Original Paper *Forma*, **19**, 97–104, 2004

Quasiperiodic Tilings Derived from Deformed Cuboctahedra

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(Received February 9, 2004; Accepted July 22, 2004)

Keywords: Quasiperiodic Tiling, Projection Method, Cuboctahedron

Abstract. 3D quasiperiodic tilings derived from deformed cuboctahedra are obtained by projection from 7D or 6D lattice space to 3D tile-space. Lattice matrices defining the projections from 7D or 6D lattice space to tile- and test-space are given by introducing a deformation parameter. Two types of lattice matrices are considered, orthonormal and row-wise orthogonal, both are corresponding to deformation of a cuboctahedron along the z-direction but the latter is corresponding to uniform deformation. Both 3D and 2D tilings are investigated though the latter is merely derived as a degenerate case of the former.

1. Introduction

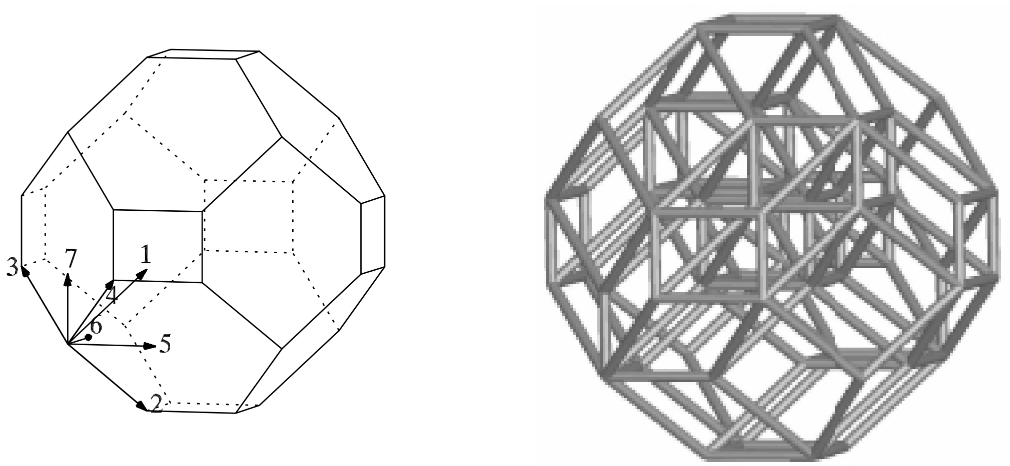
A quasiperiodic tiling generated by projection from nD lattice space is characterized by an n-star in 2D or 3D tile-space, each vector of which generally is linearly independent with respect to integer coefficient. The projection is defined by an $n \times n$ lattice matrix (for the basic definition refer to SOMA and WATANABE, 2004a). Regarding the n-star as the projection to the tile-space of n basis vectors in lattice space, vectors defined by the first 2 or 3 elements of columns constitute the n-star and those defined by the remaining elements of columns constitute an n-star in the test-space (SENECHAL, 1995; SOMA and WATANABE, 1999).

It is shown that a quasiperiodic tiling derived from a cuboctahedron is generated by projection from either 7D or 6D lattice space and that the corresponding 7- or 6-star is a mixture of a hexahedral 4- or 3-star and an octahedral 3-star (SOMA and WATANABE, 1997; WATANABE and SOMA, 2004). By deforming the cuboctahedron, expanding or contracting along the z-axis, a different quasiperiodic tiling is obtained based on the lattice matrix with deformation parameter δ. For these deformations, both orthonormal and row-wise orthogonal lattice matrices are considered. They give not only 3D quasiperiodic tilings but also 2D ones as the degenerate case of the former. For the 7-star case, the particular value of δ for the orthonormal matrix makes the star vectors flat on the xy-plane except for the one along the z-axis and the prototiles are prisms with cross sections of square and rhombus of Beenker's tiling. The condition $\delta = 0$ for the row-wise orthogonal matrix makes the 7-star as a redundant Beenker's star on the xy-plane. For the 6-star case, the degenerate 2D star becomes the mixture of two 3-stars.

2. 7-Star Derived from a Deformed Cuboctahedron

The deformation, expansion or contraction, of a cuboctahedron along the z-axis (in the direction of one of the octahedral 3-star vectors) is formulated by an orthonormal 7×7 lattice matrix A_7 derived from the deformed cuboctahedron given as,

$$A_7 = \frac{1}{\sqrt{3\lambda^2 - 2\lambda + 1}} \times \begin{pmatrix} \lambda/\sqrt{2} & \lambda/\sqrt{2} & -\lambda/\sqrt{2} & \lambda/\sqrt{2} & (1-\lambda) & 0 & 0 \\ \lambda/\sqrt{2} & \lambda/\sqrt{2} & \lambda/\sqrt{2} & -\lambda/\sqrt{2} & 0 & (1-\lambda) & 0 \\ D_1\lambda/\sqrt{2} & -D_1\lambda/\sqrt{2} & D_1\lambda/\sqrt{2} & D_1\lambda/\sqrt{2} & 0 & 0 & (1-\lambda)D \\ -(1-\lambda)/2 & -(1-\lambda)/2 & (1-\lambda)/2 & -(1-\lambda)/2 & \sqrt{2}\lambda & 0 & 0 \\ -(1-\lambda)/2 & -(1-\lambda)/2 & -(1-\lambda)/2 & (1-\lambda)/2 & 0 & \sqrt{2}\lambda & 0 \\ -D(1-\lambda)/2 & D(1-\lambda)/2 & -D(1-\lambda)/2 & -D(1-\lambda)/2 & 0 & 0 & \sqrt{2}\lambda D_1 \\ \sqrt{3}(1-\lambda)/2 & -\sqrt{3}(1-\lambda)/2 & -\sqrt{3}(1-\lambda)/2 & -\sqrt{3}(1-\lambda)/2 & 0 & 0 & 0 \end{pmatrix} \quad (1)$$



(a) (b)

Fig. 1. (a) shows the unit polyhedron, a truncated oblate rhombic dodecahedron, for $\delta = -1 + \sqrt{6/5}$ in (1) of the orthonormal case and 7-star, each vector pointing from the origin, and annotated with the column number of (1). Vectors, 5, 6 and 7 correspond to x-, y- and z-axis, respectively. (b) shows the tiling within the unit polyhedron.

where $D = \sqrt{1-4\delta-2\delta^2}$ and $D_1 = 1 + \delta$ with δ, the parameter specifying the deformation, and $0 \le \lambda \le 1$ with λ, the parameter specifying mixing of the hexahedral 4-star and the octahedral 3-star (SOMA and WATANABE, 1997), ($\lambda = 0.5$ is assumed in the following discussion). In order for D to be a real value, δ needs to be in the range $-1-\sqrt{3/2} \le \delta \le -1+\sqrt{3/2}$. The matrix shown by the upper 3 rows corresponds to the projection matrix from 7D lattice space to the 3D tile-space (x, y, z) and the 7 column vectors which form the 7-star in the tile-space. While the matrix shown by the lower 4 rows corresponds to the projection matrix to the 4D test-space (x', y', z', w') and the 7 column vectors which form the 7-star in the test-space.

2.1. Deformation based on an orthonormal lattice matrix

First, consider the deformation along the z-axis described by the lattice matrix (1) which satisfies the orthonormal condition. Figure 1(a) shows the unit polyhedron (a truncated oblate rhombic dodecahedron), the projection to the tile-space of a unit hypercube in 7D lattice space, and a 7-star, each vector with the column number of (1) for $\delta = -1 + \sqrt{6/5}$. There are 7 prototiles in the tiling. They are polyhedra, each of which is not congruent, derived from the combination of star vectors in the tile-space: (5, 6, 7) a rectangular parallelopiped C11; (5, 6, 1) and (5, 7, 1) rectangular parallelopipeds S11 and S12; (1, 2, 5), (1, 4, 5) and (1, 4, 7) rhombic parallelopipeds P11, P12 and P13; and (2, 3, 4) another rhombic parallelopiped R11. It is easy to show that, for δ given above, the x- and y-component of star vectors in the tile-space are linearly independent with respect to integer coefficient and that the z-component is multiples of $1/\sqrt{5}$. Thus the 3D tiling is quasiperiodic in the xy-plane and periodic along the z-axis.

Figure 2(a) shows the unit polyhedron and the 7-star for $\delta = -1$. Prototiles are derived

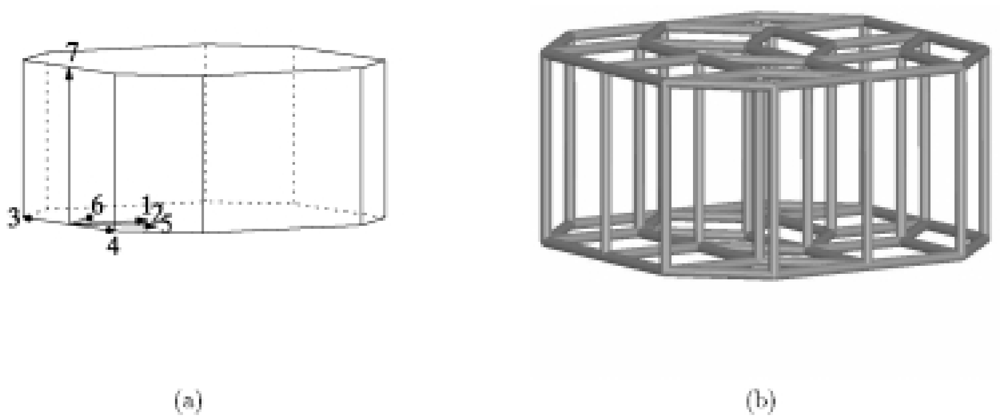


Fig. 2. (a) shows the unit polyhedron, a 4-fold symmetric octagonal prism, for $\delta = -1$ in (1) of the orthonormal case and the numbered degenerate 7-star. (b) shows the tilling within the unit polyhedron consisting of 5 square prisms and 8 rhombic prisms.

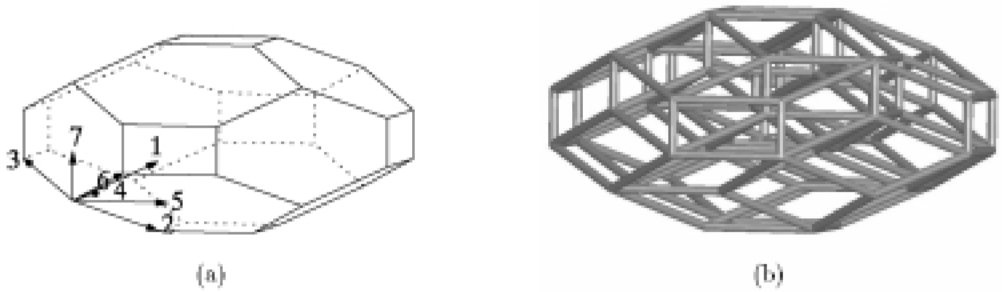


Fig. 3. (a) shows the unit polyhedron, a deformed truncated oblate rhombic dodecahedron, for $\delta = 0.5$ in (1) of the row-wise orthogonal case and the numbered 7-star. (b) shows the tilling within the unit polyhedron.

from the combination of star vectors: (1, 3, 7) a square prism; and (1, 5, 7) a rhombic prism. The cross sections of these prisms are same as the prototiles of Beenker's tiling. The star vectors in the tile-space have the similar properties as the above case and the tiling is quasiperiodic in the *xy*-plane and periodic along the *z*-axis. Since the star vectors on the *xy*-plane form a redundant BeenIker's star, the projection of the unit polyhedron to the *xy*-plane is not a regular octagon but an octagon with 4-fold symmetry.

2.2. Deformation based on a row-wise orthogonal lattice matrix

Next, consider the uniform deformation along the *z*-axis, the lattice matrix for this case is obtained by setting $D = D_1 = \delta$ or multiplying some constant δ to row 3 and 6 of (1). Such lattice matrix is no longer orthonormal but row-wise orthogonal except for the case $\delta = 1$. It represents a class of matrices whose elements can be transformed each other by affine transformation. Figure 3(a) shows the unit polyhedron (a deformed truncated oblate rhombic dodecahedron) and the numbered star vectors for $\delta = 0.5$. There are 7 prototiles as in the orthonormal case. For $\delta = 0$, 7-star in 3D reduces to 6-star in 2D tile-space and the lattice matrix is a 6×6 matrix obtained by removing line 3 and column 7 from (1) with $D = 1$. The upper 2 rows correspond to 2D tile space and the remaining 4 rows to 4D test-space. Figure 4(a) shows a unit polygon and numbered star vectors (vectors 1 and 2 are overlapping). There are 2 prototiles, each derived from the combination of star vectors: (1, 3) a square; and (1, 5) a rhombus.

It is known that star vectors in the tile-space are linearly independent with respect to integer coefficient for $\delta = 1$ (SOMA and WATANABE, 1997) and the tiling is quasiperiodic. This property is preserved by this uniform deformation including the degenerate 2D case of the 6×6 lattice matrix, which gives the modified Beenker's tiling.

3. Quasiperiodic Tiling Derived from a Deformed 7-Star

The cut-and-project method (KATZ and DUNEAU, 1986) based on the lattice matrix (1) is used to generate quasiperiodic tilings and the method of infinitesimal transfer of the test polytope (WATANABE and SOMA, 2004; PLEASANTS, private communication, 1997) is adopted for the acceptance test. Figure 1(b) shows a 3D tiling of the orthonormal

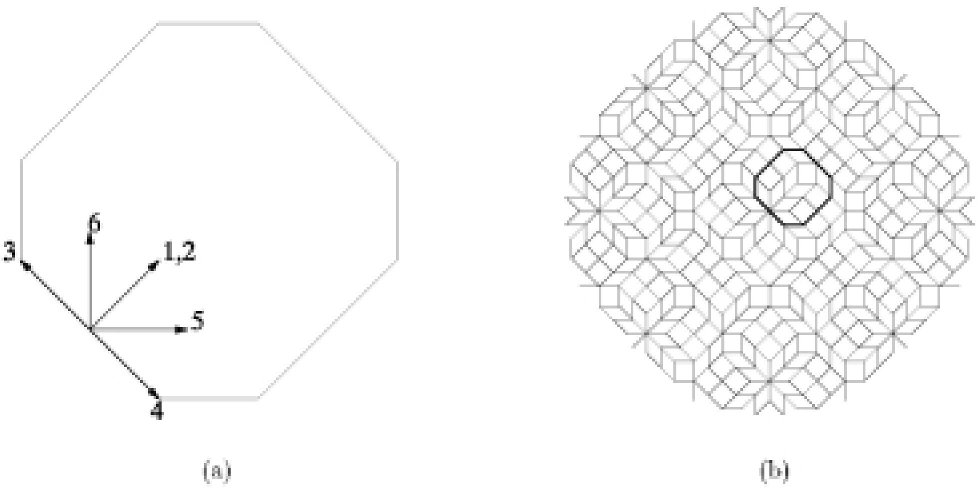


Fig. 4. (a) shows the unit polygon, a 4-fold symmetric octagon, for $\delta = 0$ in (1) of the row-wise orthogonal case and the numbered degenerate 6-star. (b) shows the tilling and the unit polygon by thick lines.

deformation type within the unit polyhedron for $\delta = -1 + \sqrt{6/5}$ obtained under the test conditions of the transfer vector (0, 0, 0, 0) and the direction of infinitesimal transfer (3, –5, –7, 2). It consists of a C11, 4 S11's, 8 S12's, 4 P11's, 4 P12's, 4 P13's and 4 R11's. Figure 2(b) shows the 3D tiling of the orthonormal deformation type within the unit polyhedron for $\delta = -1$ obtained under the same test conditions as above. It consists of 5 square prisms and 8 rhombic prisms.

Figure 3(b) shows a 3D tiling of the uniform deformation type within the unit polyhedron for $\delta = 0.5$ obtained under the same test conditions as above. It consists of prototiles of the similar types with the same numbers as in the orthonormal deformation case. Figure 4(b) shows the degenerate 2D tiling obtained under the test conditions of the transfer vector (–0.1195 ···, –0.1195 ···, 0.2886 ···, 0.5) which sets the origin at the center and the direction of infinitesimal transfer (0, 0, 0, 0). The unit polygon is shown by thick lines. The tiling is 4-fold symmetric with respect to the origin as expected from the shape of the unit polygon which is 4-fold symmetric. The unit polygon itself consists of 5 squares and 8 rhombi.

4. 6-Star Derived from a Deformed Cuboctahedron

A 6×6 lattice matrix A_6 in 6D derived from a deformed cuboctahedron is given in (2) (SOMA and WATANABE, 2004a), where C_3, S_3, C_3', S_3' and E_3 are 3 element row vectors defined as follows: $C_3 = (1\ \cos\alpha_3\ \cos 2\alpha_3)$, $S_3 = (0\ \sin\alpha_3\ \sin 2\alpha_3)$, $C_3' = (1\ \cos 2\alpha_3\ \cos\alpha_3)$, $S_3' = (0\ \sin 2\alpha_3\ \sin\alpha_3)$ and $E_3 = (1\ 1\ 1)$ with $\alpha_3 = 2\pi/3$. δ and λ are parameters specifying the deformation along the z-axis, and the mixing of the octahedral and the hexahedral star, respectively ($\lambda = 0.5$ is assumed in the following discussion).

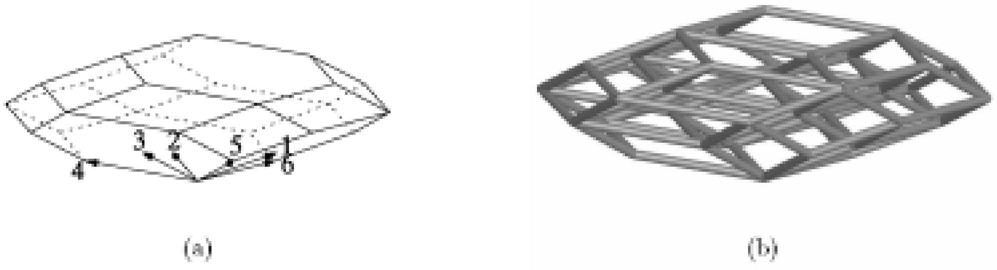


Fig. 5. (a) shows the unit polyhedron, a truncated rhombobedron, for $\delta = 0.5$ in (2) and the numbered 6-star. (b) shows the tilling within the unit polyhedron.

$$A_6 = \frac{1}{\sqrt{6\left(3\lambda^2 - 4\lambda + 2\right)}} \begin{pmatrix} 2\lambda C_3 & -2\sqrt{2}(1-\lambda)C_3 \\ 2\lambda S_3 & -2\sqrt{2}(1-\lambda)S_3 \\ \sqrt{2}\delta\lambda E_3 & \delta(1-\lambda)E_3 \\ 2\sqrt{2}(1-\lambda)C_3{}' & 2\lambda C_3{}' \\ 2\sqrt{2}(1-\lambda)S_3{}' & 2\lambda S_3{}' \\ \delta(1-\lambda)E_3 & -\sqrt{2}\delta\lambda E_3 \end{pmatrix}. \tag{2}$$

The matrix shown by the upper 3 rows corresponds to the projection matrix from 6D lattice space to the 3D tile-space (x, y, z) and the 6 column vectors of which form the 6-star in the tile-space. While the matrix shown by the lower 3 rows corresponds to the projection matrix to the 3D test-space (x', y', z') and the 6 column vectors of which form the 6-star in the test-space. With the deformation parameter δ, which specifies expansion and contraction along the z-axis, the matrix (2) represents a class of matrices whose elements can be transformed each other by affine transformation. Figure 5(a) shows the unit polyhedron and the 6-star for $\delta = 0.5$. There are 5 prototiles, each derived from the combination of star vectors in the tile-space: (1, 2, 3) a deformed rhombobedron C31; (1, 2, 6) and (1, 2, 4) rhombic parallelopipeds S31 and S32; (4, 5, 1) another rhombic parallelopiped P31; and (4, 5, 6) another deformed rhombohedron R31. For $\delta = 0$, the 6-star in 3D tile-space reduces to that in 2D tile-space. The lattice matrix is the same as (2) but the first 2 rows correspond to 2D tile-space and the remaining 4 rows to 4D test-space. Figure 6(a) shows the unit polygon and the numbered 6-star which gives degenerate 2D tiling. There are 4 prototiles derived from the combination of star vectors: (1, 2) and (4, 5), rhombi, large and small; and (1, 6) and (1, 5) parallelograms of L- and R-type.

It is easy to show that for $\lambda = 0.5$ and $\delta = 1$, the column vectors by the upper 3 rows of (2) are linearly independent with respect to integer coefficient and the tiling generated by the lattice matrix is quasiperiodic. This property is preserved so far as $\lambda = 0.5$ for the class by affine transformation including the degenerate 2D case of $\delta = 0$.

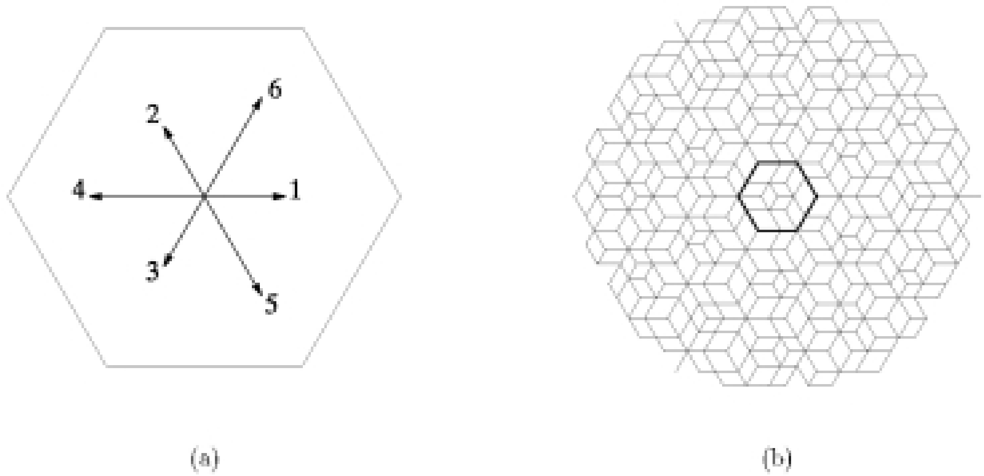


Fig. 6. (a) shows the unit polygon, a regular hexagon, for $\delta = 0$ in (2) and the numbered 6-star. (b) shows the tilling and the unit polygon by thick lines.

5. Quasiperiodic Tiling Derived from a Deformed 6-Star

The cut-and-project method based on the lattice matrix (2) is used to generate the quasiperiodic tilings and the method of infinitesimal transfer of the test polyhedron is adopted for the acceptance test as in the 7-star case. Figure 5(b) shows a 3D tiling within the unit polyhedron obtained under the test conditions of the transfer vector (0, 0, 0) and the direction of infinitesimal transfer (1, 1, 3). The tiling consists of a C31, 3 S31's, 6 S32's, 6 P31's and an R31. Figure 6(b) shows the degenerate 2D tiling obtained under the test conditions of the transfer vector $(-1.207 \cdots, 0, 0, 0.207 \cdots)$ which sets the origin at the center and the direction of infinitesimal transfer (–1, 0, 0, 1). The unit polygon is shown by thick lines. The tiling is 3-fold symmetric with respect to the origin as expected from the shape of the unit polygon which is 3-fold symmetric. The unit polygon itself consists of, large and small rhombi, L- and R-type parallelograms, 3 in number each. The tiling reported by WARRINGTON *et al.* (1997) is similar to this one but based on different formulation.

6. Concluding Remarks

It was shown that quasiperiodic tilings derived from deformed cuboctahedra can be generated by projection from 7D or 6D lattice space. The deformation along the z-axis was considered and the lattice matrices were given with the parameter specifying the deformation, from which quasiperiodic tilings were obtained by the cut-and-project method. For the case of 7D, the deformations preserving the orthonormality, and those preserving the row-wise orthogonality, of the lattice matrix were considered. For the case of 6D, only the row-wise orthogonal lattice matrix was considered which corresponds to the uniform deformation along the z-axis. Some examples of 3D tilings were shown for different values of deformation parameters including those giving the degenerate 2D tilings. The computer program generating the tilings discussed here is available elsewhere (SOMA and WATANABE, 2004b).

The authors are grateful to Dr. Himeno, the Head of the Computer and Information Division of Riken, for providing them with a comfortable research environment. Financial support for this work was provided in part by the Promotion and Mutual Aid Corporation for Private Schools of Japan. The authors also wish to thank Prof. K. Miyazaki, the reviewer, for suggestions and comments which made this paper more readable.

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