

Quasiperiodic Tilings Derived from a Cuboctahedron —Projection from 6D Lattice Space—

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Abstract. A 3D quasiperiodic tiling derived from a cuboctahedron is obtained by projection from 6D lattice space to 3D tile-space, one less dimensional lattice space than the conventional one. A lattice matrix defining projections from 6D lattice space to tile- and test-space is given and its geometric properties are investigated.

1. Introduction

A quasiperiodic tiling generated by projection from n D lattice space is characterized by an n -star in 2D or 3D tile-space, each vector of which generally is linearly independent with respect to integer coefficients. The projection is defined by an $n \times n$ lattice matrix (the basic definition is explained in Sec. 2). Regarding the n -star as the projection to the tile-space of n basis vectors in lattice space, vectors defined by the first 2 or 3 elements of columns constitute the n -star and those defined by the remaining elements of columns constitute an n -star in the test-space (SENECHAL, 1995; SOMA and WATANABE, 1999). It is shown that a quasiperiodic tiling derived from a cuboctahedron is generated by projection from 7D lattice space and that the corresponding 7-star is a mixture of a hexahedral 4-star and an octahedral 3-star (SOMA and WATANABE, 1997; WATANABE and SOMA, 2004). Since the hexahedral 4-star is linearly dependent, we would like to point out that this quasiperiodic tiling can be generated through projection from 6D lattice space (WATANABE and SOMA, 2004). This is shown in this paper by giving a 6×6 lattice matrix.

2. Basic Definitions

Figure 1 explains the projection method for generating a 1D tiling by projection from 2D lattice space (X - Y). The x -axis represents a tile-space on which the tiles are separated by \times marks. They are the projections to the tile-space of the selected lattice points (shown by filled circles). The tile itself is a line segment which is the projection to the tile-space of an edge of the lattice. There are two types of tiles in the tile-space, a long and a short one. They are called prototiles. The x' -axis represents a perpendicular-space or a test-space and

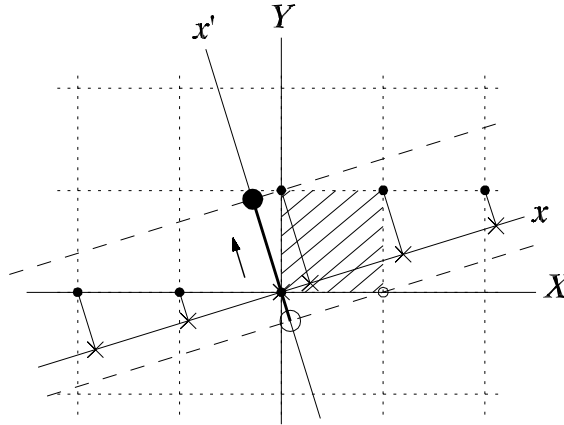


Fig. 1. 1D tiling on the x -axis consisting of two types, long and short, of tiles (line segments separated by marks \times). The separating points show the projections of selected lattice points to the x -axis. The selection is based on the projection to the x' -axis which lies inside of the shadow of the unit square shown by a thick line on the x' -axis.

the selection of lattice points is based on whether the projection to the test-space falls within the test window or not. The test window (shown by the thick line with circles at both ends) is a projection to the test-space of a unit square in lattice space (shown by hatching), while the projection to the tile-space of the unit square is called a unit line segment (a unit polygon or polyhedron in 2D or 3D tile-space). The boundary property of the test window, open or closed, affects the overlap or miss of tiles in the tiling. A method of infinitesimal transfer is known (PLEASANTS, private communication, 1997) in which the test window is transferred infinitesimally (the direction is shown by the nearby arrow). The window becomes closed at the upper end (shown by a filled circle) and open at the lower end (shown by an open circle), thus the overlapping of prototiles is avoided by excluding one of the lattice points on the X -axis (shown by an open circle). By a finite transfer of the window, different tilings in the same local isomorphism class are obtained. This projection method is called the cut-and-project method (PLEASANTS, 2000) because the selection is made by cutting the lattice space before projection.

The projection method is defined by a 2×2 matrix A_2 called a lattice matrix given by (1),

$$A_2 = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (1)$$

where, θ is the angle between axes x and X . Two 1D column vectors in the first row show the projection to the tile-space of the basis vectors of axes X and Y , and constitute a 1D 2-star in the tile-space, while those in the second row are the projection to the test-space of the basis vectors of axes X and Y , and constitute a 1D 2-star in the test-space. The condition

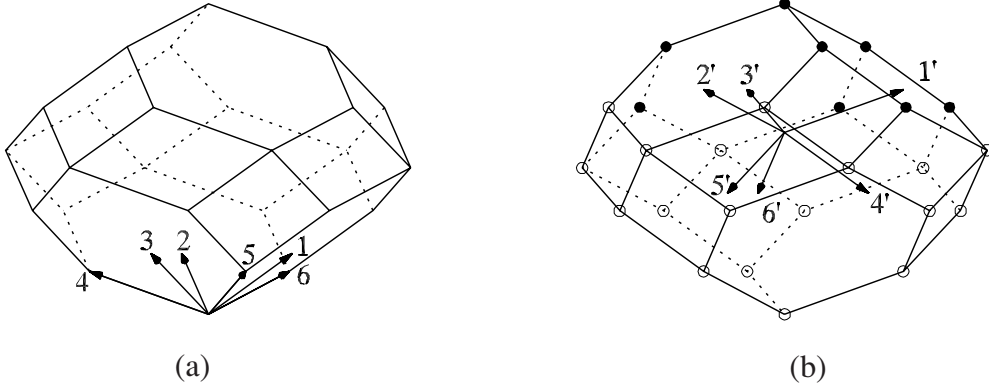


Fig. 2. Unit polyhedron (a) and test polyhedron (b) with a 6-star. Each vectors pointing from the origin are numbered according to the column number in (2). In (b), the numbers are primed indicating the test-space. As for the coordinate system, the vector numbered 1 is chosen to lie on the x - z plane with the z -axis pointing upward (same as the primed one in the test-space).

of generating a quasiperiodic tiling is that vectors of a 2-star in the tile-space are linearly independent or $\tan\theta$ is irrational. By definition, the lattice matrix is a coordinate rotation matrix and orthonormal, yet it can be row-wise orthogonal preserving the quasiperiodicity of the tiling.

The projection to 2D or 3D tile-space from an nD lattice space can generally be considered in the same way, such that the number of linearly independent vectors of the n -star is larger than the number of the dimension of the tile-space. Also the lattice space can neither be orthogonal nor equilateral (SENECHAL, 1995).

3. Cuboctahedral 6-Star

A 6×6 lattice matrix A_6 in a 6D lattice space is given in (2), where C_3 , S_3 , C_3' , S_3' and E_3 are 3 element row vectors defined as follows: $C_3 = (1 \cos\alpha_3 \cos2\alpha_3)$, $S_3 = (0 \sin\alpha_3 \sin2\alpha_3)$, $C_3' = (1 \cos2\alpha_3 \cos\alpha_3)$, $S_3' = (0 \sin2\alpha_3 \sin\alpha_3)$ and $E_3 = (1 \ 1 \ 1)$ with $\alpha_3 = 2\pi/3$ and $a_6 = 1$ (a_6 is a parameter taking any value, as is shown lately), and λ is a parameter specifying the mixing ratio of octahedral and hexahedral stars. The 6-star is composed of these stars.

$$A_6 = \frac{1}{\sqrt{6(3\lambda^2 - 4\lambda + 2)}} \begin{pmatrix} 2\lambda C_3 & -2\sqrt{2}(1-\lambda)C_3 \\ 2\lambda S_3 & -2\sqrt{2}(1-\lambda)S_3 \\ \sqrt{2}a_6\lambda E_3 & a_6(1-\lambda)E_3 \\ 2\sqrt{2}(1-\lambda)C_3' & 2\lambda C_3' \\ 2\sqrt{2}(1-\lambda)S_3' & 2\lambda S_3' \\ a_6(1-\lambda)E_3 & -\sqrt{2}a_6 E_3 \end{pmatrix}. \quad (2)$$

Table 1. Cell constants of prototiles.

	a	b	c	$\cos\alpha, (\alpha)$	$\cos\beta, (\beta)$	$\cos\gamma, (\gamma)$	V
C	1	1	1	0, (90°)	0, (90°)	0, (90°)	1
S	1	1	$\sqrt{6}/2$	0, (90°)	$\sqrt{3}/3, (54.7^\circ)$	$\sqrt{3}/3, (54.7^\circ)$	$\sqrt{2}/2$
P	$\sqrt{6}/2$	$\sqrt{6}/2$	1	$1/3, (70.5^\circ)$	$\sqrt{3}/3, (54.7^\circ)$	$\sqrt{3}/3, (54.7^\circ)$	1
R	$\sqrt{6}/2$	$\sqrt{6}/2$	$\sqrt{6}/2$	$-1/3, (109.5^\circ)$	$1/3, (70.5^\circ)$	$1/3, (70.5^\circ)$	$\sqrt{2}$

The matrix shown by the upper 3 rows corresponds to the projection matrix from 6D lattice space to the 3D tile-space (x, y, z) and the 6 column vectors of which form the 6-star in the tile-space. The matrix shown by the lower 3 rows corresponds to the projection matrix to the 3D test-space (x', y', z') and the 6 column vectors of which form the 6-star in the test-space.

Figure 2(a) shows the unit polyhedron, the projection to the tile-space of a unit hypercube in 6D lattice space and a 6-star, each vector with the column number of (2). Figure 2(b) shows the test polyhedron, the projection to the test-space of a unit hypercube in 6D lattice space and a 6-star, each vector with the primed column number. Both polyhedra are truncated rhombohedra with 18 facets formed of 6 hexagons, 6 parallelograms and 6 squares. There are four prototiles in the tiling. They are polyhedra derived from the combination of star vectors in the tile-space: (1, 2, 3) a cube (C); (1, 2, 4), a square parallelepiped (S); (1, 4, 5), a rhombic paxallelopiped (P); and (4, 5, 6), a rhombobedron (R). The cell constants of these prototiles are listed in Table 1 with approximate angles except for 90°.

It should be noted that the matrix (2) is not orthonormal but row-wise orthogonal. The lack of orthonormality comes from the fact that the component hexahedral star is not a full 4-star but a partial 3-star. To make it orthonormal, a_6 must be $\sqrt{2}$, in other words, (2) with parameter a_6 represents a class of matrices whose elements can be transformed each other by affine transformation, the orthonormal one which is the representative of the class. The parameter a_6 specifies elongation or contraction of both of the unit and test polyhedra along the direction of the z and z' axes, respectively. The particular shape for $a_6 = 1$ is said to satisfy the cuboctahedral condition because it is derived from a cuboctahedral 6-star. Additional condition, $\lambda = 1/2$, may be called Beenker's condition because the projection of the 6-star to a plane defined by vectors 1 and 2 (Fig. 2(a)) forms Beenker's 4-star (WATANABE and SOMA, 2004).

4. Quasiperiodic Tiling Based on a Cuboctahedron

The cut-and-project method (KATZ and DUNEAU, 1986) based on the lattice matrix given in the previous section is used to generate the quasiperiodic tilings. It is easy to show that, for $\lambda = 1/2$, the column vectors by the upper 3 rows of (2) are linearly independent with respect to an integer coefficient. The method of infinitesimal transfer of the test polyhedron (PLEASANTS, 1997; WATANABE and SOMA, 2004) is adopted for the selection test. The circles on the vertices of the test polyhedron in Fig. 2(b) show the boundary conditions for

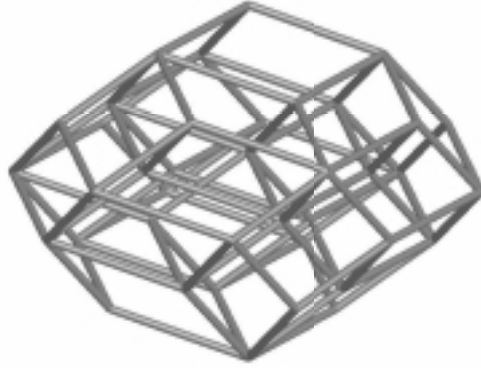


Fig. 3. 3D cuboctahedral tiling within the unit polyhedron generated by the cut-and-project method using the lattice matrix (2) with $\lambda = 1/2$ and $a_6 = 1$, and the test condition as shown in Fig. 2(b).

the finite transfer vector of $(0\ 0\ 0)$ and the infinitesimal transfer vector direction of $(\cos(\pi/10)\ \sin(\pi/10)\ 0)$; open circles show the outside points and filled ones inside points, edges with circles of the same type at both ends are of the same type as that of the circles, those with different types are inside except for the end point shown by the open circle. In Fig. 3, a 3D tiling within the unit polyhedron is shown under the test condition given in Fig. 2(b). It consists of one C, 9S's, 6P's and one R.

5. Concluding Remarks

It is shown that a quasiperiodic tiling derived from a cuboctahedron can be generated by projection from 6D lattice space. This comes from the fact that the conventional cuboctahedral 7-star is linearly dependent. Removing one of the vectors from the 7-star, a 6×6 row-wise orthogonal lattice matrix is derived and the 3D tiling is generated. This particular lattice matrix is one of the matrices in the affine transformation class. It is interesting to point out that the situation in which the quasiperiodic tilings generated by a cuboctahedral 7- or 6-star is similar to the 3D Penrose tiling which are generated by an icosahedral 6- or 5-star (RÜCK, 1987). The computer program generating tilings discussed here is available elsewhere (SOMA and WATANABE, 2004).

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