by computer, and feature values were calculated.

1. Introduction

A three dimensional digital picture (3D picture) is a set of gray values stored on a 3D array of cubic cells which are called voxels. It is conceptually a straight forward extension of a 2D digital picture which is a set of gray values stored in a 2D array of square pixels (Fig. 1).

3D pictures are now used in medical fields more and more, because of recent progress in imaging. High quality of 3D images of the human body are taken relatively easily nowadays and application to screening began to be studied.

Computer processing of 3D pictures has been studied actively in recent years, especially in the fields of computer aided diagnosis (CAD) of pictures and computer aided surgery (CAS) in medicine (TORIWAKI and MORI, 1999; TORIWAKI, 2002b). Many algorithms of picture analysis and recognition have also been developed (KONG and ROSENFELD, 1989; NIKOLARIDIS, 2001; TORIWAKI, 2002a). However, they are not so

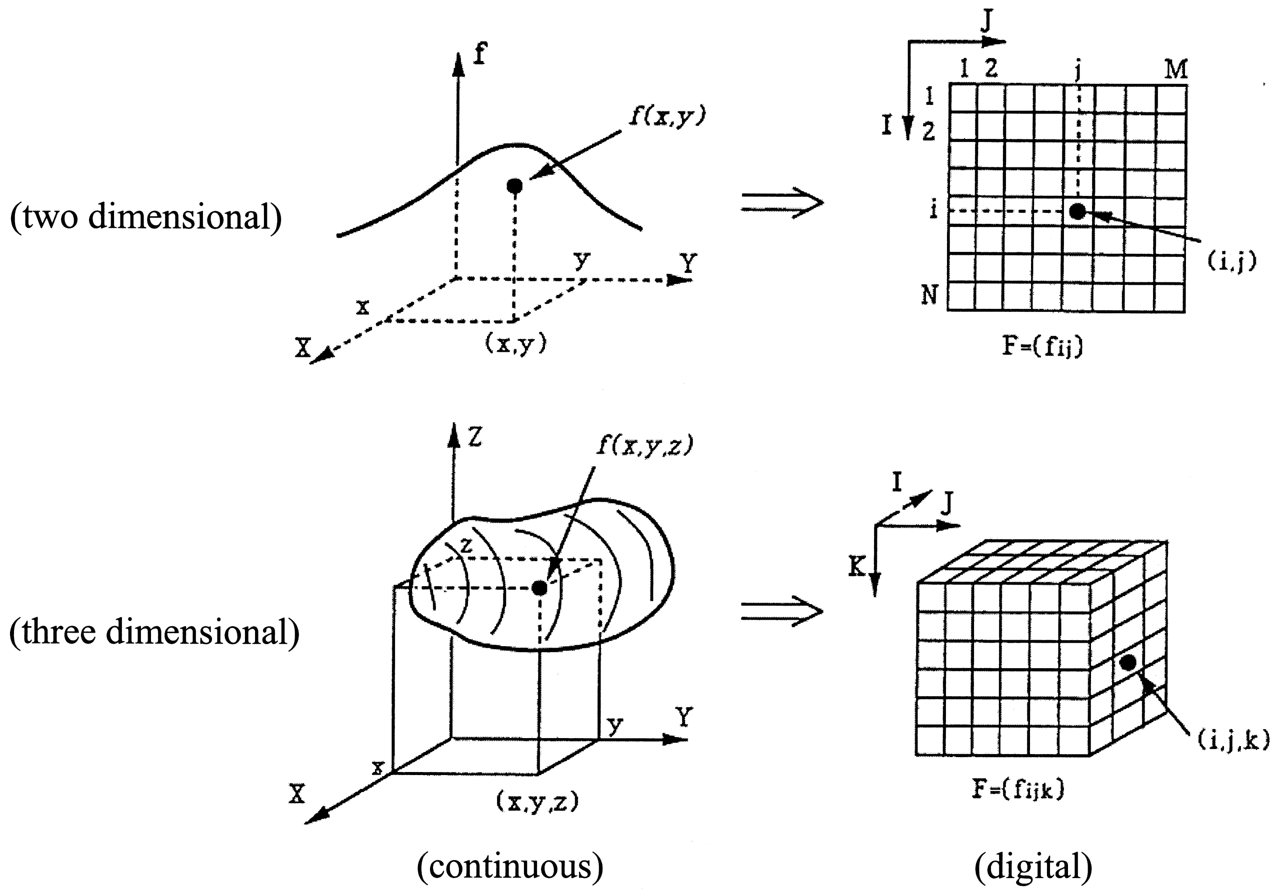

Fig. 1. Digital pictures. Upper row: two dimensional picture. Lower row: three dimensional picture. Left: continuous picture. Right: digital picture.

advanced, being compared with those of 2D pictures (ROSENFELD and KAK, 1982; TORIWAKI, 1988; WATT and POLICARPO, 1998).

One important basic issue in 3D image analysis is the geometry of 3D pictures. Geometrical properties of 3D digital pictures are extremely different from 2D pictures. For example, existence of handles (holes), linkage of two loops, and knot of a line figure are phenomena which are not seen in 2D figures (TORIWAKI, 2002a).

The field of study on the geometrical property of digital pictures is called discrete geometry or digital geometry (KONG and ROSCOE, 1985, KONG and ROSENFELD, 1989; KLETTE *et al.*, 1998; HERMAN, 1998; BERTRAND *et al.*, 2001; TORIWAKI, 2002a, TORIWAKI and YONEKURA, 2002; BRAQUELAIRE *et al.*, 2002). In spite of recent increase in the interest to this field, many important problems still remain unsolved. One of them is basic properties of local patterns. Even for the minimum local area of $3 \times 3 \times 3$ voxels, properties of all possible patterns have not been completely revealed.

From the viewpoint of algorithms one important class of picture processing algorithms is a local operation based on a $3 \times 3 \times 3$ neighborhood. They are designed by considering possible $3 \times 3 \times 3$ patterns carefully. In this article we present basic properties of local patterns on the $3 \times 3 \times 3$ local area by investigating all possible arrangements of 0's and 1's exhaustively. We calculate four indexes of the topological features for all of $3 \times 3 \times 3$ patterns. We show numbers of different local patterns for each values of features. Four features above are called the connectivity index, the hole index, the cavity index, and the connectivity number (the last one is derived from the others) and are explained in detail in authors previous review article (TORIWAKI and YONEKURA, 2002). These results will be useful for designing algorithms of the image analysis such as thinning and shrinking (YONEKURA *et al.*, 1980c; SAITO and TORIWAKI, 1996), and other topology-preserving transformations.

2. Basic Definitions

A 3D picture (three dimensional digital picture) is formally defined as a mapping

$$\mathrm{I} \times \mathrm{I} \times \mathrm{I} \to \mathrm{R},$$

where I is the set of all integers, and R is the set of all real numbers.

In particular, the picture defined as $\mathrm{I} \times \mathrm{I} \times \mathrm{I} \to \{0, 1\}$ is called a (3D) binary picture. A binary picture represents a digitized picture consisting of a cubic element (called the voxel) whose density values are 0 or 1. A voxel which has a value 0 and 1 is called the 0-voxel and 1-voxel, respectively.

[Definition 1] (neighborhood) For an arbitrary voxel x, sets of the following voxels are called the 6-neighborhood, the 18-neighborhood, and the 26-neighborhood of x, and denoted by $\mathrm{N}^{[m]}(\mathrm{x})$, where m = 6, 18, 26, respectively (Fig. 2).

Intuitively, they can be described as follows.

6-neighborhood = Set of voxels which share one surface with x.

18-neighborhood = Set of voxels which share at least one edge with x.

26-neighborhood = Set of voxels which share at least one vertex with x.

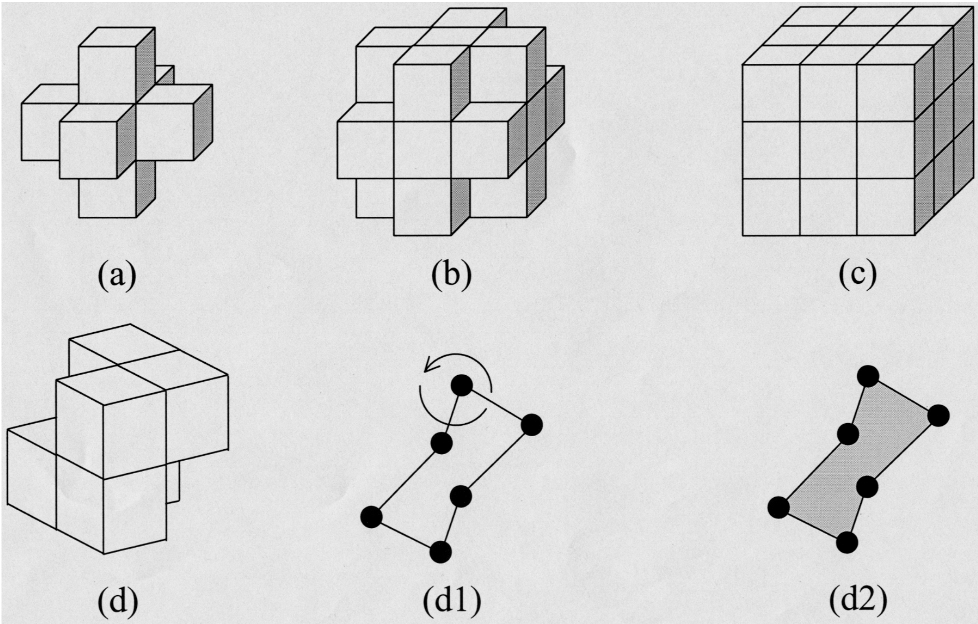

Fig. 2. Three kinds of neighborhood. The 18′-connectivity is topologically the same as the 6-connectivity except for only one specific arrangement shown in d). This arrangement of 0- and 1-voxels is regarded as a loop in the 6-connectivity case (d1), and is treated as a surface in the 18′-connectivity case (d2). Therefore the 18′-neighborhood is not considered.

A 1-voxel in the m-neighborhood of the 1-voxel x is said to be m-adjacent to x (or m-connected to x). Those relations between 0-voxels are also defined in the same way.

[Definition 2] (connectivity) For two voxels x_1 and x_2 of the same value, if the sequence of voxels

$$y_0\ (=x_1),\ y_1,\ y_2,\ \ldots,\ y_n\ (=x_2),$$

exists, such that

$$y_i \in N^{[m]}\ (y_{i-1}),$$

$m = 6, 18, 18'$, and 26 ($1 \leqq i \leqq n$) and density values are the same as x, then voxels x_1 and x_2 are said to be m-connected each other.

Connectivity is an equivalence relation among voxels. Each equivalence class of voxels is called a connected component. Connectivity for 1-voxels and that of 0-voxels are defined in the same way.

When we consider connected components of 1-voxels and those of 0-voxels simultaneously, only specific pairs of connectivities as shown below are admissible so that we may avoid the contradiction specific to digital pictures.

$$\begin{aligned}&(\text{connectivity of 1-voxels, that of 0-voxels}) = (m, m')\\&\qquad = (6, 26), (26, 6), (18, 18'), (18', 18).\end{aligned} \tag{1}$$

The 18′-connectivity here is almost the same as the 6-connectivity and introduced specifically to keep consistency of the 18-connectivity. For details see (TORIWAKI, 2002a; TORIWAKI and YONEKURA, 2002).

Let us denote the $3 \times 3 \times 3$ neighborhood of a voxel x as $N_{333}(x)$, that is,

$$N_{333}(x) = N^{[26]}(x) \cup x.$$

The neighborhood $N_{333}(x)$ is very important in practical picture processing, particularly in the processing of binary picture. Large parts of processing algorithms used in practice are based on $N_{333}(x)$. For example, many of linear spatial filters are defined by weight values given on $N_{333}(x)$. Important operations for binary pictures such as thinning are designed considering arrangement of 0's and 1's of an input picture on $N_{333}(x)$. Including them, a class of operators such that the value of the output at each voxel x is determined only from input values on $N_{333}(x)$ is called "local operation based on $N_{333}(x)$" or simply "local operation".

A local operation is said to be "topology preserving", if the topology of an input picture does not change by its execution. In other words, an operator O for a digital picture is topology preserving if, by its execution, none of the following occurs:

- generation and extinction of connected components,
- generation and extinction of holes (handles),
- generation and extinction of cavities.

Changing a 1-voxel x into a 0-voxel is called the deletion of the voxel x. If the deletion of a voxel x does not change the topology of an input picture, the voxel x is said to be deletable. The deletable 1-voxel may be often called a simple point. To examine whether a given voxel x is deletable or not is called the deletability test (or simple point test). Because of simplicity and effectiveness, the deletable test realizable by a local operation is important (SAITO and TORIWAKI, 1995; TORIWAKI and MORI, 2001). Concrete forms of equations (or local operations) to test deletability depend on the type of connectivity. Authors showed such equations of the pseudo-Boolean type for all of four kinds of connectivity (TORIWAKI and YONEKURA, 2002).

3. Connectivity Index

3.1. The connectivity number and the connectivity index

We use in this paper local features defined below (TORIWAKI, 2002a; TORIWAKI and YONEKURA, 2002).

[Definition 3] (Connectivity number) The connectivity number $N_c^{[m]}(x)$ at a 1-voxel x for the *m*-connectivity case (*m*-c. case) is defined as

$$N_c^{[m]}(x) = E^{[m]}(x) - E^{[m]}(\bar{x}) + 1,$$

where $E^{[m]}(x)$ and $E^{[m]}(\bar{x})$ are the Euler numbers of the whole of a given binary picture before and after the 1-voxel x is deleted, respectively.

Next we define a set of three local features called the connectivity index as follows.

[Definition 4] (Connectivity index) At each 1-voxel x and the 3 × 3 × 3 neighborhood $N_{333}(x)$ centered at the voxel x, we define the following set of local features ($R^{(m)}(x)$, $H^{(m)}(x)$, $Y^{(m)}(x)$) and call it the connectivity index, where

(a) Component index $R^{(m)}(x)$ = the number of *m*-connected components which are connected to x and exist in the 18-neighborhood for $m = 6$ and 18′ (in the 26-neighborhood for $m = 18$ and 26).

(b) Hole index $H^{(m)}(x)$ = the number of holes which are newly generated by the deletion of x, or equivalently the decrease in the number of $\overline{m}$-connected components of 0-voxels in this local area caused by the deletion of x ($\overline{m}$ means the type of connectivity determined from *m* by Eq. (1),

(c) Cavity index $Y^{(m)}(x)$ = the number of cavities which are newly generated by the deletion of the voxel x.

3.2. Properties of local features

Let us summarize here important properties of local features introduced above to show the significance of them. Proofs and further detail are given in TORIWAKI (2002a), TORIWAKI and YONEKURA (2002).

[Property 1] The connectivity number $N_c^{(m)}(x)$ at a 1-voxel x is equal to the following value,

$N_c^{(m)}(x)$ = (change in the number of connected components
– change in the number of holes + change in the number of cavities) + 1 (2)

caused in the whole picture by replacing the 1-voxel x by a 0-voxel (by deleting x).

[Property 2] The following relation holds among the connectivity number and the connectivity index ($R^{(m)}(x)$, $H^{(m)}(x)$, $Y^{(m)}(x)$) at a 1-voxel x.

$$N_c^{(m)}(x) = R^{(m)}(x) - H^{(m)}(x) + Y^{(m)}(x) \tag{3}$$

where *m* ($m = 6$, 18, 18′, 26) denotes the type of connectivity.

[Property 3] Assume that density values of a 1-voxel x and its 26-neighborhood are given. Then the 1-voxel x is deletable if and only if the connectivity index ($R^{(m)}(x)$, $H^{(m)}(x)$, $Y^{(m)}(x)$) = (1, 0, 0), $m = 6$, 18, 18′, 26

[Property 4] A 1-voxel x is deletable if and only if

$$N_c^{(m)}(x) = 1 \quad \text{and} \quad R^{(m)}(x) = 1, \quad m = 6, 18, 18', 26. \tag{4}$$

All of these were referred from (TORIWAKI and YONEKURA, 2002). Derivation and proofs of them are given there with the method of calculation. As was stated there, the feature corresponding to the hole (or the handle) above does not exist in a 2D picture. Therefore preservation of the Euler number exactly means the preservation of topology. In the case of 3D pictures, however, the set of three features ($R^{(m)}(x)$, $H^{(m)}(x)$, $Y^{(m)}(x)$), or the pair of two features ($N_c^{(m)}(x)$, $R^{(m)}(x)$) is required for testing the deletability of a 1-voxel x.

4. Counting Local Patterns

Presently the most important use of these features is the deletability test. They are also expected to be useful as shape features of 3D figures. However, detailed properties of these features have not been studied. For instance, possible values of features have not been reported except for a few original papers (YONEKURA *et al.*, 1980b) and only one book in Japanese (TORIWAKI, 2002a). It has not been reported which type of local arrangements of 1-voxels take which values of features. How many different patterns exist for a given set of values of the connectivity index has not been published except one paper written in Japanese (TORIWAKI *et al.*, 1995).

In this paper we provide the following data on the values of the connectivity index:

(1) List of all possible values of the connectivity index.

(2) Examples of local patterns on $N_{333}(x)$ which take specific values of the connectivity index.

(3) Numbers of different patterns on $N_{333}(x)$ for all possible values of the connectivity index.

All feature values were calculated according to the methods given in YONEKURA *et al.* (1980a, b). All arrangements of 0 and 1 on $N_{333}(x)$ (local patterns) were systematically generated by computer, and transferred to the program to calculate feature values. Local patterns which are symmetry to other patterns with respect to the points, lines and planes in the $3 \times 3 \times 3$ space were detected and excluded from the procedure of feature value calculation.

Let us introduce parts of the detailed procedure briefly.

(1) Consider one of $3 \times 3 \times 3$ local patterns and embed it in the center of a small 3D picture of the size $5 \times 5 \times 5$ voxels. All the background voxels are assumed to be 0. Let us denote this picture as F for the convenience of explanation.

(2) Calculate the connectivity number $N_c^{(m)}$ of F by counting simplexes or by utilizing pseudo-Boolean equations as shown in TORIWAKI (2002a) and TORIWAKI and YONEKURA (2002).

(3) Calculate the component index $R^{(m)}$ of F by counting the number of connected components by the labeling algorithm.

(4) Calculate the cavity index $Y^{(m)}$ of F by simple pattern matching.

Table 1. Possible values of connectivity indexes.

6 connectivity				26 connectivity				18 connectivity				18' connectivity			
R	H	Y	Nc	R	H	Y	Nc	R	H	Y	Nc	R	H	Y	Nc
1	7	0	-6	1	5	0	-4	1	5	0	-4	1	5	0	-4
1	6	0	-5	1	4	0	-3	1	4	0	-3	1	4	0	-3
1	5	0	-4	1	3	0	-2	1	3	0	-2	1	3	0	-2
1	4	0	-3	1	2	0	-1	1	2	0	-1	1	2	0	-1
1	3	0	-2	2	3	0	-1	0	0	0	0	0	0	0	0
2	4	0	-2	0	0	0	0	1	1	0	0	1	1	0	0
1	2	0	-1	1	1	0	0	2	2	0	0	2	2	0	0
2	3	0	-1	2	2	0	0	1	0	0	1	1	0	0	1
0	0	0	0	1	0	0	1	2	1	0	1	2	1	0	1
1	1	0	0	2	1	0	1	1	0	1	2	1	0	1	2
2	2	0	0	3	2	0	1	2	0	0	2	2	0	0	2
1	0	0	1	1	0	1	2	3	1	0	2	3	1	0	2
2	1	0	1	2	0	0	2	3	0	0	3	3	0	0	3
3	2	0	1	3	1	0	2	4	0	0	4	4	0	0	4
1	0	1	2	3	0	0	3	5	0	0	5	5	0	0	5
2	0	0	2	4	1	0	3	6	0	0	6	6	0	0	6
3	1	0	2	4	0	0	4								
3	0	0	3	5	1	0	4								
4	1	0	3	5	0	0	5								
4	0	0	4	6	0	0	6								
5	0	0	5	7	0	0	7								
6	0	0	6	8	0	0	8								

Notations R, H, Y: component index, hole index, cavity index.
N_c: connectivity number.

(5) Obtain the value of the hole index $H^{(m)}$ of F from the connectivity number $N_c{}^{(m)}$, the component index $R^{(m)}$ and the cavity index $Y^{(m)}$ calculated above.

(6) Count the number of $3 \times 3 \times 3$ patterns for each values of indexes ($R^{(m)}$, $H^{(m)}$, $Y^{(m)}$) with iterating the procedure (2)~(5) for all possible $3 \times 3 \times 3$ patterns. The values of B in Table 2 are obtained by doing this procedure.

(7) The values of A in the same table are found by applying the above steps, but excluding patterns which are symmetry to any other pattern.

(8) Table 1 is derived immediately by the above results.

5. Results and Discussion

(1) On the local area of $3 \times 3 \times 3$ voxels $N_{333}(x)$, the number of possible patterns of 0- and 1-voxels is $2^{27} = 134{,}217{,}728$. Without loss of generality we assume that the center voxel of $N_{333}(x)$ is a 1-voxel. Still $2^{26} = 6{,}710{,}864$ patterns may exist. In the case of a 2D picture, the 3×3 local area is the basic neighborhood, and $2^9 = 512$ patterns may exist on it.

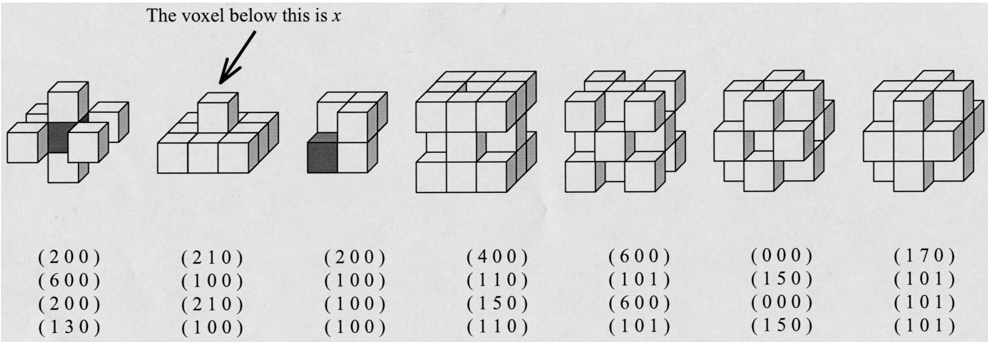

Fig. 3. Examples of local patterns with values of connectivity indexes. Numbers in parenthesis are (component index, hole index, cavity index) at the voxel x which is located at the center of a $3 \times 3 \times 3$ voxel array, or at the voxel which is shaded in the figure.

(2) Consider the symmetry relation on $N_{333}(x)$. Then 48 different types of symmetry can be possible. All of them are given in TORIWAKI and YONEKURA (2002). By excluding patterns which are symmetric to other patterns the number of all possible patterns is reduced to be 2,852,288. This number was derived theoretically by SAKABE *et al.* (1981), and was confirmed by exhaustive enumeration of all 2^{27} patterns here.

(3) All possible values of the connectivity index are shown in Table 1. The same result was already obtained in YONEKURA *et al.* (1980a, b) through the heuristic considerations. These were also confirmed here by the exhaustive procedure.

(4) In Table 2, numbers of different patterns are shown for each of all possible values of the connectivity index for all of four kinds of connectivity.

(5) Several examples of patterns on $N_{333}(x)$ are illustrated in Fig. 3 with their connectivity indexes.

(6) For fixed values of the connectivity index, maximum numbers of patterns are more than seven millions and more than one hundred and sixty thousands even for excluding all symmetry cases. For example, 26,073,184 patterns exist for (R, H, Y) = (2, 0, 0) (6-c. case). Such numbers are far beyond the possibility of the heuristic design of local operations. Because of the progress in computer technology, however, it is not difficult to store more than a few megabytes of binary patterns in local memory nowadays. Therefore execution of local pattern matching for $3 \times 3 \times 3$ patterns will be possible enough, if we could develop suitable operations.

(7) The number of patterns satisfying the delectability condition (that is, (R, H, Y) = (1, 1, 0)) is 26,073,184 (26-c. case). This is also too large for finding better 3D thinning algorithms by the heuristics of engineers or cut and try procedures. In the case of two dimensional pictures, a thinning algorithm could be designed successfully by considering less than 30 local patterns (TORIWAKI and YOKOI, 1985; TORIWAKI, 1988).

(8) There is only one case that the cavity index Y = 1 for each connectivity, and Y = 0 for other patterns. Concrete patterns for Y = 1 will be obvious.

Table 2. Numbers of possible local patterns classified according to values of connectivity indexes.

6c				No. of	No. of patterns			26c				No. of	No. of patterns		
R	H	Y	Nc	1-voxels	A	B	A/B	R	H	Y	Nc	1-voxels	A	B	A/B
2	0	0	2	3	2	15	13.3	1	1	0	0	5	7	99	7.1
				4	10	288	3.5					6	47	1680	2.8
				5	75	2670	2.8					7	347	13990	2.5
				6	371	15888	2.3					8	1648	74088	2.2
				7	1531	68121	2.2					9	6034	276741	2.2
				8	4848	224040	2.2					10	16456	771464	2.1
				9	12603	587520	2.1					11	35347	1663548	2.1
				10	26738	1260120	2.1					12	60078	2845944	2.1
				11	47595	2247318	2.1					13	83145	3938614	2.1
				12	70936	3363888	2.1					14	94327	4479192	2.1
				13	89406	4237956	2.1					15	89406	4237956	2.1
				14	94327	4479192	2.1					16	70936	3363888	2.1
				15	83145	3938614	2.1					17	47595	2247318	2.1
				16	60078	2845944	2.1					18	26738	1260120	2.1
				17	35347	1663548	2.1					19	12603	587520	2.1
				18	16456	771464	2.1					20	4848	224040	2.2
				19	6034	276741	2.2					21	1531	68121	2.2
				20	1648	74088	2.2					22	371	15888	2.3
				21	347	13990	2.5					23	75	2670	2.8
				22	47	1680	2.8					24	10	288	3.5
				23	7	99	7.1					25	2	15	13.3
				TOTAL	551551	26073184	2.1					TOTAL	551551	26073184	2.1

6c				No. of	No. of patterns			26c				No. of	No. of patterns		
R	H	Y	Nc	1-voxels	A	B	A/B	R	H	Y	Nc	1-voxels	A	B	A/B
1	0	0	1	2	1	6	16.7	1	0	0	1	2	3	26	11.5
				3	5	120	4.2					3	6	132	4.5
				4	35	1152	3.0					4	30	844	3.6
				5	174	7068	2.5					5	127	4924	2.6
				6	721	31158	2.3					6	565	23928	2.4
				7	2321	105276	2.2					7	2054	93136	2.2
				8	6174	284288	2.2					8	6239	287932	2.2
				9	13543	632528	2.1					9	15155	711480	2.1
				10	25332	1188042	2.1					10	30214	1422586	2.1
				11	40756	1922536	2.1					11	49359	2336836	2.1
				12	57719	2725528	2.1					12	67795	3209124	2.1
				13	72266	3422796	2.1					13	79194	3755736	2.1
				14	80647	3817962	2.1					14	80647	3817962	2.1
				15	79194	3755736	2.1					15	72266	3422796	2.1
				16	67795	3209124	2.1					16	57719	2725528	2.1
				17	49359	2336836	2.1					17	40756	1922536	2.1
				18	30214	1422586	2.1					18	25332	1188042	2.1
				19	15155	711480	2.1					19	13543	632528	2.1
				20	6239	287932	2.2					20	6174	284288	2.2
				21	2054	93136	2.2					21	2321	105276	2.2
				22	565	23928	2.4					22	721	31158	2.3
				23	127	4924	2.6					23	174	7068	2.5
				24	30	844	3.6					24	35	1152	3.0
				25	6	132	4.5					25	5	120	4.2
				26	3	26	11.5					26	1	6	16.7
				TOTAL	550435	25985144	2.1					TOTAL	550435	25985144	2.1

Table 2. (continued).

6c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B	26c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B
3 0 0 3	4	2	20	10.0	1 2 0 -1	6	2	24	8.3
	5	12	352	3.4		7	15	500	3.0
	6	80	2996	2.7		8	124	4756	2.6
	7	377	16324	2.3		9	616	27380	2.2
	8	1421	63668	2.2		10	2365	107244	2.2
	9	4064	188356	2.2		11	6526	304500	2.1
	10	9368	436840	2.1		12	13912	651432	2.1
	11	17172	808988	2.1		13	22805	1077084	2.1
	12	25602	1206460	2.1		14	29698	1400736	2.1
	13	30655	1450692	2.1		15	30655	1450692	2.1
	14	29698	1400736	2.1		16	25602	1206460	2.1
	15	22805	1077084	2.1		17	17172	808988	2.1
	16	13912	651432	2.1		18	9368	436840	2.1
	17	6526	304500	2.1		19	4064	188356	2.2
	18	2365	107244	2.2		20	1421	63668	2.2
	19	616	27380	2.2		21	377	16324	2.3
	20	124	4756	2.6		22	80	2996	2.7
	21	15	500	3.0		23	12	352	3.4
	22	2	24	8.3		24	2	20	10.0
	TOTAL	164816	7748352	2.1		TOTAL	164816	7748352	2.1

6c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B	26c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B
1 1 0 0	7	1	8	12.5	2 0 0 2	3	11	193	5.7
	8	5	128	3.9		4	34	1212	2.8
	9	29	1023	2.8		5	163	6160	2.6
	10	131	5380	2.4		6	566	24652	2.3
	11	480	20884	2.3		7	1738	77350	2.2
	12	1394	63420	2.2		8	4129	190524	2.2
	13	3382	155284	2.2		9	7965	369188	2.2
	14	6638	309692	2.1		10	12022	564068	2.1
	15	10733	501532	2.1		11	14560	681590	2.1
	16	13914	654108	2.1		12	13914	654108	2.1
	17	14560	681590	2.1		13	10733	501532	2.1
	18	12022	564068	2.1		14	6638	309692	2.1
	19	7965	369188	2.2		15	3382	155284	2.2
	20	4129	190524	2.2		16	1394	63420	2.2
	21	1738	77350	2.2		17	480	20884	2.3
	22	566	24652	2.3		18	131	5380	2.4
	23	163	6160	2.6		19	29	1023	2.8
	24	34	1212	2.8		20	5	128	3.9
	25	11	193	5.7		21	1	8	12.5
	TOTAL	77895	3626396	2.1		TOTAL	77895	3626396	2.1

Table 2. (continued).

6c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B	26c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B
0 0 0 0	1	1	1	100	1 0 1 2	7	1	1	100
	2	2	20	10.0		8	2	20	10.0
	3	10	190	5.3		9	10	190	5.3
	4	37	1140	3.2		10	37	1140	3.2
	5	136	4845	2.8		11	136	4845	2.8
	6	376	15504	2.4		12	376	15504	2.4
	7	907	38760	2.3		13	907	38760	2.3
	8	1746	77520	2.3		14	1746	77520	2.3
	9	2811	125970	2.2		15	2811	125970	2.2
	10	3695	167960	2.2		16	3695	167960	2.2
	11	4078	184756	2.2		17	4078	184756	2.2
	12	3695	167960	2.2		18	3695	167960	2.2
	13	2811	125970	2.2		19	2811	125970	2.2
	14	1746	77520	2.3		20	1746	77520	2.3
	15	907	38760	2.3		21	907	38760	2.3
	16	376	15504	2.4		22	376	15504	2.4
	17	136	4845	2.8		23	136	4845	2.8
	18	37	1140	3.2		24	37	1140	3.2
	19	10	190	5.3		25	10	190	5.3
	20	2	20	10.0		26	2	20	10.0
	21	1	1	100		27	1	1	100
	TOTAL	23520	1048576	2.2		TOTAL	23520	1048576	2.2

6c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B	26c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B
2 1 0 1	8	1	24	4.2	2 1 0 1	6	6	150	4.0
	9	9	336	2.7		7	42	1680	2.5
	10	56	2334	2.4		8	219	9344	2.3
	11	233	10416	2.2		9	729	33024	2.2
	12	724	33048	2.2		10	1763	80922	2.2
	13	1678	77832	2.2		11	3081	143464	2.1
	14	2969	138194	2.1		12	4040	188400	2.1
	15	3968	185664	2.1		13	3968	185664	2.1
	16	4040	188400	2.1		14	2969	138194	2.1
	17	3081	143464	2.1		15	1678	77832	2.2
	18	1763	80922	2.2		16	724	33048	2.2
	19	729	33024	2.2		17	233	10416	2.2
	20	219	9344	2.3		18	56	2334	2.4
	21	42	1680	2.5		19	9	336	2.7
	22	6	150	4.0		20	1	24	4.2
	TOTAL	19518	904832	2.2		TOTAL	19518	904832	2.2

Table 2. (continued).

6c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B	26c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B
4 0 0 4	5	2	15	13.3	1 3 0 -2	7	1	3	33.3
	6	8	228	3.5		8	3	60	5.0
	7	50	1668	3.0		9	22	588	3.7
	8	186	7716	2.4		10	92	3524	2.6
	9	589	25074	2.3		11	350	14210	2.5
	10	1337	60228	2.2		12	920	40740	2.3
	11	2435	109700	2.2		13	1928	85948	2.2
	12	3336	153308	2.2		14	2978	136372	2.2
	13	3629	164970	2.2		15	3629	164970	2.2
	14	2978	136372	2.2		16	3336	153308	2.2
	15	1928	85948	2.2		17	2435	109700	2.2
	16	920	40740	2.3		18	1337	60228	2.2
	17	350	14210	2.5		19	589	25074	2.3
	18	92	3524	2.6		20	186	7716	2.4
	19	22	588	3.7		21	50	1668	3.0
	20	3	60	5.0		22	8	228	3.5
	21	1	3	33.3		23	2	15	13.3
	TOTAL	17866	804352	2.2		TOTAL	17866	804352	2.2

6c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B	26c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B
1 2 0 -1	10	1	12	8.3	3 0 0 3	4	20	544	3.7
	11	5	156	3.2		5	73	3060	2.4
	12	31	1104	2.8		6	277	11652	2.4
	13	123	5208	2.4		7	721	32752	2.2
	14	413	17976	2.3		8	1531	69408	2.2
	15	1013	46236	2.2		9	2404	111660	2.2
	16	1938	88572	2.2		10	2966	136652	2.2
	17	2726	126792	2.1		11	2726	126792	2.1
	18	2966	136652	2.2		12	1938	88572	2.2
	19	2404	111660	2.2		13	1013	46236	2.2
	20	1531	69408	2.2		14	413	17976	2.3
	21	721	32752	2.2		15	123	5208	2.4
	22	277	11652	2.4		16	31	1104	2.8
	23	73	3060	2.4		17	5	156	3.2
	24	20	544	3.7		18	1	12	8.3
	TOTAL	14242	651784	2.2		TOTAL	14242	651784	2.2

6c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B	26c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B
1 3 0 -2	13	2	48	4.2	4 0 0 4	5	29	707	4.1
	14	12	456	2.6		6	77	3240	2.4
	15	64	2457	2.6		7	221	8972	2.5
	16	181	8016	2.3		8	386	17296	2.2
	17	389	16836	2.3		9	552	24042	2.3
	18	528	23984	2.2		10	528	23984	2.2
	19	552	24042	2.3		11	389	16836	2.3
	20	386	17296	2.2		12	181	8016	2.3
	21	221	8972	2.5		13	64	2457	2.6
	22	77	3240	2.4		14	12	456	2.6
	23	29	707	4.1		15	2	48	4.2
	TOTAL	2441	106054	2.3		TOTAL	2441	106054	2.3

Table 2. (continued).

6c	No. of	No. of patterns			26c	No. of	No. of patterns		
R H Y Nc	1-voxels	A	B	A/B	R H Y Nc	1-voxels	A	B	A/B
3 1 0 2	9	1	24	4.2	2 2 0 0	8	1	24	4.2
	10	7	264	2.7		9	10	303	3.3
	11	35	1407	2.5		10	41	1656	2.5
	12	105	4584	2.3		11	126	5244	2.4
	13	225	9876	2.3		12	242	10752	2.3
	14	324	14592	2.2		13	341	15018	2.3
	15	341	15018	2.3		14	324	14592	2.2
	16	242	10752	2.3		15	225	9876	2.3
	17	126	5244	2.4		16	105	4584	2.3
	18	41	1656	2.5		17	35	1407	2.5
	19	10	303	3.3		18	7	264	2.7
	20	1	24	4.2		19	1	24	4.2
	TOTAL	1458	63744	2.3		TOTAL	1458	63744	2.3

6c	No. of	No. of patterns			26c	No. of	No. of patterns		
R H Y Nc	1-voxels	A	B	A/B	R H Y Nc	1-voxels	A	B	A/B
2 2 0 0	11	1	24	4.2	3 1 0 2	7	5	120	4.2
	12	6	240	2.5		8	22	912	2.4
	13	33	1264	2.6		9	83	3316	2.5
	14	93	4080	2.3		10	168	7488	2.2
	15	197	8508	2.3		11	262	11376	2.3
	16	264	11888	2.2		12	264	11888	2.2
	17	262	11376	2.3		13	197	8508	2.3
	18	168	7488	2.2		14	93	4080	2.3
	19	83	3316	2.5		15	33	1264	2.6
	20	22	912	2.4		16	6	240	2.5
	21	5	120	4.2		17	1	24	4.2
	TOTAL	1134	49216	2.3		TOTAL	1134	49216	2.3

6c	No. of	No. of patterns			26c	No. of	No. of patterns		
R H Y Nc	1-voxels	A	B	A/B	R H Y Nc	1-voxels	A	B	A/B
5 0 0 5	6	1	6	16.7	1 4 0 -3	10	1	6	16.7
	7	3	72	4.2		11	3	72	4.2
	8	14	408	3.4		12	14	408	3.4
	9	38	1416	2.7		13	38	1416	2.7
	10	87	3306	2.6		14	87	3306	2.6
	11	131	5424	2.4		15	131	5424	2.4
	12	158	6384	2.5		16	158	6384	2.5
	13	131	5424	2.4		17	131	5424	2.4
	14	87	3306	2.6		18	87	3306	2.6
	15	38	1416	2.7		19	38	1416	2.7
	16	14	408	3.4		20	14	408	3.4
	17	3	72	4.2		21	3	72	4.2
	18	1	6	16.7		22	1	6	16.7
	TOTAL	706	27648	2.6		TOTAL	706	27648	2.6

Table 2. (continued).

6c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B	26c R H Y Nc	No. of 1-voxels	No. of patterns A	B	A/B
1 4 0 -3	14	1	6	16.7	5 0 0 5	6	17	454	3.7
	15	2	48	4.2		7	38	1584	2.4
	16	15	388	3.9		8	79	3032	2.6
	17	36	1424	2.5		9	86	3672	2.3
	18	77	2904	2.7		10	77	2904	2.7
	19	86	3672	2.3		11	36	1424	2.5
	20	79	3032	2.6		12	15	388	3.9
	21	38	1584	2.4		13	2	48	4.2
	22	17	454	3.7		14	1	6	16.7
	TOTAL	351	13512	2.6		TOTAL	351	13512	2.6
2 3 0 -1	14	2	48	4.2	4 1 0 3	8	2	48	4.2
	15	7	264	2.7		9	7	264	2.7
	16	17	624	2.7		10	17	624	2.7
	17	20	816	2.5		11	20	816	2.5
	18	17	624	2.7		12	17	624	2.7
	19	7	264	2.7		13	7	264	2.7
	20	2	48	4.2		14	2	48	4.2
	TOTAL	72	2688	2.7		TOTAL	72	2688	2.7
1 5 0 -4	17	4	66	6.1	6 0 0 6	7	8	142	5.6
	18	6	240	2.5		8	9	336	2.7
	19	14	396	3.5		9	14	396	3.5
	20	9	336	2.7		10	6	240	2.5
	21	8	142	5.6		11	4	66	6.1
	TOTAL	41	1180	3.5		TOTAL	41	1180	3.5
4 1 0 3	10	1	8	12.5	2 3 0 -1	11	1	8	12.5
	11	3	56	5.4		12	3	56	5.4
	12	6	168	3.6		13	6	168	3.6
	13	10	280	3.6		14	10	280	3.6
	14	10	280	3.6		15	10	280	3.6
	15	6	168	3.6		16	6	168	3.6
	16	3	56	5.4		17	3	56	5.4
	17	1	8	12.5		18	1	8	12.5
	TOTAL	40	1024	3.9		TOTAL	40	1024	3.9

(9) Local patterns that the hole index (=H) is non-zero are usually complicated patterns. There are 12 kinds of (R, H, Y) combinations for the 26 connectivity case, and 14 kinds for the 6 = connectivity case (Table 1). Such patterns will be detected by pattern matching if we have a list of patterns four which $H \neq 0$. Presently it will be difficult to find such patterns except for exhaustive test of all $3 \times 3 \times 3$ patterns like this paper.

6. Conclusion

In this paper we enumerated all possible arrangements (which we call local patterns) of 0 and 1 on the $3 \times 3 \times 3$ local area of a 3D binary picture. We showed how many different local patterns exist for each values of the connectivity indexes. The results will provide most basic data for analyzing a 3D picture and developing algorithms of picture analysis. For instance, we have an idea of how large memory is required to implement feature detection procedures by local pattern matching. We counted the number of patterns excluding symmetry patterns, but counting patterns for each values of connectivity indexes excluding symmetry patterns remains unsolved for future study.

Table 2. (continued).

6c				No. of	No. of patterns			26c				No. of	No. of patterns		
R	H	Y	Nc	1-voxels	A	B	A/B	R	H	Y	Nc	1-voxels	A	B	A/B
3	2	0	1	12	1	12	8.3	3	2	0	1	10	1	12	8.3
				13	2	72	2.8					11	2	72	2.8
				14	6	180	3.3					12	6	180	3.3
				15	6	240	2.5					13	6	240	2.5
				16	6	180	3.3					14	6	180	3.3
				17	2	72	2.8					15	2	72	2.8
				18	1	12	8.3					16	1	12	8.3
				TOTAL	24	768	3.1					TOTAL	24	768	3.1
6	0	0	6	7	1	1	100	1	5	0	-4	13	1	1	100
				8	1	8	12.5					14	1	8	12.5
				9	3	28	10.7					15	3	28	10.7
				10	3	56	5.4					16	3	56	5.4
				11	6	70	8.6					17	6	70	8.6
				12	3	56	5.4					18	3	56	5.4
				13	3	28	10.7					19	3	28	10.7
				14	1	8	12.5					20	1	8	12.5
				15	1	1	100					21	1	1	100
				TOTAL	22	256	8.6					TOTAL	22	256	8.6
2	4	0	-2	15	1	6	16.7	5	1	0	4	9	1	6	16.7
				16	1	24	4.2					10	1	24	4.2
				17	2	36	5.6					11	2	36	5.6
				18	1	24	4.2					12	1	24	4.2
				19	1	6	16.7					13	1	6	16.7
				TOTAL	6	96	6.3					TOTAL	6	96	6.3
1	6	0	-5	18	1	12	8.3	7	0	0	7	8	2	20	10.0
				19	1	24	4.2					9	1	24	4.2
				20	2	20	10.0					10	1	12	8.3
				TOTAL	4	56	7.1					TOTAL	4	56	7.1
1	0	1	2	27	1	1	100	0	0	0	0	1	1	1	100
1	7	0	-6	19	1	1	100	8	0	0	8	9	1	1	100

Notations 6c etc.: 6 connectivity case etc.
R, H, Y: component index, hole index, cavity index.
No. of patterns A: Symmetry patterns excluded; B: Symmetry patterns included; A/B: Rate of A in percent.
No. of 1 voxels: Number of 1-voxels in the $3 \times 3 \times 3$ local pattern.

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