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Free Surface Density Instead of Volume Fraction in the Bone Remodeling Equation: Theoretical Considerations

Gholamreza ROUHI[1], Walter HERZOG[1,2], Les SUDAK[1], Keikhosrow FIROOZBAKHSH[3] and Marcelo EPSTEIN[1]*

[1]*Department of Mechanical and Manufacturing Engineering, University of Calgary, 2500 University Drive NW Calgary, Canada T2N 1N4*
[2]*Department of Kinesiology, Human Performance Laboratory, University of Calgary, 2500 University Drive NW Calgary, Canada T2N 1N4*
[3]*Department of Orthopaedics Health Sciences Center, University of New Mexico, Albuquerque NM 87131, U.S.A.*
**E-mail address: mepstein@ucalgary.ca*

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Abstract. In this paper we propose a new set of constitutive equations for bone remodeling that uses the specific surface instead of volume fraction. The theory of small-strain adaptive elasticity proposed by COWIN and HEGEDUS and a surface remodeling equation are derived to develop the remodeling theory. A relationship between net bone cell activity, bone material property and mechanical stimuli is derived. For illustration, the rate of change of trabecular remodeling is derived for selected geometries. With this model, the effect of bone micro-structure and mechanical stimuli on the rate of remodeling can be studied.

1. Introduction

Bone is continuously remodeled through a coupled process of bone resorption and bone formation, and this process is called bone remodeling. An early hypothesis about the dependence of the structure and form of bones, and the mechanical loads they carry was proposed by Galileo in 1638 (ASCENZI, 1993). The nature of this dependence was first described in a semi-quantitative manner by WOLFF (1982), who stated that every change in the form or function of a living bone is followed by adaptive changes in its internal architecture and its external shape. The remodeling process is generally viewed as a material response to functional demands that is governed by an intricate relationship between bone apposition and resorption. It is accepted that bone growth, maintenance, degeneration and remodeling are biochemically regulated processes influenced by mechanical function.

Many theories, mostly phenomenological, have been proposed to explain the process of bone remodeling (MARTIN, 1972, 1974; GJELSVIK, 1973a, b; COWIN and HEGEDUS, 1976; HEGEDUS and COWIN, 1976; CARTER and HAYES, 1977; COWIN and VAN BUSKIRK, 1978, 1979; COWIN and FIROOZBAKHSH, 1981; FIROOZBAKHSH and COWIN, 1981a, b; CARTER, 1987; CARTER *et al.*, 1987, 1996; COWIN, 1987, 1993; HUISKES *et al.*, 1987; FIROOZBAKHSH and ALEYASSIN, 1989; HART and DAVY, 1989; BEAUPRE *et al.*, 1990a, b; FIROOZBAKHSH *et al.*, 1992; WEINANS *et al.*, 1992; MULLENDER *et al.*, 1994; PRENDERGAST and TAYLOR, 1994; MULLENDER and HUISKES, 1995; PRENDERGAST and HUISKES, 1996; SIFFERT *et al.*, 1996; HUISKES, 1997; JACOBS *et al.*, 1997; SMITH *et al.*, 1997; RAMTANI and ZIDI, 2001). A widely accepted phenomenological model was proposed by COWIN and HEGEDUS (1976) and HEGEDUS and COWIN (1976). In this paper, we will modify the COWIN and HEGEDUS model by introducing a new variable, the so-called 'free surface density' instead of 'volume fraction' in the constitutive equations.

Two important features of the internal structure of bone are its porosity and specific surface (MARTIN, 1984). Porosity is defined as the void volume per unit volume of the whole bone or the fractional part of bone occupied by soft tissues. The specific surface is defined as the internal surface area per unit volume of the whole bone (MARTIN, 1984). Bone making cells (Osteoblasts) and resorbing cells (Osteoclasts) lie on the free surfaces of bone, thus, all bone resorption and apposition is thought to occur at these sites (MARTIN, 1972, 1984; CARTER and BEAUPRE, 2001). Therefore, it seems appropriate to consider the specific surface rather than volume fraction in the constitutive equations of bone remodeling. Using the standard conservation equations and the entropy inequality, we propose a new formulation for the remodeling process of bone in the small-strain regime, including the effects of mechanical stimuli and bone micro-structural geometry.

It is well accepted that bone remodeling is a surface phenomenon, and from a cellular point of view there is no difference between remodeling on different types of surfaces of bones (MARTIN, 1972, 1984; CARTER and BEAUPRE, 2001). However, there is no theory that includes surface and internal remodeling simultaneously.

Here, we derive a unique formulation for bone remodeling which can be used for both surface and internal remodeling. In this model, bone geometry effects can be seen explicitly. The surface remodeling equation proposed by COWIN and VAN BUSKIRK (1979) is derived using this new model by considering some assumptions and restrictions.

2. Fundamental Assumptions

In order to construct a general framework for the description of strain-induced bone remodeling processes, we make the following assumptions: (1) the porosity of the bone matrix depends on the ambient long-term mechanical stimuli (here, strain tensor) history, (2) the transfer of mass, energy and entropy occurs as a result of biochemical reactions between the bone actor cells (osteoblasts and osteoclasts) and the matrix, (3) the extra-cellular fluid is in contact with the blood plasma that supplies the materials for the synthesis of bone matrix, and (4) the characteristic time of chemical reactions for resorption and apposition is several orders of magnitude greater (months) than the characteristic time associated with a complete perfusion of the blood plasma in the bone. Therefore, any excess heat generated by chemical reactions is quickly carried away by circulation and the remodeling processes can be considered iso-thermal and quasi-static.

Mechanical stimuli that have been considered in bone remodeling include strain (COWIN and HEGEDUS, 1976; HEGEDUS and COWIN, 1976), stress (KUMMER, 1972; WOLFF, 1982), effective stress (CARTER, 1987; BEAUPRE *et al.*, 1990b), strain energy (HUISKES *et al.*, 1987) and strain rate (HART and DAVY, 1989). However, it is not known which of these factors or combination of factors is the mechanical stimulus for bone remodeling. We will consider strain as the basic mechanical signal for bone remodeling, because it represents the immediate local effect of external bone loading, and it is a primary and directly measurable quantity.

3. Kinematics

The motion of material points is described in a Cartesian system of coordinates by $x_i = x_i(X_j, t)$ where $x_i(X_j, t)$ gives the location of the material particle X_j at time t, and where i, j = 1, 2, 3 are indices for the three coordinate directions in the spatial and reference configurations, respectively. The deformation function, $x_i = x_i(X_j, t)$, is assumed to be one to one, continuous, and invertible. The velocity field of the continuum at time t is given by:

$$V_i = \frac{\partial x_i}{\partial t}. \tag{1}$$

The deformation gradient, F_{ij}, the local volume change, J, and the velocity gradient, L_{ij}, are defined as:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{2}$$

$$J = \det F \tag{3}$$

and

$$L_{ij} = \frac{\partial V_i}{\partial X_j}. \tag{4}$$

Let υ denote the volume fraction of the solid phase (bone matrix) in the current configuration. Then, the bulk density, ρ, of the porous structure in the current configuration is given by:

$$\rho = \gamma\upsilon \tag{5}$$

where γ is the density of the bone matrix.

If ζ denotes the volume fraction of the matrix in an unstrained reference configuration, then one might imagine a cube of porous elastic material with the eight vertices marked for the purpose of measuring strain. When porosity changes, material is added or taken away

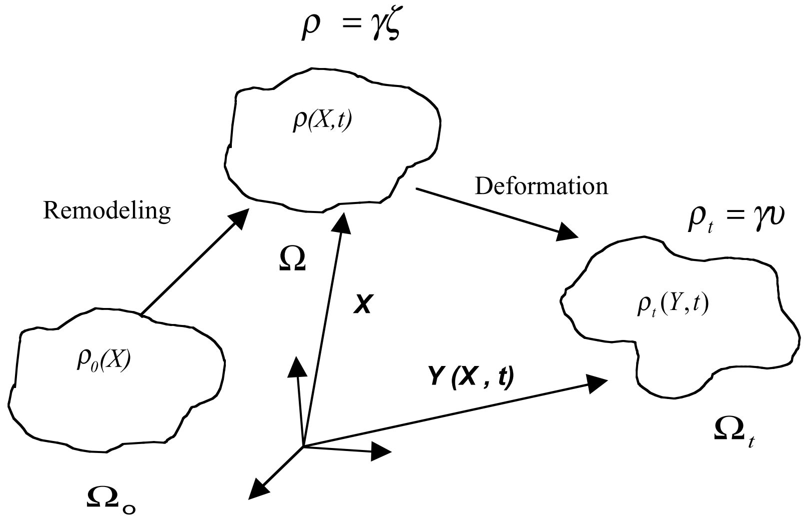

Fig. 1. Lagrangian description for the remodeling problem.

from the pores, but if there is no deformation, the distance between the vertices of the cube does not change. Therefore, ζ can change in the zero-strain reference state. Usually, the Lagrangian description is used to describe the kinematics of solids. The undeformed (stress free) configuration is used as the reference frame to describe the stress, the strain and to formulate the equilibrium of the deformed configuration. But, as pointed out by SKALAK *et al*. (1981) for volumetric growth problems, this classical approach must be modified, and a current stress free configuration must be used as a reference (Fig. 1). Thus, three different configurations are used: the initial stress free configuration (Ω_0), the current stress free configuration (Ω), and the actual configuration (Ω_t). Now, one can find a relation between ζ and υ:

$$\zeta = J\upsilon. \tag{6}$$

In other words, ζ and υ are the same, but are viewed from the effective free deformation reference state and the current deformed state, respectively. Note that ζ may change without changing the strain reference state if we assume that, at constant temperature and zero body force, there exists a unique zero-strain reference state for all values of ζ (COWIN and HEGEDUS, 1976). Thus, by combining Eqs. (5) and (6), the bulk density ρ of the porous structure is given by:

$$\rho = \frac{\gamma\zeta}{J}. \tag{7}$$

4. Field Equations

The control volume for writing the conservation equations and entropy inequality is the porous structure without the fluid. Conservation of mass for the porous structure can be written as:

$$\frac{D}{Dt}\iiint \gamma\upsilon dv = \iiint C dv \tag{8}$$

where D/Dt is the material time derivative, C is the rate of apposition or resorption of bone material per unit volume of bone, dv is an element of volume, and the domain of the integrals is the porous matrix structure. Equation (8) can be rewritten in differential form as:

$$\gamma\dot{\upsilon} + \gamma\upsilon V_{k,k} = C \tag{9}$$

where $V_{k,k}$ is equivalent to the divergence of the velocity vector.

Introducing Eq. (7) into Eq. (9), a new relation can be found:

$$\dot{\zeta} = \frac{CJ}{\gamma} \tag{10}$$

where, the dot $(\cdot)$ represents the material time derivative. Equation (10) is the conservation of mass equation for the porous structure.

The conservation of momentum for the porous structure is:

$$D/Dt\iiint \gamma\upsilon V_i dv = \iint T_{ij} n_j ds + \iiint \gamma\upsilon B_i dv + \iiint \left(P_i + C_i\right) dv \tag{11}$$

where T_{ij} is the stress tensor, n_i is the i-th element of unit normal vector, B_i is the body force, and P_i is the force exerted by the fluid phase on the porous structure.

Conservation of energy for the porous structure is given by:

$$D/Dt\iiint \gamma\upsilon\left(\varepsilon + \frac{1}{2}CV_iV_i\right)dV$$
$$= \iint \left(T_{ij}n_jV_i - q_in_i\right)dS + \iiint \gamma\upsilon\left(b_iV_i + r\right)dV + \iiint\left(P_iV_i + \frac{1}{2}CV_iV_i + C\varepsilon + \overset{*}{h}\right)dV \tag{12}$$

where, ε is the specific internal energy; r is the specific heat per unit time; q is the heat flux vector, and $\overset{*}{h}$ is the energy transferred between the matrix and bone fluid (compensation term).

The entropy inequality for the porous structure is:

$$D/Dt\iiint \gamma\upsilon\eta dv \geq \iiint \frac{\gamma\upsilon r}{\theta}dv - \iint \frac{q_i n_i}{\theta}dv + \iiint \left(\frac{\overset{**}{h}}{\theta} + C\eta\right)dv \tag{13}$$

where η is the specific entropy; θ is the absolute temperature and $\overset{**}{h}$ is the entropy production term caused by the interaction between the matrix and the perfusant which is not accounted for in other terms. We distinguish between $\overset{*}{h}$ and $\overset{**}{h}$ to indicate that not all the energy transferred from the perfusant to the matrix needs to contribute to the entropy production (COWIN and HEGEDUS, 1976).

By using the divergence theorem, Eqs. (11)–(13) can be rewritten as:

$$\gamma\upsilon\dot{V}_i = T_{ij,j} + \gamma v b_i + p_i \tag{14}$$

$$\gamma\upsilon\dot{\varepsilon} = T_{ij}L_{ij} + \gamma v r - q_{i,i} + \overset{*}{h} \tag{15}$$

$$\gamma\upsilon\dot{\eta} \geq \frac{\gamma\upsilon r}{\theta} - \left(\frac{q_i}{\theta}\right)_{,i} + \frac{\overset{**}{h}}{\theta} \tag{16}$$

where, i indicates the partial derivative with respect to x, y and z (or x_1, x_2 and x_3) in the Cartesian coordinate system.

The specific free energy (ψ) is given as:

$$\psi = \varepsilon - \eta\theta \tag{17}$$

and

$$h = \overset{*}{h} - \overset{**}{h}. \tag{18}$$

Then Eq. (16) can be converted to:

$$-\gamma\upsilon\dot{\psi} - \gamma\upsilon\eta\dot{\theta} + T_{ij}L_{ij} - \frac{1}{\theta}q_i\theta_{,i} + h \geq 0 \tag{19}$$

Inequality (19) is a reduced form of the entropy inequality which is obtained by using the energy field equation (Eq. (15)) to eliminate the radiation supply (r) from the entropy inequality (Eq. (16)). Then by introducing the Helmohltz free energy (ψ), and Eq. (18), the inequality (19) follows naturally.

5. Constitutive Assumptions

Bone can only be added or removed on the specific surfaces by osteoblasts and osteoclasts, respectively (MARTIN, 1984). Thus, the rate of change of porosity (or the rate of remodeling) is influenced by the amount of internal surface that is available for physiologic activity. A bone whose specific surface is 5 mm^2/mm^3 has a greater potential for remodeling than one whose specific surface is 1 mm^2/mm^3 (MARTIN, 1972, 1984).

The relation between volume fraction and free surface density depends on the pattern of mass distribution in the bone. We can have two bones with equal porosity but different specific surfaces (Fig. 2). For the same mechanical stimuli, the rate of remodeling is greater for the bone with the greater specific surface. Roughly, the magnitude of the specific surface is proportional to the potential for bone remodeling (MARTIN, 1972, 1984; RECKER, 1983; BRONNER and WORRELL, 1999).

Therefore, the important quantities in the constitutive equations for the specific free energy, specific entropy, stress tensor, heat generation, enthalpy and the rate of remodeling are the temperature, temperature gradient, specific surface and the deformation gradient. Since bone remodeling is considered an iso-thermal process, we do not need the temperature gradient in the essential assumptions.

The free surface density of bone is defined as:

$$S_v = \frac{S_b}{V_t} \tag{20}$$

where S_b is the total area of the solid phase which is in contact with soft tissue.

Assuming that only a portion of the free surface is active, we can define an effective free surface as:

$$S_\lambda = \lambda \frac{S_b}{V_t} \tag{21}$$

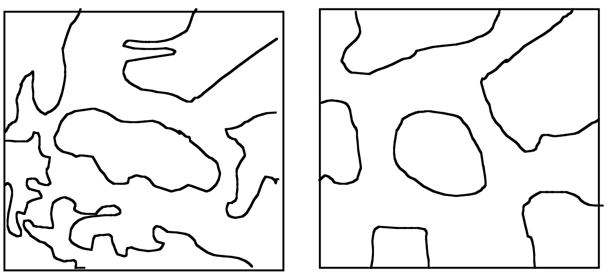

Fig. 2. Samples of bone with identical porosity but different specific surfaces.

where λ represents the ratio of the surface that is active for remodeling.

When considering the role of the free surface density in bone remodeling, we assume that the specific free energy, specific entropy, stress tensor, heat generation, enthalpy, and the rate of remodeling are functions of the temperature, temperature gradient, free surface density, and the deformation gradient.

For example, for the specific free energy we can write:

$$\psi = \psi\left(\theta, \theta_{,i}, S_\lambda, F_{ij}\right). \tag{22}$$

Taking the time derivative of the free energy function Eq. (22), substituting it into Eq. (19), and making use of standard arguments (COLEMAN and CURTIN, 1967), we obtain the following relations:

$$\eta = -\frac{\partial \psi}{\partial \theta} \tag{23}$$

$$T_{ij} = \frac{\gamma\zeta}{J}\frac{\partial \psi}{\partial F_{ik}} F_{jk} \tag{24}$$

$$\gamma\upsilon\psi - \gamma\upsilon\frac{\partial \psi}{\partial S_\lambda}\dot{S}_\lambda - \frac{1}{\theta} q_i \theta_{,i} + h \geq 0. \tag{25}$$

We assume that, for constant temperature and zero body force, there exists a unique zero-strain reference state for all values of ζ which satisfies:

$$T_{ij}\left(\theta_0, 0, S_\lambda, \delta_{ik}\right) = 0. \tag{26}$$

6. Small-Strain Approximation

Strains in bone during everyday physiological activities are small (FRITTON *et al*., 2000). Therefore, we can use a small strain approach for the remodeling of bone exposed to physiological loads. In the small strain domain, ζ and F change to e and E, respectively (HEGEDUS and COWIN, 1976) where:

$$e = \zeta - \zeta_0 \tag{27}$$

and ζ_0 is the reference value for the volume fraction, and E_{ij} is:

$$E_{ij} = \frac{1}{2}\left(U_{i,j} + U_{j,i}\right) \tag{28}$$

where $\boldsymbol{U}$ is the displacement vector.

Using the above assumptions, we obtain:

$$C = C(S_\lambda, E) \tag{29}$$

$$\psi = \psi(S_\lambda, E). \tag{30}$$

Equation (10) can be rewritten as:

$$\frac{de}{dt} = \Phi\left(S_\lambda, E\right) \tag{31}$$

where:

$$\Phi\left(S_\lambda, E\right) = \frac{C\left(S_\lambda, E\right)\sqrt{\det(I + 2E)}}{\gamma} \tag{32}$$

$\Phi(S_\lambda, E)$ and $\Psi(S_\lambda, E)$ may be approximated by neglecting higher order terms in a Taylor series expansion, thus:

$$\Phi\left(S_\lambda, E\right) \cong a\left(S_\lambda\right) + A^*_{ij}\left(S_\lambda\right)E_{ij} + \frac{1}{2}B_{ijkm}\left(S_\lambda\right)E_{ij}E_{km} \tag{33}$$

where

$$a\left(S_\lambda\right) = \Phi\left(S_\lambda, 0\right) \tag{34}$$

$$A^*_{ij}\left(S_\lambda\right) = \left[\frac{\partial \Phi\left(S_\lambda, E_{ij}\right)}{\partial E_{ij}}\right]_{E_{ij}=0} \tag{35}$$

and

$$B_{ijkm}\left(S_\lambda\right) = \left[\frac{\partial^2 \Phi\left(S_\lambda, E_{ij}\right)}{\partial E_{ij}\partial E_{km}}\right]_{E_{ij}=0}. \tag{36}$$

Here $a(S_\lambda)$, $A^*_{ij}(S_\lambda)$, and $B_{ijkm}(S_\lambda)$ are the material properties of bone. Substituting Eq. (33) into Eq. (31), and neglecting the third term in Eq. (33), we obtain:

$$\frac{de}{dt} = \left\{ a\left(S_\lambda\right) + A^*_{ij}\left(S_\lambda\right)E_{ij} \right\}. \tag{37}$$

Equation (37) describes the remodeling process. In order to find the required coefficients for Eq. (37), a method of approximation is used:

$$\frac{de}{dt} = \left\{ a_0 + a_1 S_\lambda + a_2 S_\lambda^2 + \left(A^0_{ij} + A_{ij}S_\lambda\right)E_{ij} \right\} \tag{38}$$

where $A_{ij}{}^0$ and A_{ij} can be defined as follows:

$$A^0_{ij} = A^*_{ij}\left(0, E_{ij}\right) \tag{39}$$

and

$$A_{ij} = \left[\frac{\partial A^*_{ij}\left(S_\lambda, E_{ij}\right)}{\partial S_\lambda} \right]_{S_\lambda = 0}. \tag{40}$$

The following assumptions are made: (1) when S_λ is zero, de/dt must be zero. (2) when $E_{ij} = E_{ij}{}^*$ then $de/dt = 0$; where $E_{ij}{}^*$ is the strain state corresponding to the equilibrium state as proposed by BEAUPRE *et al*. (1990a, b). The first assumption emphasizes that remodeling can only occur in the presence of a free surface area. The second assumption implies that when the mechanical stimuli lie in the lazy zone; there is no net remodeling. Applying these assumptions to Eq. (38), the governing equation for the remodeling process becomes:

$$\frac{de}{dt} = -\frac{dP_v}{dt} = A_{ij}S_\lambda\left(E_{ij} - E^*_{ij}\right) \tag{41}$$

where:

$$A_{ij} = \left[\frac{\partial^2 \Phi\left(S_\lambda, E_{ij}\right)}{\partial S_\lambda \partial E_{ij}} \right]^{S_\lambda = 0}_{E_{ij} = 0}. \tag{42}$$

P_v is the porosity fraction ($e + P_v = 1$).

Equation (41) defines the remodeling process of bone and it shows that the rate of remodeling depends strongly on both the micro-structural pattern (factor S_λ) and mechanical stimuli (E_{ij}). The specific surface can be defined using mathematical relations for samples of cortical bone. For cancellous bone, the specific surface can be determined experimentally with micro-computed tomography (Micro-CT).

In order to use this model, regions with nearly uniform bone mass distribution must be identified, thus the specific surface in each region can be expressed uniquely. Then, the mechanical stimuli (e.g. strain) distribution in each region must be determined. Finally, once A_{ij} is known, Eq. (41) can be used to derive the rate of remodeling for a sample of bone.

7. Some Examples

Below, we will show some example applications of the current bone remodeling theory.

7.1. Surface remodeling equation

FROST (1964) divided bone remodeling into surface and internal remodeling. Surface remodeling refers to the remodeling of bone on the external surfaces (Endosteal and Periosteal surfaces). Internal remodeling refers to remodeling on the Haversian canals and trabecular surfaces.

Surface remodeling has been assumed to be linearly proportional to the strain tensor (COWIN and VAN BUSKIRK, 1979) for a hollow compact cylindrical bone:

$$\frac{dR}{dt} = C_{ij}(\mathbf{n}, Q)\left(E_{ij}(Q) - E_{ij}^{*}(Q)\right) \tag{43}$$

where R is the radius of the endosteum or periosteum, $E_{ij}(Q)$ and $E_{ij}{}^*(Q)$ are the Cartesian components of the strain tensor at point Q and the reference value of the strain where no remodeling occurs, respectively, and $C_{ij}(\boldsymbol{n}, Q)$ are surface remodeling rate coefficients

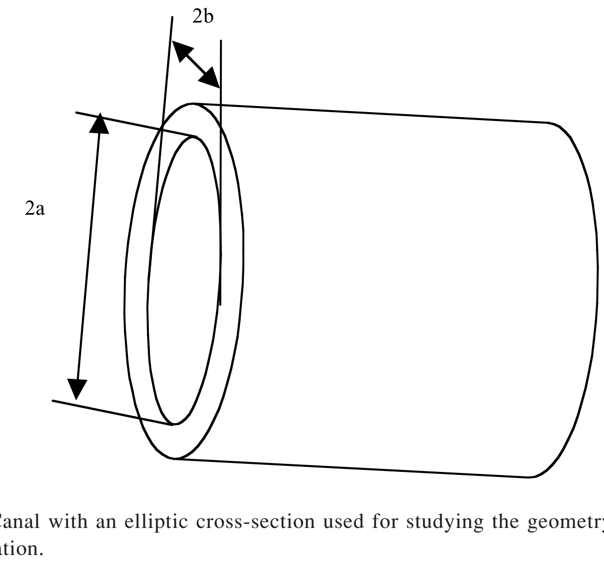

Fig. 3. Haversian Canal with an elliptic cross-section used for studying the geometry effect on the surface remodeling equation.

which are, in general, dependent upon the point Q and the normal vector, $\boldsymbol{n}$, to the surface at Q. Equation (43), can be found using Eq. (41) for the remodeling on the inner and outer walls of a hollow cylindrical bone, only and only if we assume that there is remodeling either on the outer or the inner surface. However, it is well accepted that there is remodeling on both surfaces simultaneously, except possibly in the Haversian canals where there is remodeling only in the Haversian canal. Remodeling equations for the Haversian canals can be derived using Eq. (41) for different geometries of the canal. For example, for a Haversian canal with an elliptic cross section (Fig. 3), the remodeling equation can be found using Eq. (41) if we assume that the rate of remodeling is the same for all points of the canal:

$$\frac{dr}{dt} = f(a,b)C_{ij}\left(E_{ij}(Q) - E_{ij}^{*}(Q)\right) \tag{44}$$

where:

$$f(a,b) = 1 - \frac{1}{4}\left[1-\left(\frac{b}{a}\right)^2\right] - \frac{3}{64}\left[1-\left(\frac{b}{a}\right)^2\right]^2 - \frac{5}{256}\left[1-\left(\frac{b}{a}\right)^2\right]^3 - \frac{175}{16384}\left[1-\left(\frac{b}{a}\right)^2\right]^4 - \frac{441}{65536}\left[1-\left(\frac{b}{a}\right)^2\right]^5. \tag{45}$$

a and b are minor and major radii of the elliptic cross-section, respectively (Fig. 3).

In Eq. (44), the effect of the cross-sectional geometry is captured in $f(a, b)$. Surface remodeling rate coefficients in Eq. (43) are negative of the corresponding ones in Eq. (41); i.e., $C_{ij} = -A_{ij}$.

For example, the rate of bone remodeling for a Haversian canal with an elliptic cross-section (e.g. $a = 5b$) (Fig. 3) is about 69% of that of a canal with a circular cross-section ($R = a$).

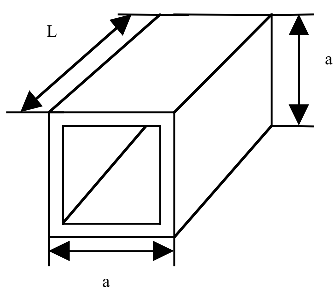

Fig. 4. A hypothetical shape of the micro-structure of a cancellous bone, all thicknesses = T.

Comparing the percentage of the free surface of an elliptic and circular Haversian canal 60% and 69%, respectively, with the experimental rate of remodeling shows good agreement (RECKER, 1983; BRONNER and WORRELL, 1999). If the factor $f(a, b)$ in front of Eq. (44) is not considered for non-circular cross-sections, the effect of geometry on the rate of remodeling can not be seen.

7.2. Rate of change of trabecular bone thickness

For simple asymptotical geometries Figs. 4–6, one can derive the rate of change in trabecular thickness (T) using Eq. (41). The geometries are chosen for the purpose of illustrating the potential of the proposed model, and they are not meant to be realistic representations of bone. For the sake of simplicity, the mechanical stimuli are assumed to be uniform. The rate of change in thickness of each trabeculae in Figs. 4–6 is given by:

Fig. (4):

$$\frac{dT}{dt} = A_{ij}\left(E_{ij} - E_{ij}^{*}\right) \tag{46}$$

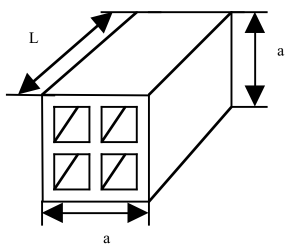

Fig. 5. A hypothetical shape of the micro-structure of a cancellous bone, all thicknesses = T.

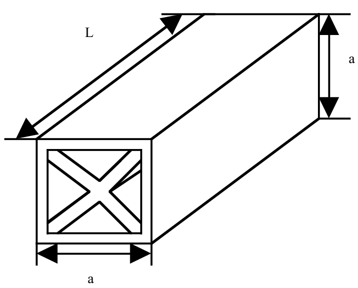

Fig. 6. A hypothetical shape of the micro-structure of a cancellous bone, all thicknesses = T.

Fig. (5):

$$\frac{dT}{dt} = \frac{4}{3} A_{ij} \left(E_{ij} - E_{ij}^* \right) \tag{47}$$

Fig. (6):

$$\frac{dT}{dt} = \sqrt{2} A_{ij} \left(E_{ij} - E_{ij}^* \right). \tag{48}$$

Thus, by knowing the microstructure of bone, one can use Eq. (41) to analyze the remodeling process.

7.3. Cell activity (Martin's) model

Martin proposed a model for bone remodeling based on bone cell activity (MARTIN, 1972):

$$\frac{de}{dt} = \left(a_b \delta_b \lambda_b - a_c \delta_c \lambda_c \right) S_\lambda \tag{49}$$

where S_λ, a_b, a_c, λ_b, λ_c, δ_c and δ_b are the free surface density, osteoblast and osteoclast activities, the fraction of S_λ occupied by osteoblasts and osteoclasts, and the osteoblast and osteoclast densities, respectively.

Comparing Eqs. (41) and (49), the following relation is obtained:

$$\left(a_b \delta_b \lambda_b - a_c \delta_c \lambda_c \right) = A_{ij} \left(E_{ij} - E_{ij}^* \right). \tag{50}$$

Equation (50) states that the net cell activity (rate of apposition – rate of resorption) is linearly proportional to the mechanical stimuli (here, strain tensor).

MARTIN proposed an empirical formula that relates bone porosity and the effective free surface density (MARTIN, 1984):

$$S_\lambda = 32.2 P_v - 93.9 P_v^2 + 134 P_v^3 - 101 P_v^4 + 28.8 P_v^5. \tag{51}$$

Assuming that Eq. (51) is valid for all bone samples and the entire free surface is active for remodeling; Eq. (41) can be written as:

$$\frac{de}{dt} = -\frac{dP_v}{dt} = A_{ij} \left(32.2 P_v - 93.9 P_v^2 + 134 P_v^3 - 101 P_v^4 + 28.8 P_v^5 \right) \left(E_{ij} - E_{ij}^* \right). \tag{52}$$

Equation (52) can be solved for bone remodeling if A_{ij} and the mechanical stimuli are known.

8. Discussion

COWIN and HEGEDUS (1976), HEGEDUS and COWIN (1976) assumed that the specific free energy (ψ), entropy (η), heat flux (q_i), stress tensor (T_{ij}), energy transferred between matrix and fluid phase (h), and the rate of resorption and apposition (C) are functions of the volume fraction, deformation gradient, and temperature for the isothermal process of bone remodeling. However, experimental evidence suggests that resorption and apposition can only take place on the free surfaces of bone (MARTIN, 1984). Thus, the rate of bone remodeling is proportional to the specific surface area. Hence, we have replaced volume fraction with the specific surface area in the constitutive equations. In other words, the Helmholtz free energy, rate of resorption and/or apposition, enthalpy, entropy, stress tensor, and heat flux vector were assumed to be functions of temperature, specific surface, and deformation gradient. The novelty of our approach is that one can observe the effect of bone geometry and mass distribution on the rate of remodeling (RECKER, 1983; BRONNER and WORRELL, 1999). Using our model, a single equation (Eq. (41)) can be found for surface and internal remodeling, thus the effects of mechanical stimuli and bone geometry on bone remodeling, can be studied simultaneously. Using Eq. (41), the effects of the geometry and the rate of remodeling of Haversian canals can be tackled. Geometric feedback in the bone remodeling process, as proposed by MARTIN (1972), can also be explored.

The specific surface area is always non-zero, thus, in accordance with Eq. (41), the rate of remodeling is only zero when the mechanical stimuli are in the lazy zone (BEAUPRE *et al.*, 1990a). The rate of remodeling can be different for the same mechanical stimuli, same volume fraction, but different mass distribution.

Considering the obvious effect of the specific surface on the bone remodeling equation (Eq. (41)), we can conclude that two people with an equal average mass density, similar shape of bones (macroscopically), similar mechanical stimuli, similar hormonal stimuli and same form of nutrition can experience different rates of osteoporosis because of the micro-structure of the bones. Thus, for evaluating risks for fracture in osteoporotic bones, besides measuring the volume fraction (solid phase volume per total volume), the microstructure of bone and the magnitude of the specific surface must be quantified. Further research is needed to evaluate Eq. (41) and determine the rate of remodeling of healthy and osteoporotic bone.

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NOMENCLATURE

x and X	Coordinate in the spatial and reference configurations, respectively
t	Time
V_i	Velocity
F_{ij}	Deformation gradient
J	Determinant of the deformation gradient tensor
L_{ij}	Velocity gradient tensor

υ Volume fraction in the current configuration
ρ Apparent density of bone
γ Bulk density
ζ Volume fraction in unstrained reference state
Ω_0 Initial stress free configuration of the body
Ω Current stress free configuration of the body
Ω_t Actual configuration of the body at time t
D/Dt or •
Material time derivative
C Rate of resorption (or apposition)
dv Element of volume
$V_{k,k}$ Divergence of the velocity vector
T_{ij} Stress tensor
$\boldsymbol{n}$ Unit normal vector
B_i Body force
P_i Force exerted by the fluid phase on the porous structure
ε Specific internal energy
r Specific heat per unit volume
q Heat flux vector
h^* Energy transferred between the bone matrix and bone fluid
η Specific entropy
θ Absolute temperature
h^{**} Entropy production term caused by the interaction between the matrix and the perfusant
ψ Helmholtz free energy
S_v Free surface density
S_b Total area of interface between the solid and the porosity
V_t Total volume of bone
S_λ Active free surface density
λ Ratio of the surface that is active for remodeling
$\theta_{,i}$ Gradient of absolute temperature
δ_{ij} Kronecker delta
e Difference between the unstrained volume fraction at time t and before remodeling
ζ^o Unstrained volume fraction at $t = 0$ (before remodeling)
E_{ij} Strain tensor
E_{ij}^* Strain at the remodeling equilibrium state
$\boldsymbol{U}$ Displacement vector
$U_{i,j}$ Gradient of the displacement vector
I Identity tensor
$a_0, a_1, a_2, a(S_\lambda), A_{ij}^*(S_\lambda), A_{ij}^0, A_{ij}, B_{ijkm}(S_\lambda)$
Material constants of bone
P_v Void fraction
R Radius of Endosteum or Periosteum
C_{ij} Surface remodeling rate coefficients
Q An arbitrary point on the bone surface
r Radius of endosteum on an elliptic cross section
a, b Minor and major radii of an ellipse
$f(a, b)$ A function of a &b cause by the geometry of the cross section
T Thickness of trabeculae
a_b, a_c Osteoblast and osteoclast activity, respectively
δ_b, δ_c Fraction of S_λ occupied by osteoblast and osteoclast, respectively
P_v Porosity (void fraction)

REFERENCES

ASCENZI, A. (1993) Biomechanics and Galileo Galilei, *J. Biomech.*, **26**, 95–100.

BEAUPRE, G. S., ORR, T. E. and CARTER, D. R. (1990a) An approach for time-dependent bone modeling and remodeling-theoretical development, *J. Orthop. Res.*, **8**, 651–661.

BEAUPRE, G. S., ORR, T. E. and CARTER, D. R. (1990b) An approach for time-dependent bone modeling and remodeling—a preliminary remodeling simulation, *J. Orthop. Res.*, **8**, 662–670.

BRONNER, F. and WORRELL, R. V. (eds.) (1999) *Orthopaedics, Principles of Basic and Clinical Science*, CRC Press, Boca Raton, FL.

CARTER, D. R. (1987) Mechanical loading history and skeletal biology, *J. Biomech.*, **20**, 1095–1109.

CARTER, D. R. and BEAUPRE, G. S. (2001) *Skeletal Function and Form, Mechanobiology of Skeletal Development, Aging, and Regeneration*, Cambridge University Press.

CARTER, D. R. and HAYES, W. C. (1977) Compact bone fatigue damage—I. Residual strength and stiffness, *J. Biomech.*, **10**, 325–337.

CARTER, D. R., FYHRIE, D. P. and WHALEN, R. T. (1987) Trabecular bone density and loading history: regulation of connective tissue biology by mechanical energy, *J. Biomech.*, **20**(8), 785–794.

CARTER, D. R., VAN DER MEULEN, M. C. and BEAUPRE, G. S. (1996) Mechanical factors in bone growth and development, *Bone*, **18** (Suppl. 1), S5–S10.

COLEMAN, B. D. and CURTIN, M. (1967) Thermodynamics with internal variables, *J. Chem. Phys.*, **47**, 597–613.

COWIN, S. C. (1987) Bone remodeling of diaphyseal surfaces by torsional loads: theoretical predictions, *J. Biomech.*, **19**(1), 1111–1120.

COWIN, S. C. (1993) Bone stress adaptation models, *J. Biomech. Eng.*, **115**(4B), 528–533.

COWIN, S. C. and FIROOZBAKHSH, K. (1981) Bone remodeling of diaphysial surfaces under constant load: Theoretical predictions, *J. Biomech.*, **7**, 471–484.

COWIN, S. C. and HEGEDUS, D. M. (1976) Bone Remodeling I: A theory of adaptive elasticity, *J. Elasticity*, **6**, 313–325.

COWIN, S. C. and VAN BUSKIRK, W. C. (1978) Internal bone remodeling induced by a medullary pin, *J. Biomech.*, **11**(5), 269–275.

COWIN, S. C. and VAN BUSKIRK, W. C. (1979) Surface bone remodeling induced by a medullary pin, *J. Biomech.*, **12**(4), 269–276.

COWIN, S. C., HART, R. T., BALSER, J. R. and KOHN, D. H. (1985) Functional adaptation in long bones: establishing in vivo values for surface remodeling rate coefficients, *J. Biomech.*, **18**(9), 665–684.

FIROOZBAKHSH, K. and ALEYASSIN, M. (1989) The effect of stress concentration on bone remodeling: Theoretical predictions, *J. Biomech. Eng. (ASME)*, **3**, 355–360.

FIROOZBAKHSH, K. and COWIN, S. C. (1981a) Devolution of inhomogeneities in bone structure: Predictions of adaptive elasticity theory, *J. Biomech. Eng. (ASME)*, **102**, 287–293.

FIROOZBAKHSH, K. and COWIN, S. C. (1981b) An analytical model of Pauwels' functional adaptation mechanism in bone, *J. Biomech. Eng. (ASME)*, **103**, 246–252.

FIROOZBAKHSH, K., ALEYASSIN, M. and MONEIM, M. S. (1992) Evolution of bone inhomogeneity around a hole in an orthotropic plate of bone, *J. Biomech.*, **25**(4), 387–394.

FRITTON, S. P., MCLEOD, K. J. and RUBIN, C. T. (2000) Quantifying the strain history of bone: spatial uniformity and self similarity of low magnitude strains. *J. Biomech.*, **33**(3), 317–325.

FROST, H. M. (ed.) (1964) Dynamics of bone remodeling, in *Bone Biodynamics*, Little and Brown, Boston, 316 pp.

GJELSVIK, A. (1973a) Bone remodeling and piezoelectricity—I, *J. Biomech.*, **6**, 69–77.

GJELSVIK, A. (1973b) Bone remodeling and piezoelectricity—II, *J. Biomech.*, **6**, 187–193.

HART, R. T. and DAVY, D. T. (1989) Theories of bone modeling and remodeling, in *Bone Mechanics* (ed. S. C. Cowin), Chap. 11, pp. 253–277, CRC Press, Boca Raton, FL.

HEGEDUS, D. M. and COWIN, S. C. (1976) Bone Remodeling II: Small strain adaptive elasticity, *J. Elasticity*, **6**, 337–352.

HUISKES, R. (1997) in *Simulation of Self-Organization and Functional Adaptation in Bone*, pp. 299–319, Springer, Berlin.

HUISKES, R., WEINANS, H., GROOTENBOER, H. J., DALSTRA, M., FUDALA, B. and SLOOF, T. J. (1987) Adaptive bone remodeling theory applied to prosthetic-design analysis, *J. Biomech.*, **20**(11–12), 167–182.

JACOBS, C. R., SIMO, J. C., BEAUPRE, G. S. and CARTER, D. R. (1997) Adaptive bone remodeling incorporating simultaneous density and anisotropy considerations, *J. Biomech.*, **30**(6), 603–613.

KUMMER, B. K. F. (1972) Biomechanics of bone: mechanical properties, functional structure, functional adaptation, in *Biomechanics* (ed. Y. C. Fung), pp. 237–271, Englewood Cliffs, Prentice Hall.

MARTIN, R. B. (1972) The effects of geometric feedback in bone remodeling and osteoporosis. *J. Biomech.*, **5**, 447–455.

MARTIN, R. B. (1974) The specific surface-porosity relationship in bone: refinement of the concept of geometric feedback, *Orthopaedic Research Society Meeting*, Dallas, January 14–17.

MARTIN, R. B. (1984) in *Porosity and Specific Surface of Bone*, pp. 179–222, CRC Critical Reviews in Biomedical Engineering, Boca Raton, FL.

MULLENDER, M. G. and HUISKES, R. (1995) Proposal for the regulatory mechanism of Wolff's law, *J. Orthop. Res.*, **13**(4),503–512.

MULLENDER, M. G., HUISKES, R. and WEINANS, H. (1994) A physiological approach to the simulation of bone remodeling as a self-organizational control process, *J. Biomech.*, **27**(11), 1389–1394.

PRENDERGAST, P. J. and HUISKES, R. (1996) Microdamage and osteocyte-lacuna strain in bone. A micro structural finite element analysis, *J. Biomech. Eng.*, **118**, 240–246.

PRENDERGAST, P. J. and TAYLOR, D. (1994) Prediction of bone adaptation using damage accumulation, *J. Biomech.*, **27**, 1067–1076.

RAMTANI, S. and ZIDI, M. (2001) A theoretical model of the effect of continuum damage on a bone adaptation model, *J. Biomech.*, **34**, 471–479.

RECKER, R. R. (ed.) (1983) *Bone Histomorphometry: Techniques and Interpretation*, CRC Press, Boca Raton, FL.

SIFFERT, R. S., LUO, G. M., COWIN, S. C. and KAUFMAN, J. J. (1996) Dynamics relationship of trabecular bone density, architecture and strength in a computational model of osteopenia, *Bone*, **18**(2), 197–206.

SKALAK, R., DASGUPTA, G. and MOSS, M. (1981) Analytical description of growth, *J. Theoretical Biology*, **94**, 555–577.

SMITH, T. S., MARTIN, R. B., HUBBARD, M. and BAY, B. K. (1997) Surface remodeling of trabecular bone using a tissue level model, *J. Orthop. Res.*, **15**(4), 593–600.

WEINANS, H., HUISKES, R. and GROOTENBOER, H. J. (1992) The behavior of adaptive bone-remodeling simulation models, *J. Biomech.*, **25**(12), 1425–1441.

WOLFF, J. (1982) *The Law of Bone Remodeling* (original publication 1892 translated in 1986 by P. Maquet and R. Furlong), Springer, Berlin.