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Golden Gray

Dirk HUYLEBROUCK

Sint-Lucas Department for Architecture, Paleizenstraat 65, 1030 Brussels, Belgium
E-mail address: huylebrouck314@hotmail.com

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Abstract. Literature describes the golden number $\phi = 1.618$ ... and related golden angle of 36° extensively, including statements of doubtful verification. Still, the authors propose to introduce a new "golden" concept, since $\phi - 1 = 1/\phi = 0.618$ ... occurs in color theory. This unusual field of mathematical application uses the "gray value" of a transparent layer, that is, its part w of white (transparent) pixels (the remaining $1 - w$ pixels being black). If a rotating disk is divided in a ρth-part of a gray tone and $(1 - \rho)$th-part of a double layer of that gray tone, the result can be surprising. Indeed, when a disk divided into halves ($\rho = 1/2$) and when the resulting gray is half white ($w = 1/2$), the golden number provides the gray tone to be used. The concept provides an experimental determination for the golden number.

1. The Golden Number

1.1. The golden section: a particular proportion of lengths

Classically, the golden number ϕ (or golden section, mean, ratio, divine proportion, etc.) arises when a line segment of length x (>1) is divided into two pieces of lengths 1 and $x - 1$. This division must be such that the ratio $x - 1$ equals the ratio $1/x$. This produces the equation $x^2 - x - 1 = 0$, of which $(1 + \sqrt{5})/2 = 1.6180 ... = \phi$ is the positive solution (see WALSER (2001) for a sound introduction about the mathematical properties of the number). The rectangle of width 1 and length ϕ corresponds to the optimal solution of the extra-area added when extending the sides of a given square to get a larger square, with respect to the sum of the areas on the diagonal squares (see Fig. 1).

More generally, the equality of ratios $x - n = 1/x$ transforms into the quadratic $x^2 - n{\cdot}x - 1 = 0$ whose the positive roots yield the family of "metallic means", for different values of $n \in N$. For $n = 2$, it is the silver mean, $1 + \sqrt{2}$, for $n = 3$ the bronze mean, $(3 + \sqrt{13})/2$, etc. They too correspond to an optimal solution when extending the side of a square in comparison to other areas, such as the rectangles on one side (see HUYLEBROUCK (2001) and HUYLEBROUCK and LABARQUE (2002)).

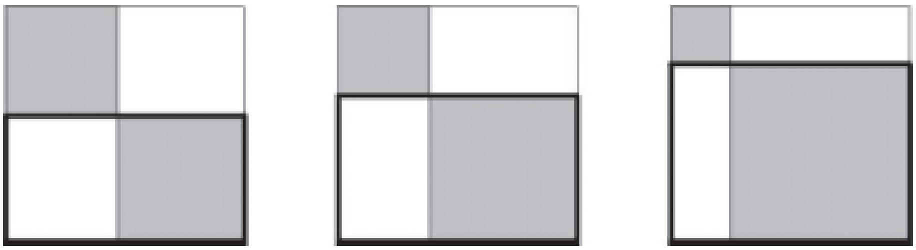

Fig. 1. The golden rectangle (in thick black lines) in the middle is of optimal size, when an initial square of side, say, 1, is extended to get another square of side x, and the added area $2x + x^2$ (in gray) is compared to the sum of the areas $1 + x^2$ of the squares on the diagonal: $f(x) = (2x + x^2)/(1 + x^2)$ is maximal for $x = \phi = 1.618$ ...

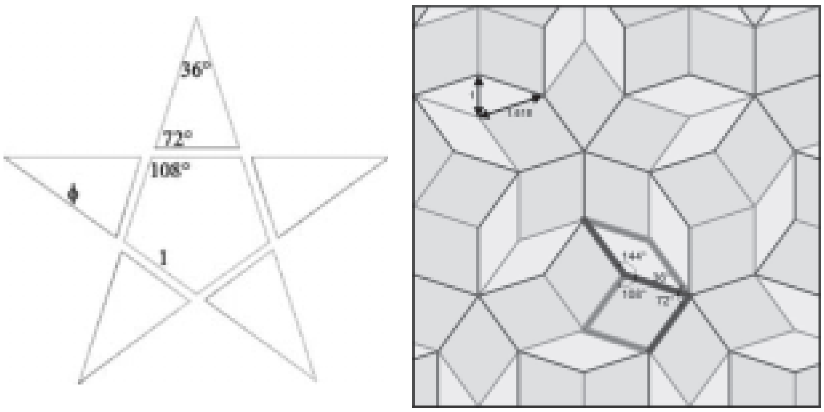

Fig. 2. The regular pentagonal star and the Penrose tiles show "the golden angle", of 36°.

1.2. The golden angle: a particular angle

Lengths in a pentagonal star are related to the golden section as well. One third of the angle of the regular pentagon, 36°, is called "*the golden angle*" (see web site).

A motivation for that special name is the large cluster of graphical applications in which the 36° angle occurs, from quasi-crystals up to the DNA-structure (see Fig. 2).

1.3. Other applications of golden results: handle with care

A part from lengths or angles, some have tried to reveal the golden number to music, poetry (i.e. Virgil's Aeneid), or anatomy (i.e. the human body), but several authors have shown these applications are mere fruits of frantic imagination lacking scientific justification (NÉROMAN, 1983; MARKOWSKY, 1992; NEVEUX, 1995). The golden section misconceptions have been divulged so energetically in literature that a true "*golden section myth*" was created. Matila Ghyka was one of its most prominent advocates and he influenced not the least, such as Le Corbusier. Most of today's schoolbooks are infected by the golden section

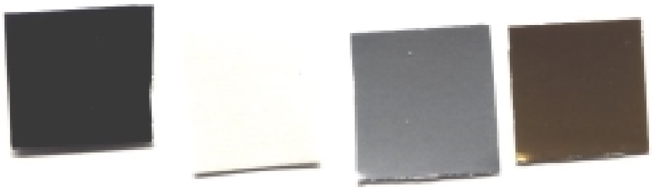

Fig. 3. These glossy and metallic covers on flat cards are called "glossy black, shimmer pearl-champagne iridescent linen, silver metallic and gold metallic". This "golden metallic gray" has nothing to do with the presently introduced "golden gray".

epidemic as well. It seems a mathematical equivalent of what astrology is to astronomy or creationism to biology. Yet, these pseudo-scientific statements are not the present topic, but being aware of them can be important when proposing a new concept, as will be done below.

Furthermore, it is important here to note the absence of reliable statistics and experiments where the golden number would be the output. For instance, drawing a rectangle around the Parthenon is so arbitrary that it is a mere subjective matter to get a golden section rectangle as result. Choosing it as most aesthetic form among a list of rectangles or squares, as proposed by Fechner in 1864, appeared to be a non-reproducible experiment, though it had a considerable effect on numerous authors, even up to today. Here, a conventional physics experiment is proposed, and the number 0.618 ... is indeed the verifiable result.

1.4. A still different application: golden gray

The applications in the theory of proportions (the golden "*section*"), and in pentagonal related items (the golden "*angle*"), are supplemented here by an application in color theory. The obtained tone with special 1.618-properties was thus given the nickname of "*golden gray*". In color or photography literature, the term "*golden gray*" may be encountered, but it then means gray with a "golden" aspect and this has nothing to do with the present notion of "golden gray". In Fig. 3 for instance, some glossy and metallic covers are shown on flat cards, called "glossy black, shimmer pearl-champagne iridescent linen, silver metallic and gold metallic" (see web sites given in the references).

2. Mixing Gray Tones

In most graphical computer software, the user can numerically determine the kind of gray in shading surfaces. The K-value of a gray surface indicates the amount of white (or, more precisely, of "*pure light*"). The letter K refers to the last letter of the word black, as in the well-known abbreviation used in describing color components, *CMYK* (Cyan, Magenta, Yellow, blacK). A K-value of 0 corresponds to black, while a K-value of 1 means white.

Unfortunately, the value given by the software does not necessarily guarantee an identical result on the printer. The machine may have some defaults, or been used, or the

Fig. 4. Comparison with gray cards is used to determine the gray scale of an unknown tone. Left: gray cards (see web site); right: a gray card book (see web site).

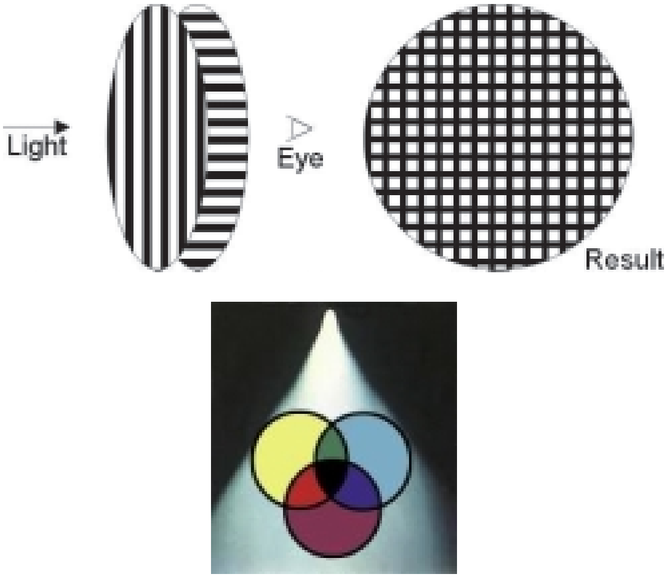

Fig. 5. Subtractive mixture of gray tones (left) is the special only-only case of well-known more general subtractive mixture of colors (right; see web site).

ink may be of a bad quality, and be more or less absorbed by the paper. The determination of the exact gray value of a drawing is important in for many graphical applications. Designers use all kinds of techniques, such as comparison with standard gray cards (see Fig. 4).

The application of several transparent layers on top of each other is another graphical technique. It yields surface with at least as many black elements as the initial two layers. The probability of seeing a white pixel corresponds to having a white pixel on the first AND on the second layer. Thus, if K_1 is the K-value of the first layer, and K_2 the value of the second, their product provides the K-value of the double layer. Figure 5 illustrates such a "*subtractive*" mixture.

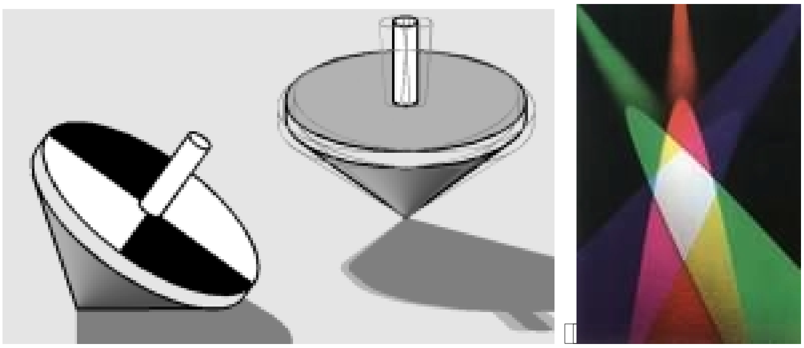

Fig. 6. Partitive mixture on rotating disks (left) and the different additive color mixing (right; see web site).

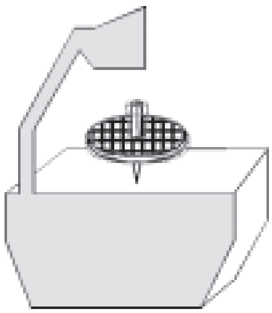

Fig. 7. A didactic device for illustrating partitive mixing: transparent layers on a rotating disk, placed on an overhead projector.

Rapidly rotating disks with different gray tones provide another technique to judge the final impression of a black and white mixture. At each moment, a pixel from one gray tone OR from the other gray tone appears. Thus, the probability of seeing such a pixel in the rotating result will correspond to a weighted sum. If the K_1 gray covers a ρ_1-part of the disk, and the K_2 gray a ρ_2-part of the disk ($\rho_1 + \rho_2 = 1$), the rotation will show a gray with value $\rho_1 K_1 + \rho_2 K_2$. Designers call this procedure a *partitive* mixture. Often, the two gray tones cover the disks in sectors, and thus their share is more easily measured. Although partitive mixture on rotating disks uses an addition sign in its mathematical formulation, it is different from so-called additive color mixing. For instance, the latter mixing method allows the observation of several colors simultaneously, in contrast to the partitive method. This third way of mixing colors, i.e. the additive method, will not be used here (see Fig. 6).

The above properties can be combined using tops turning on a light source, such as light box used for technical drawings or an overhead projector. Transparent layers thus project their subtractive mixture, eventually rotating to get a partitive mixture (see Fig. 7).

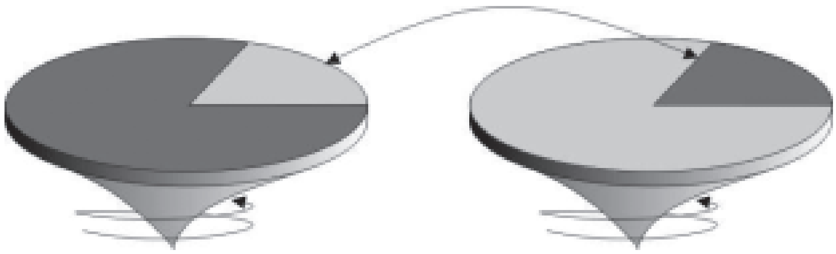

Fig. 8. Exchanging parts of gray layers on 2 rotating disks.

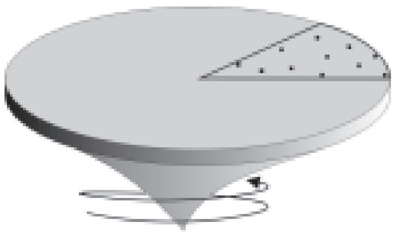

Fig. 9. Adding black spots on a part of the layer of a rotating disk.

Example 1

Suppose that two disks with gray tones K_1 and K_2 are to be cut in parts and placed on two tops. Say that from each disk we interchange one ρth part ($\rho \neq 0.5$ to avoid identical tops), and put the newly configured disks on the tops. Since no layers are place on top of each other, only subtractive mixing is used, twice:

$$\begin{cases} (1-\rho)K_1 + \rho K_2 = w_1 & \text{for top 1;} \\ \rho K_1 + (1-\rho)K_2 = w_2 & \text{for top 2.} \end{cases}$$

The system is easily solved for K_1 and K_2:

$$K_1 = \frac{\rho w_1 + \rho w_2 - w_1}{-1+2\rho} \quad \text{and} \quad K_2 = \frac{\rho w_1 + \rho w_2 - w_2}{-1+2\rho}, \quad \text{with } \rho \neq 0.5.$$

For example, if one quarter is cut of from each disk ($\rho = 0.25$), and if a comparison of the rotating tops with gray cards shows that $w_1 = 0.5$ and $w_2 = 0.7$, then it is easily computed that $K_1 = 0.4$ and $K_2 = 0.8$ (see Fig. 8).

Example 2

Suppose gray with value K_1 covers a disk but that this value is not black enough. How

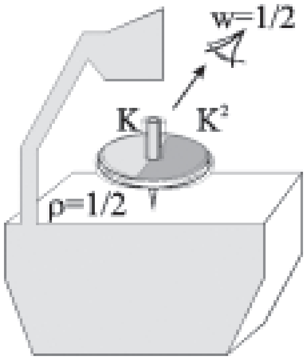

Fig. 10. A layer with gray value K, and a double layer, with value K_2, each covering half of a rotating disk ($\rho = 1/2$). The resulting gray value should be $w = 1/2$.

to correct this by adding just a few pixels, with an extra layer cut in a small sector? More precisely, suppose that another transparent layer with value K_2 has to be added, on just $1/\rho$ of the complete surface, to correct the mistake. Thus:

$$\left(1-\frac{1}{\rho}\right)\cdot K_1 + \frac{1}{\rho}\cdot K_1 K_2 = w.$$

This yields: $K_2 = \rho\cdot(w - (1 - 1/\rho)K_1)/K_1$.

For example, suppose a $K_1 = 0.53$ has to be corrected to a $w = 0.5$ value by adding a slice on 1/8 of the disk's surface. The extra layer should have $K_2 = 0.547$, or 54.7% white (see Fig. 9).

3. The Golden Gray Case

In order to yield different tones of gray, the transparent coverings could consist of same gray, in a single or a double layer. On a rotating disk, they can produce any gray in between that single and double gray tone. Mathematically formulated, if the single layer of K-gray covers a $1 - \rho$-part of the disk, and the double K layer a ρth part, their subtractive mixture corresponds to:

$$(1 - \rho)K + \rho K_2 = w.$$

This is the quadratic equation

$$K_2 - (1 - 1/\rho)K - w/\rho = 0.$$

There are very straightforward questions related to this set-up. For instance, will a given covering rate on the disk yield a gray with the same number of gray level? These cases

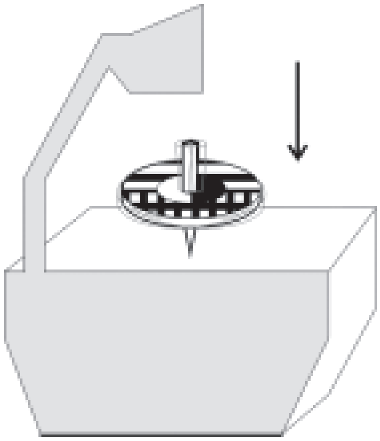

Fig. 11. A device for an experimental confirmation of the golden section. Observing the disk in rotation on an overhead projector, yields an image as given below under the heading "rotating disks".

where $w = \rho = 1/n$, provide a family of interesting particular cases, for arbitrary $n \in N$. It transforms the quadratic into

$$K_2 - (1 - n)K - 1 = 0.$$

For $n = 2$, the equation is $K^2 + K - 1 = 0$, and its solutions are $-(1 \pm \sqrt{5})/2$, or 0.618 ... and –1.618 ... The first is the inverse of the golden number and the second its opposite. Rejecting the negative value, it shows that an approximate 61.8% gray provides the solution. Summarizing, if a disk is covered for 1/2 of a 61.8% gray and for the other half of a double layer of this gray, then the rotation yields a 1/2 white and 1/2 black result (see Fig. 10).

Similarly, if a disk is covered for 1/3 of a $\sqrt{2} - 1$ or 41% white gray and for the other 2/3 of a double layer of this gray, then the rotation yields a 1/3 white result. Here, $\sqrt{2} - 1$ is the inverse of the silver section a $1 + \sqrt{2}$, just as $(\sqrt{5} - 1)/2$ is the inverse of ϕ. For 1/4 coverage by a $(\sqrt{13} - 3)/2$ or 30% gray and for the other 3/4 of a double layer of this gray, then the rotation yields a 25% white result. Now $(\sqrt{13} - 3)/2$ is the inverse of $(\sqrt{13} + 3)/2$, the bronze number.

4. Experimental Search for ϕ

The first paragraphs of this paper explained the difficulty in observing the golden number through an experiment. Well now, here is a test to confirm the value of the golden section.

We take a transparent circular disk that is uniformly gray with 70% white (and thus 30% black). Another layer of the same gray with 70% white covers half of the disk, the two forming a $0.70^2 = 0.49$ white layer for one half of the disk. Rotating the disk, which is

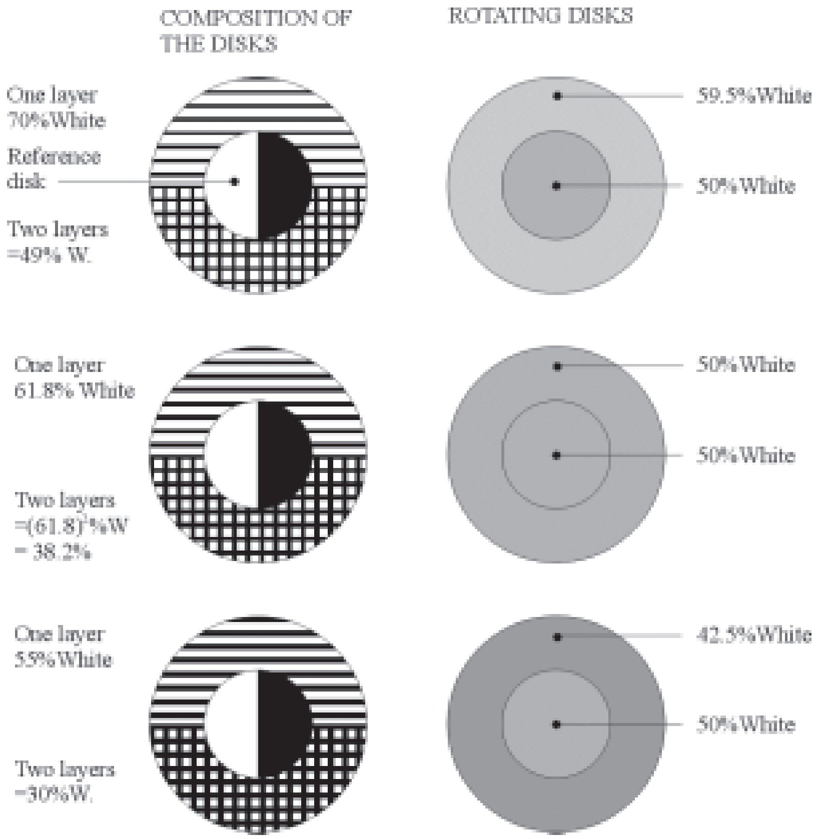

Fig. 12. Several disks different gray tones, to be compared to a 50% black card, for an experimental search of ϕ. The disk above is too white; the one below too black, since only the one in the middle is such that rotation yields a 50% gray.

composed of the semicircle with 0.69 gray and that of 0.4761% (=0.69^2) gray, again a gray tone is observed, and it is compared to a 50% black card held close to it. Since the tone will not be dark enough (the computation is 0.49/2 + 0.70/2 = 0.595 of white), we add another percentage of black to the initial transparent circular disk, and again form a double layer for one half of the disk. Rotating the half 69% and half 47.61% white disk, again a gray tone is obtained and compared to the 50% disk (see Fig. 11).

Continuing in this way, one should finally stop at an approximate 61%, and this is an estimate for $\phi - 1 = 0.618$ ... Thus, Fig. 12 can be considered as the gray equivalent of Fig. 1.

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