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The Square, the Circle and the Golden Proportion: A New Class of Geometrical Constructions

Janusz KAPUSTA

Brooklyn, NY
E-mail address: kapusta@earthlink.net

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Abstract. This article is a picture essay showing how new geometrical properties and constructions of the golden mean are derived from a circle and a square. A new discovery of the relationship between the entire family of silver means is presented.

1. Introduction

I am an artist with an interest in philosophy and mathematics, and so my search for the golden proportion was different from the approaches taken by mathematicians. The main reason for my taking a fresh look at the number phi is due to its overwhelming appearance in art, nature and mathematics. I felt that an entity with such a power must have a deeper basis. During 2000 I discovered a new world of geometrical relationships residing within the square and the circle. To my knowledge it was the first visual construction connecting the golden and silver mean proportions in a single diagram. This discovery took me on a long journey to an unknown terrain of beauty, mathematics, and philosophy. But it is philosophy that I consider the main stimulus for my investigations. For me, geometry provides the visual language for what we see in our observation of nature, and it helps us to gain a global understanding of the Universe so that we see all of its relationships in a single glance. This picture essay is a portion of a book about the golden mean that I am preparing for publication. Most of my results cannot found elsewhere. In other books, the source of the golden mean's power arises from:

1. Geometry, through the division of a line into the extreme and mean ratio as discovered in ancient Greece and found in the writings of Plato.
2. Number, as the ratio of successive terms from a Fibonacci sequence as discovered by Johannes Kepler.

I have created a third visual approach in which the golden mean is seen as an element of a larger system of relationships in which geometry and number are woven together into a single fabric.

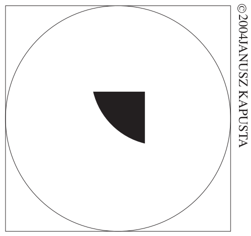


Fig. 1. The black form in the middle of the square and the circle (visible) arose from superimposing another square and circle (invisible) in a golden proportion to the previous ones. From collection of Getulio Alviani (2004).

2. Beginning

My search for the golden mean begins with a unit square. On a number line placed along the side of the square, any positive number outside the unit interval has its inverse image inside the interval. A second insight came from my artistic background. I did not like the usual approach to the golden proportion in which a line is divided by a point into two segments in such a way that shorter segment *a* to the longer one *b* is the same as the longer one to both of them $a + b$. I found it more natural to use a circle with a radius one. Now the division takes the following form: On a diameter of a circle made up of a pair of radii, divide one of the radii in such a way that the shorter segment to the longer one equals the longer one to the second radius. The beauty off this approach is seen when circles with diameters equal to the divided segments are constructed. As a consequence, these two shapes—the square and the circle—brought me to a surprising chain of discoveries.

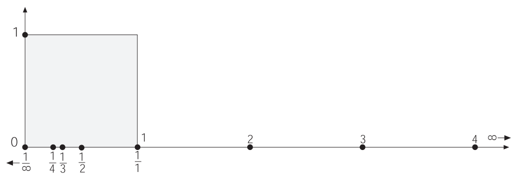


Fig. 2. For this unit square, the inverse image of any number placed on a number line along a side of the square and outside the square, lies within the unit interval that defines the edge of the square.

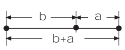


Fig. 3. The usual construction of the golden proportion.

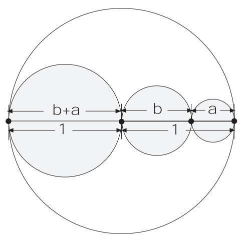


Fig. 4. Circle with unit radius.

3. The Square, the Circle, and the Pyramid

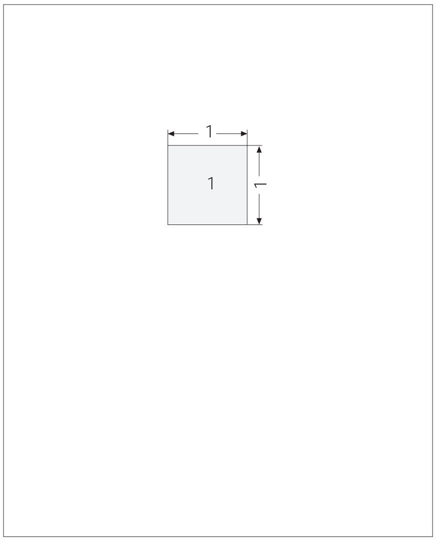


Fig. 5. A square.

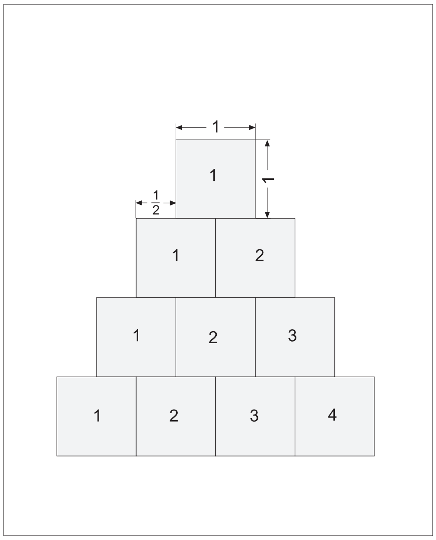


Fig. 6. Ten squares seen as the projection of a three-dimensional pyramid-like structure.

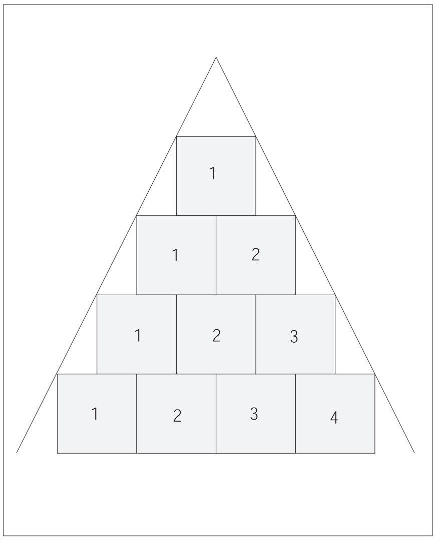


Fig. 7. Two lines touching the corners of the 10 squares.

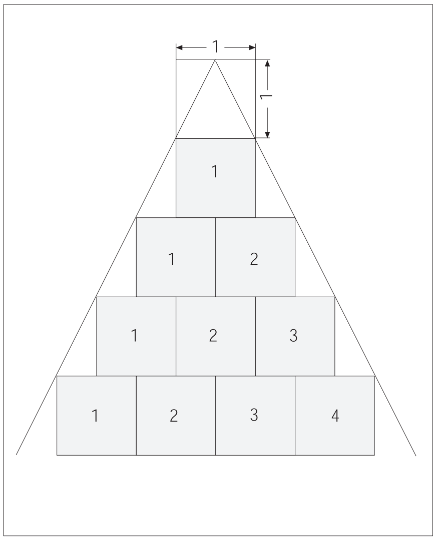


Fig. 8. The appearance of a new "virtual square" with an inscribed upward pointed triangle.

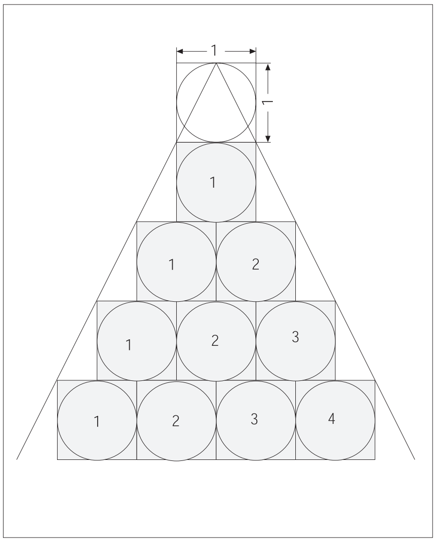


Fig. 9. Circles are inscribed within the squares.

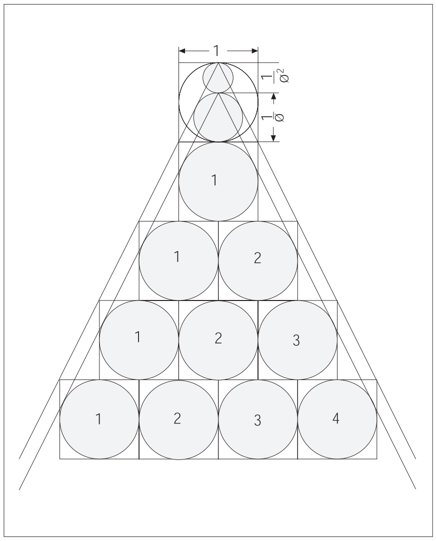


Fig. 10. Two lines tangent to the circles define a pair of circles within the virtual square with diameters in the golden proportion.

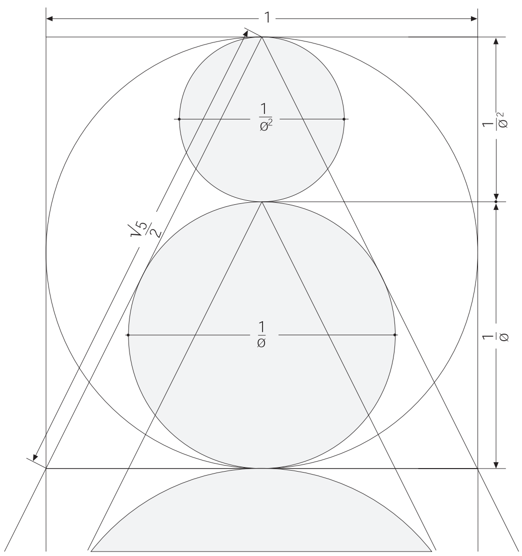


Fig. 11. The virtual square is seen in exploded view.

4. A Study of Relations

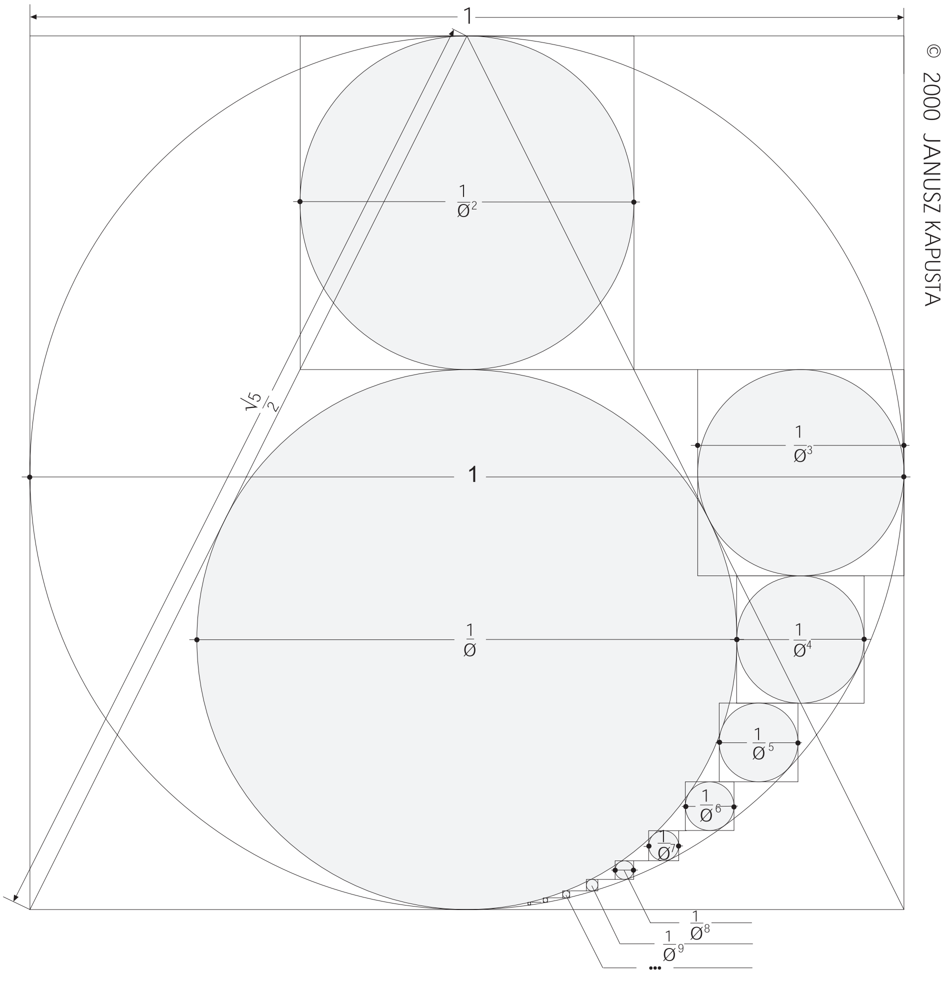


Fig. 12. A sequence of kissing (tangent) circles are created with the inverse powers of the golden mean as their diameters.

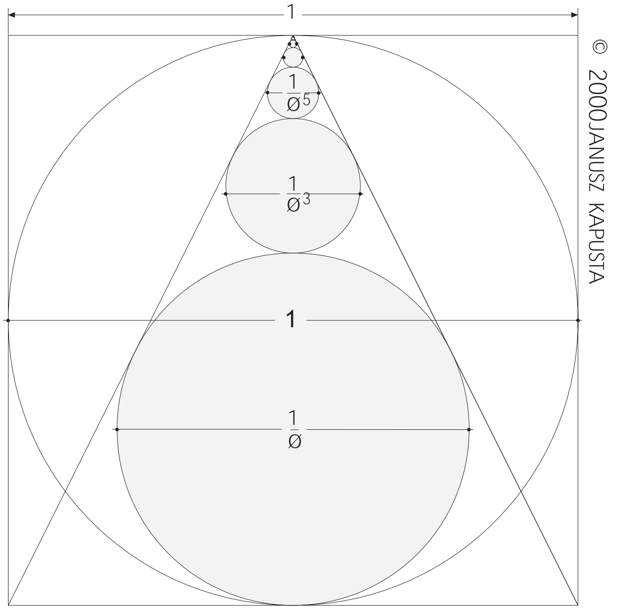


Fig. 13. Odd, inverse powers of the golden mean sum to unity.

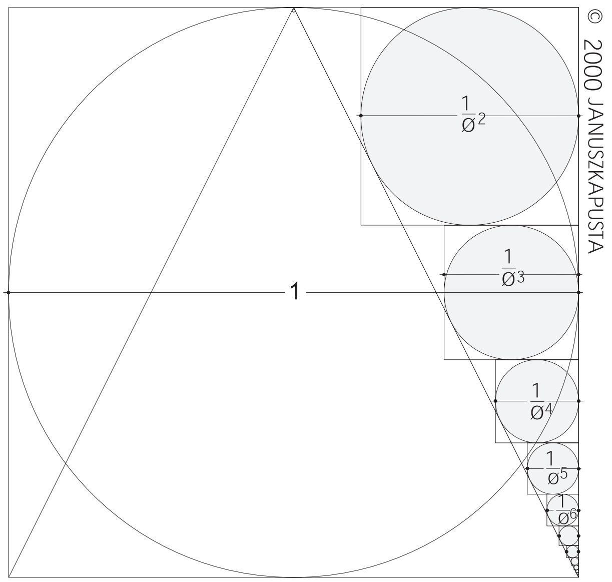


Fig. 14. All the inverse powers of the golden mean with the exception of $1/\phi$ sum to unity.

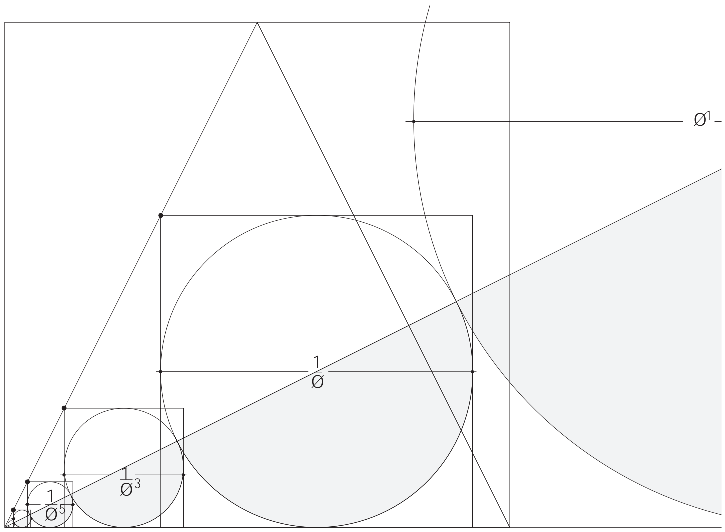


Fig. 15. Another surprising relation of odd, inverse powers. Notice that the squares that circumscribe the sequence of the golden circles, touch the side of the upward pointed triangle.

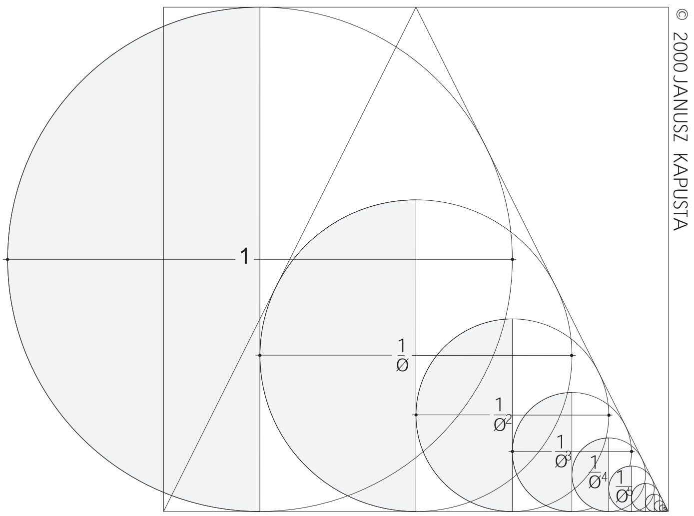


Fig. 16. An infinite sequance of a half golden circles tangent to their diameters and to the side of a upward pointed triangle.

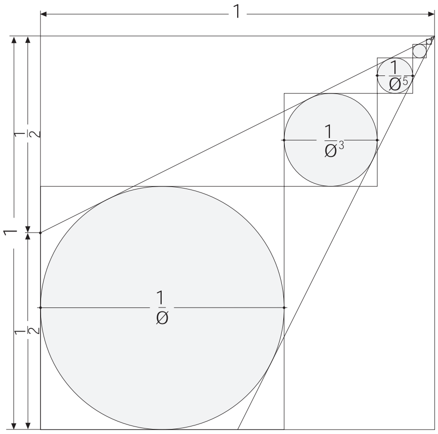


Fig. 17. Another way to view the odd, inverse powers of the golden mean as a sequence of circles.

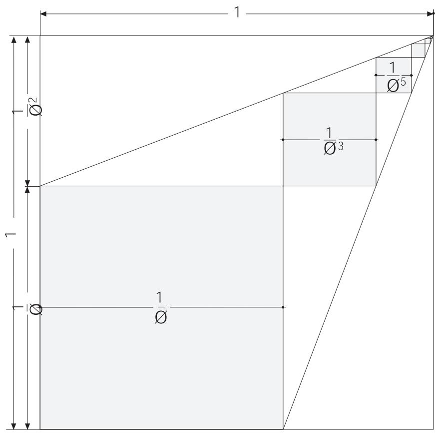


Fig. 18. They can also be seen as a sequence of squares.

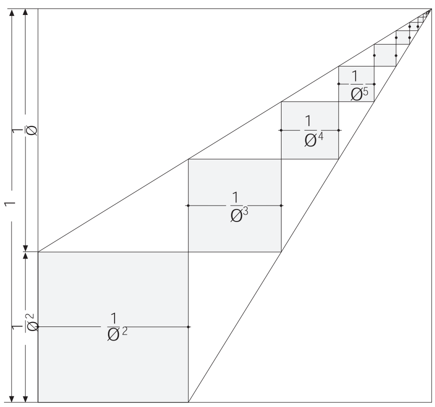


Fig. 19. An infinite sequence is seen to be a geometric sequence of squares of decreasing size.

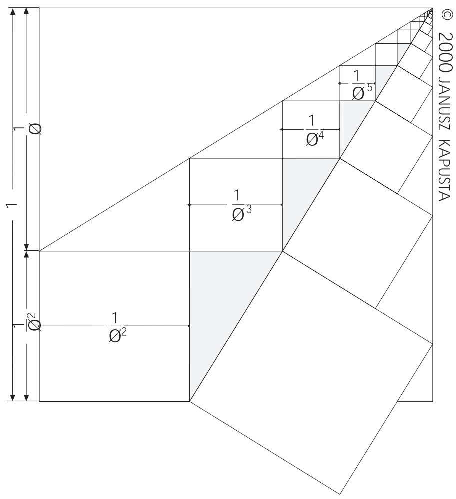


Fig. 20. The Pythagorean theorem is expressed by this sequence of squares. Notice how a sequence of vertices of the squares upon the hypotenuse lie against the right edge of the framing square.

5. Constructions

As a result of these findings, I have come upon some new constructions of the golden mean based on the relationship between the circle and the square.

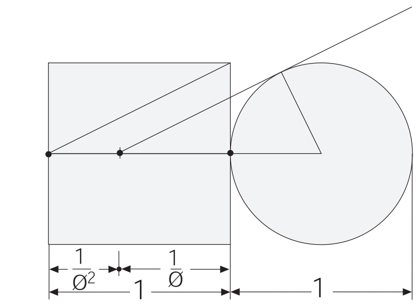


Fig. 21. A line tangent to the circle and parallel to the diagonal of two squares divides a line in the golden section.

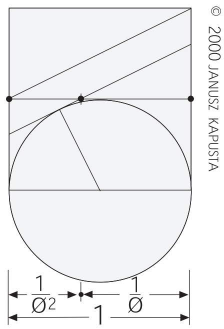


Fig. 22. A similar construction with a different configuration.

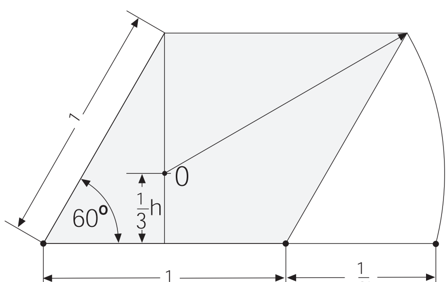


Fig. 23. Draw a circle with center at O (one-third the altitude of a rhombus) to meet a line extended from the base of the rhombus. The resulting length, along with the base of the rhombus is divided in the golden section.

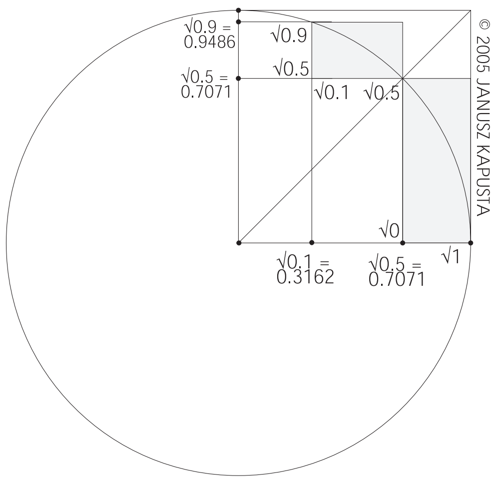


Fig. 24. Unusual appearance of a golden and a silver mean rectangle (see Fig. 44) on the circumference of a circle. Notice that the pairs of numbers under the root sign ($\sqrt{0.1}$, $\sqrt{0.9}$), ($\sqrt{0.5}$, $\sqrt{0.5}$) and ($\sqrt{0}$, $\sqrt{1}$) sum to one.

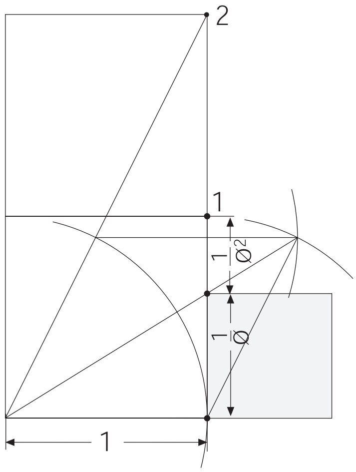


Fig. 25. The golden section of the side of a unit square gotten by bisecting the angle between a diagonal of a double square and its base.

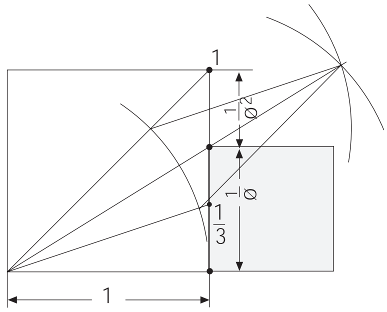


Fig. 26. A golden section of the side of a unit square gotten by bisecting the angle between a diagonal of the square and a line connecting a vertex of the square with the 1/3-point of its opposite side.

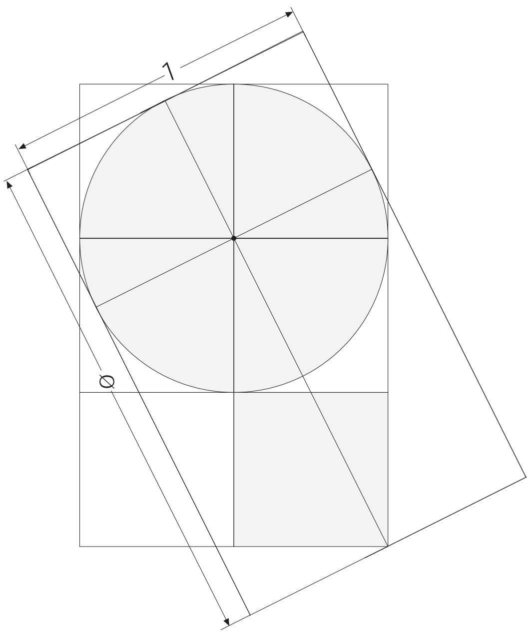


Fig. 27. A golden rectangle built upon 6 squares.

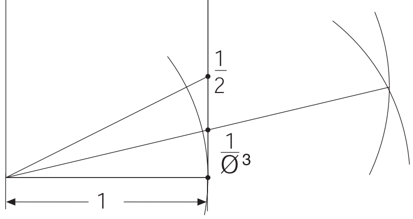


Fig. 28. A bisection of the angle between the midpoint of the opposite side of the unit square and its base divides the side in a proportion $1/\phi^3$:1.

6. Roots

Root-Circles Here I present two constructions showing an intimate connection between the golden ratio and circles with integer square root radii. I have discovered many other relations between them, but the work is still in progress.

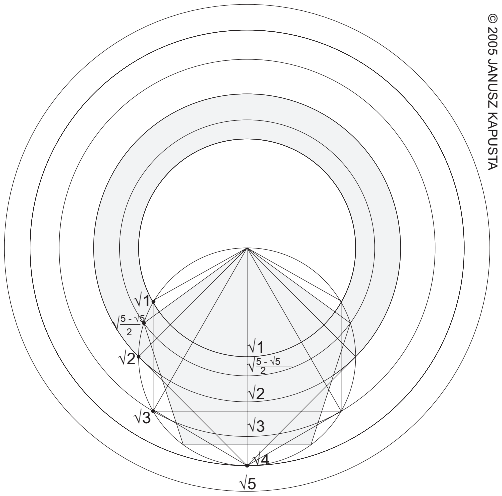


Fig. 29. Starting from a pair of Vesica circles, create a sequence of concentric circles to one of them with radii equal to $\sqrt{4}$, $\sqrt{3}$, $\sqrt{2}$ and $\sqrt{1}$. The concentric circles demark a sequence of points on the second circle. These are the locations of the points that enable a circle to be divided into 2, 3, 4 and 6 equal parts, i.e. regular polygons inscribed within the circle. The missing pentagon corresponds to a circle with radius between the $\sqrt{2}$ and $\sqrt{1}$ circles. Its radius is $\sqrt{\left(5-\sqrt{5}\right)/2}$. This circle divides the area of the radial segment between $\sqrt{2}$ and $\sqrt{1}$ in the golden proportion.

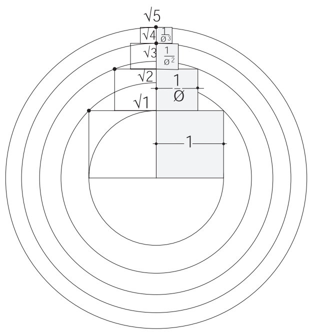


Fig. 30. Double squares touching the $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ and $\sqrt{5}$ circles with widths exhibiting a golden sequence.

7. The Golden Ratio as Human Scale

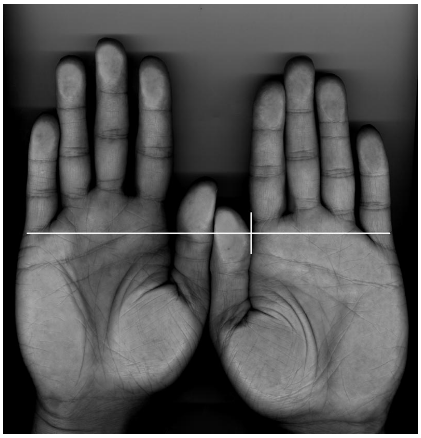


Fig. 31. A line between thumb and forefinger naturally divides the ten finger distance in the golden proportion.

8. Discovery

Despite the beauty of my previous findings, in Fig. 32 I step towards infinity, within the upward pointed triangle, in a different way. This leads me to the real breakthrough in my discoveries which come in the next image where I have found for the first time unknown geometrical relations between golden and silver means (see KAPPRAFF and ADAMSON in another article in this issue).

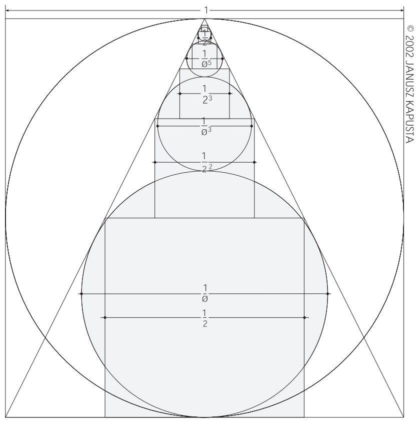
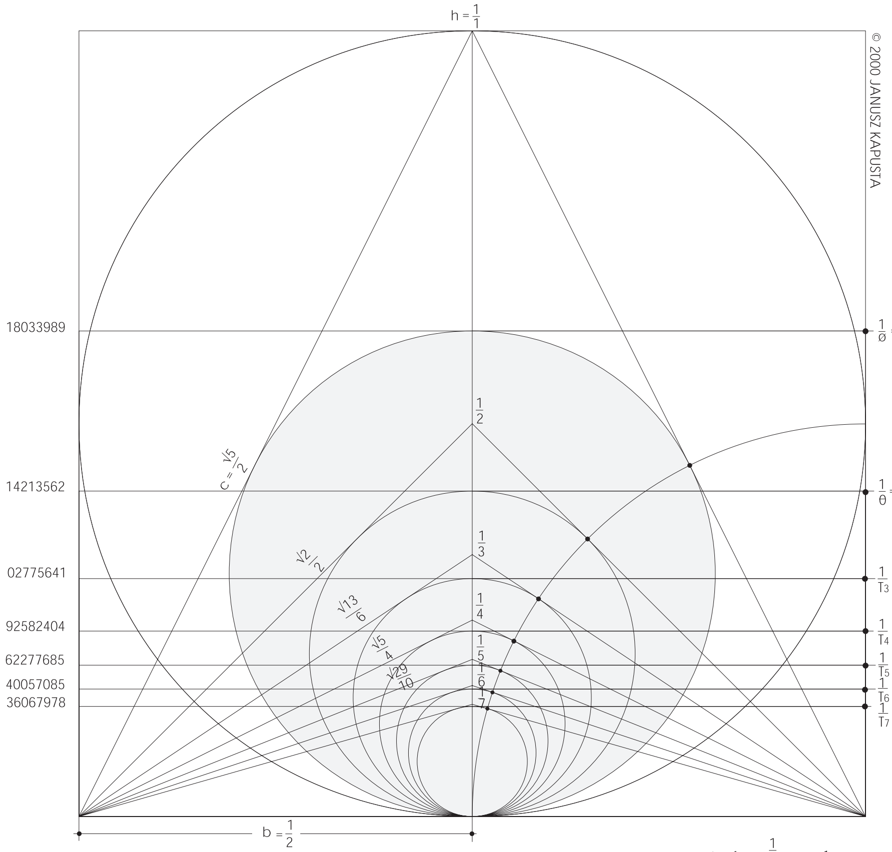


Fig. 32. Inscribe in the upward pointed triangle within the virtual square, circles with diameters equal to odd, inverse powers of the golden mean: $1/\phi$, $1/\phi^3$, $1/\phi^5$, ... The diameters of these circles sum to side of the unit square. Alternatively, squares with sides in the geometric sequence: $1/2$, $1/2^2$, $1/2^3$, ... touching the same upward triangle also sum to side of the unit square.



$$\frac{1}{\varnothing} = \frac{1}{T} = \frac{\frac{1}{1}}{\frac{\sqrt{5}}{2}+\frac{1}{2}} = \cfrac{1}{1+\cfrac{1}{1+\dots}} = 0.618033989$$

$$\frac{1}{\theta} = \frac{1}{T_2} = \frac{\frac{1}{2}}{\frac{\sqrt{2}}{2}+\frac{1}{2}} = \cfrac{1}{2+\cfrac{1}{2+\dots}} = 0.414213562$$

$$\frac{1}{T_3} = \frac{\frac{1}{3}}{\frac{\sqrt{13}}{6}+\frac{1}{2}} = \cfrac{1}{3+\cfrac{1}{3+\dots}} = 0.302775641$$

$$\frac{1}{T_4} = \frac{\frac{1}{4}}{\frac{\sqrt{5}}{4}+\frac{1}{2}} = \cfrac{1}{4+\cfrac{1}{4+\dots}} = 0.236067978$$

$$\frac{1}{T_5} = \frac{\frac{1}{5}}{\frac{\sqrt{29}}{10}+\frac{1}{2}} = \cfrac{1}{5+\cfrac{1}{5+\dots}} = 0.192582404$$

Fig. 33. Create a sequence of upward pointed triangles with base on a unit square and height equal to the harmonic sequence: 1/1, 1/2, 1/3, 1/4, 1/5, ... The diameters of the inscribed circles within these triangles are the sequence of odd, inverse silver means $1/T_N$ where $T_1 = \phi = 1 + \sqrt{5}/2$ and $T_2 = \theta = 1 + \sqrt{2}$. Just as for the golden mean in the previous figure (Fig. 32), the diameters of the sequence of circles with odd, inverse powers of T_N sum to the height of its triangle, 1/N (not shown). Notice that all tangent points of the silver mean circles to their respective upward triangles lie on the circumference of a unit circle. The diameters of the inscribed circles are determined by a simple theorem of geometry that I independently discovered. Theorem: For the inscribed circle of an isosceles triangle, $D = h/(c + b/2)$ where D is the diameter of the inscribed circle, c is the hypothenuse, b is the base, and h is the altitude. This theorem is applied to the circles inscribed within the harmonic triangles of Fig. 33. Note the elegant forms that they take when expanded as continued fractions:

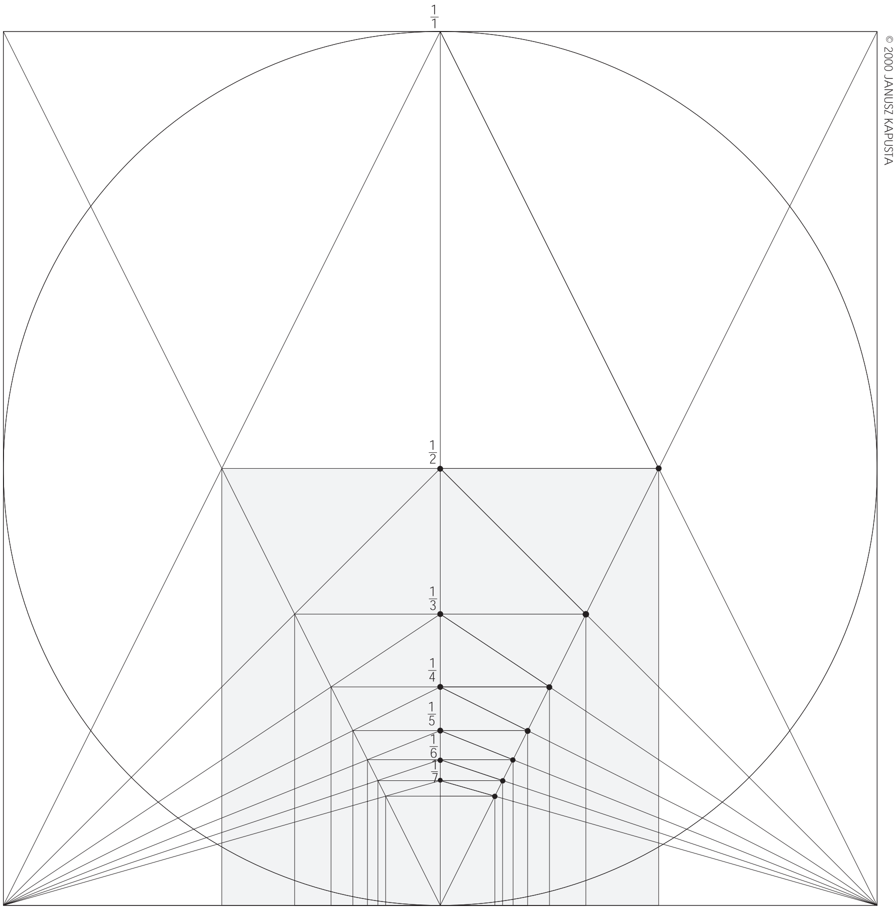


Fig. 34. Dual to the previous figure we have a downward pointed triangle within the virtual square. Where this downward triangle crosses the upward triangles, this is the basis for the projective construction of an harmonic sequence of squares, where the points of projection are the vertices on the base of the unit square. These are the initial squares of families of geometric sequences of squares analogous to ones the shown in Fig. 32 where as $1/2 + 1/2^2 + 1/2^3 + ... = 1$. Now we find that: $1/3 + 1/3^2 + 1/3^3 + ... = 1/2$, $1/4 + 1/4^2 + 1/4^3 + ... = 1/3$, etc. Combining three figures: 14, 32 and 34 we can write the surprising equation: $1/2^2 + 1/2^3 + 1/2^4 + ... + 1/3^2 + 1/3^3 + 1/3^4 + ... + 1/4^2 + 1/4^3 + 1/4^4 + ... = 1/\phi^2 + 1/\phi^3 + 1/\phi^4 + ...$

9. The Mysterious Triangle

(see next page)

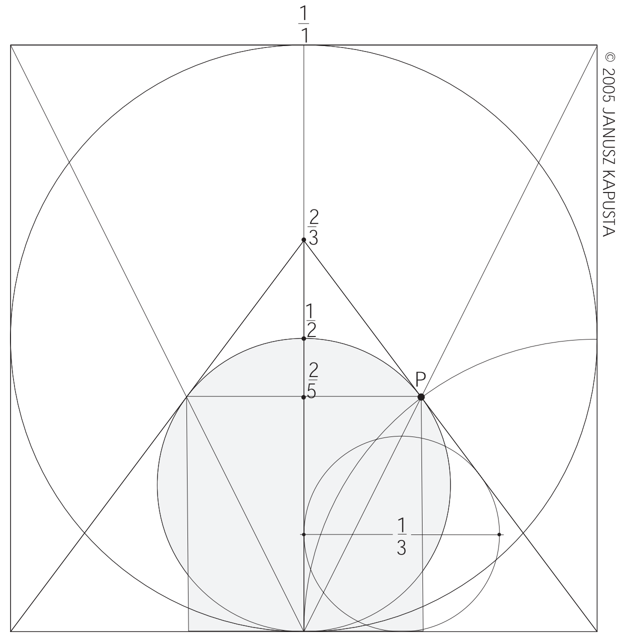


Fig. 35. In Fig. 33, all tangent points of the silver mean circles to their respective upward pointed triangles lie on the circumference of a unit circle. In Fig. 34, all squares touch the downward pointed triangle. Where the circumference of this unit circle and the downward pointed triangle intersect determines a point P, the only point at which the tangent circle and touching square meet. An upward pointed triangle, with height 2/3 and 2/5-square where: $2/5 + (2/5)^2 + (2/5)^3 + \ldots = 2/3$, is constructed from this intersection. The inscribed circle within the 2/3 triangle has a diameter, $D = 1/2$.

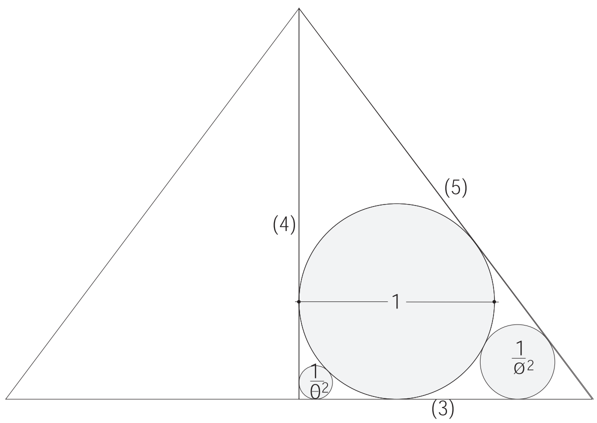


Fig. 36. The circle inscribed within half of this triangle (right-angled triangle) in Fig. 35 has a diameter, $D = 1/3$ (rescaled to 1). This right-angle triangle is the famous 3,4,5-Egyptian triangle. In this triangle two circles tangent to the unit circle and the base are inverse squares of the golden and silver proportions.

10. A Golden Rotation

If the unit square is rotated both clockwise and counterclockwise through the base angle of one of the upward pointed triangles, its points of intersection with the initial square divides the edge of this square in the inverse of the silver mean, $1/T_N$, associated with that triangle.

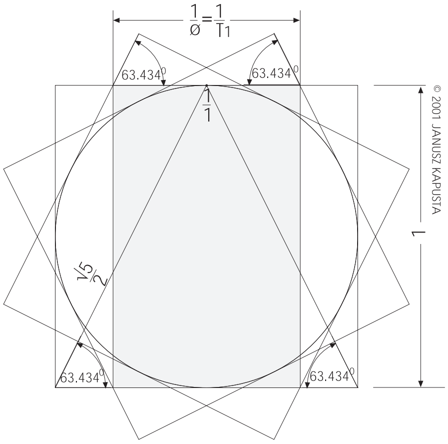


Fig. 37. A golden rectangle (T_1) as the rotation of two squares by angle 63.434°.

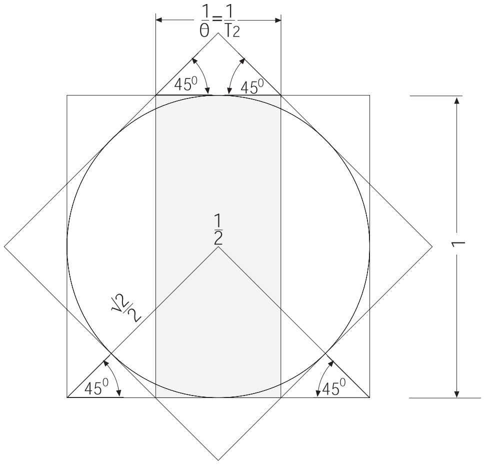


Fig. 38. A silver rectangle (T_2) as the rotation of two squares by angle 45°.

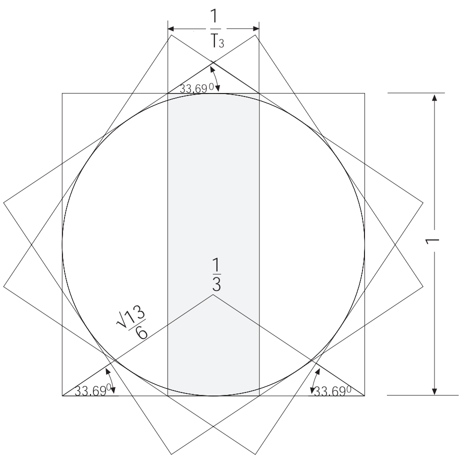


Fig. 39. A silver rectangle (T_3) as the rotation of two squares by angle 33.69°.

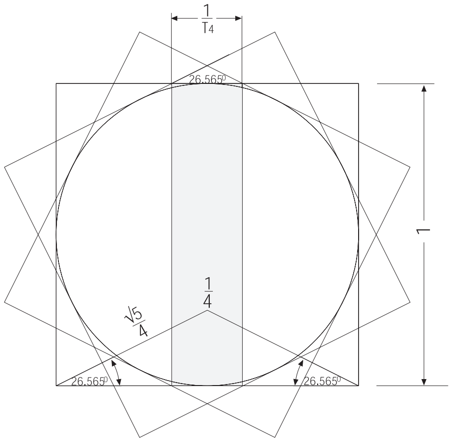


Fig. 40. A silver rectangle (T_4) as the rotation of two squares by angle 26.565°.

11. The Fibonacci Sequence and Its Generalizations

My knowledge of Fibonacci and Pell sequences served as the basis for the intuitions which led to my discovery of the geometrical relation between the golden and silver means in Fig. 33, on the one hand, and a set of general equations between them, on the other. The Fibonacci sequence has the property that each term is the sum of the two preceding terms: $a_n = a_{n-2} + a_{n-1}$, where $a_0 = 0$, and $a_1 = 1$, and the ratios of successive terms approaches ϕ. The Pell sequence has the property that each term is the sum of twice the previous term and the term before that: $a_n = a_{n-2} + 2a_{n-1}$ where $a_0 = 0$ and $a_1 = 1$, and the ratio of successive terms approaches the silver mean, θ. Sequences approaching the k-th silver mean, T_k, have the analogous properties:

$$a_n = a_{n-2} + ka_{n-1} \text{ where } a_0 = 0,\ \ a_1 = 1 \text{ and } T_k = \lim_{n\to\infty} \frac{a_{n+2}}{a_n} = \frac{k + \sqrt{4+k^2}}{2}.$$

I refer to this sequence as the T_k - sequence. This general approach is surprisingly visible when unit squares are constructed representing numbers from the T_k - sequence as I demonstrate in Fig. 41 for k = 1, 2, and 3.

k = 1	$a_n = 1a_{n-2} + 1a_{n-1}$
k = 2	$a_n = 1a_{n-2} + 2a_{n-1}$
k = 3	$a_n = 1a_{n-2} + 3a_{n-1}$

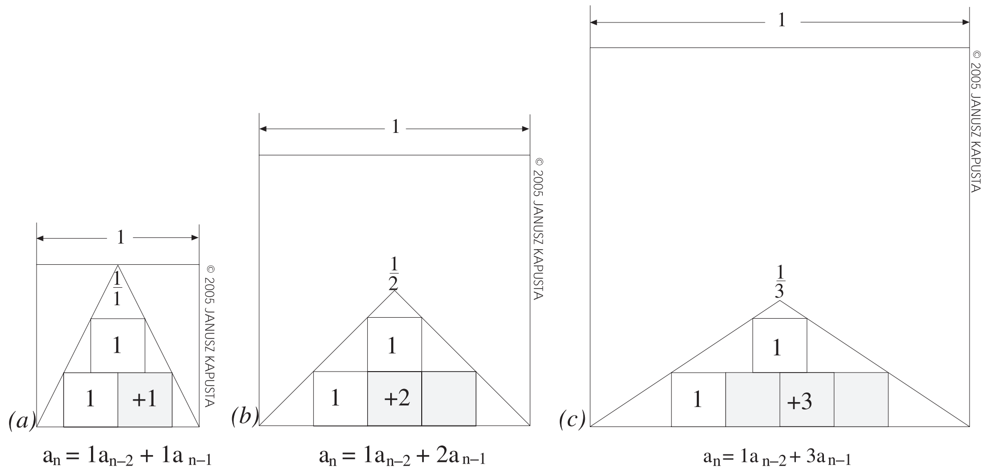


Fig. 41. A construction of upward pointed triangles built on a unit square showing one square in the upper row, and a number of squares equal to the number *k* of the T_k -sequence in the lower row, (two lines touch the corners of successive squares) generates the same sequence of upward pointed triangles as in Fig. 33 in which the diameters of the inscribed circles is the sequence of odd, inverse powers of the silver means. (a) with *k* = 1, the construction represents a Fibonacci sequence (T_1); (b) with *k* = 2, the construction represents a Pell sequence (T_2); (c) with *k* = 3, the construction represents a (T_3)- sequence.

12. Revisiting the Square and the Circle

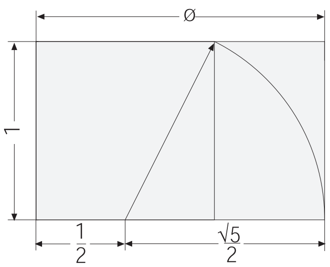


Fig. 42. This is the most often used construction of a golden rectangle. I use this construction as the starting point for a new world of ideas.

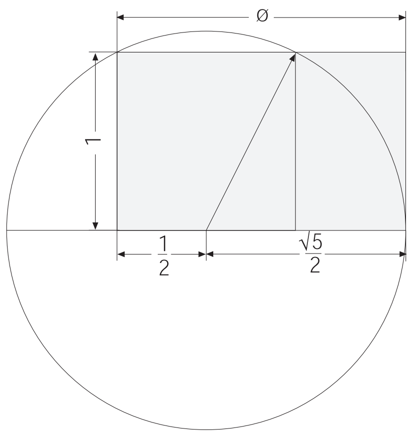


Fig. 43. I complete this construction to a full circle. Notice a single square inscribed within the half circle. What if I inscribe two squares?

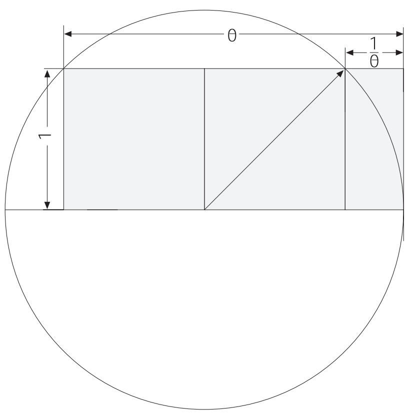


Fig. 44. The well-known silver mean, $\theta = 1 + \sqrt{2}$, emerges.

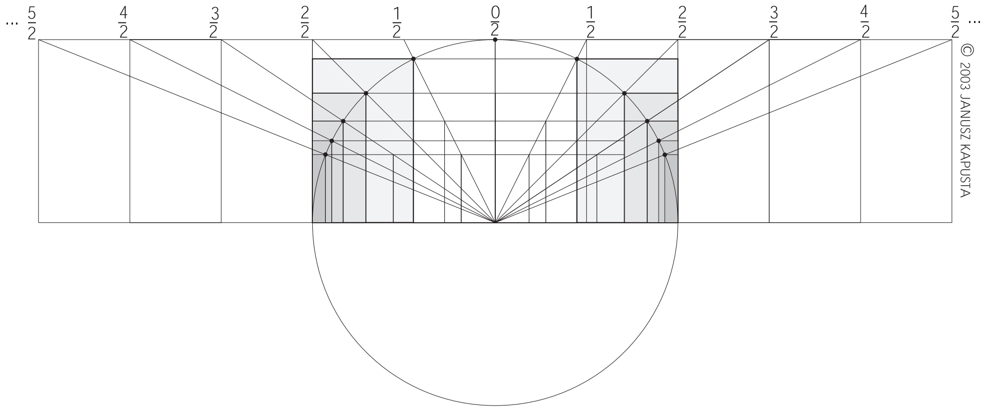


Fig. 45. Following in this manner, and based on a sequence of half-integers, I construct of the infinite family of silver means, which we denote by T_N (i.e. $\phi = T_1$ and $\Theta = T_2$), satisfying the equation x - 1/x = N shown in the next figure.

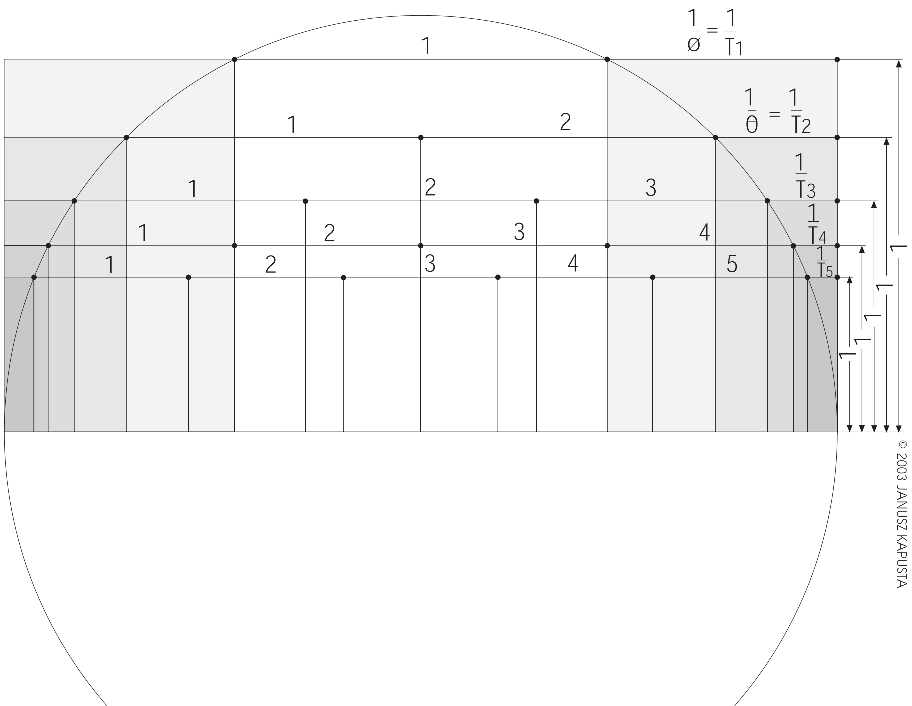


Fig. 46. I leave it to the reader to discover the essence of this construction through quiet contemplation.

13. Golden Art

The compositions, shown below, were made for Getulio Alviani, an Italian artist, art critic and art collector, in 2004. Each image is a study of the golden proportion.

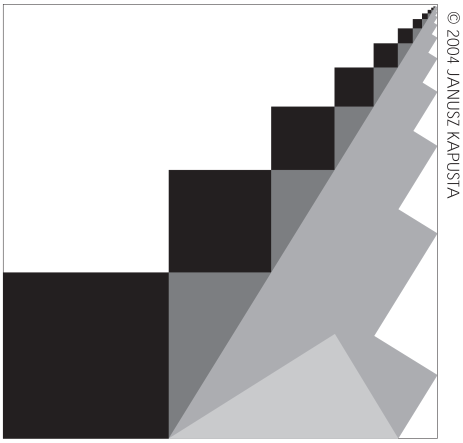


Fig. 47. "A Pythagorean Theorem". This artwork is based on a study presented in Fig. 20.

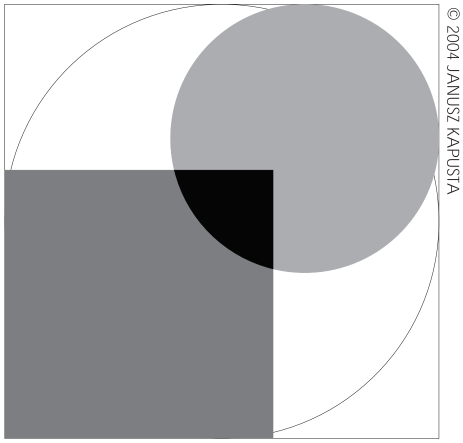


Fig. 48. "A Golden Joint". This artwork explains the idea of Fig. 1.

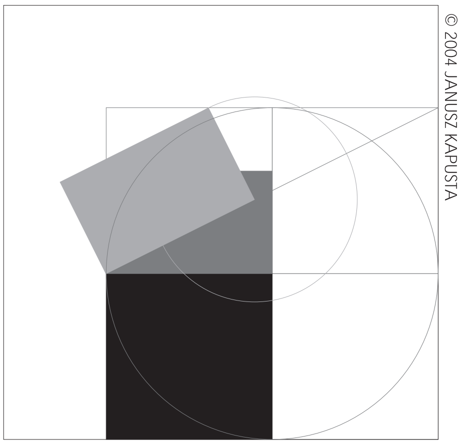


Fig. 49. "A Golden Rotation". Notice how the vertex of a rotated golden rectangle lies on the upper framing square and its side on the diagonal.

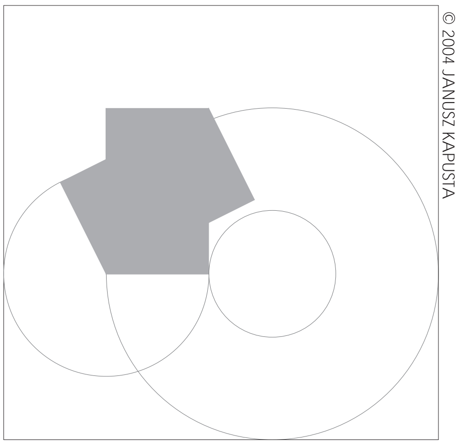


Fig. 50. "K-Dron Golden Circles" (see KAPUSTA, 2005).

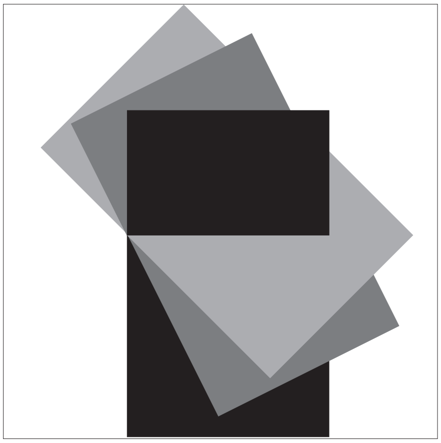


Fig. 51. "A Golden Penetration". The degree of greyness of these three golden rectangles is in golden proportion as well.

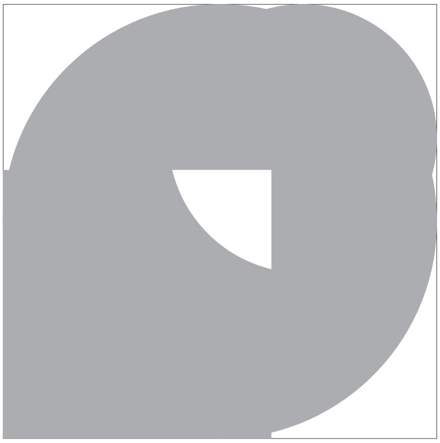


Fig. 52. "The White Form". Another variation of the work presented in Fig. 1.

14. Conclusions

My findings have shown that the golden ratio is unique and complex, and that when it is seen as the FIRST in an infinite sequence of proportions, it emerges as a fundamental relationship in the construction of the Universe. The golden proportion is no longer an isolated island of beauty. Because it is FIRST, and because of its intimate relation to the number One, the golden mean makes a significant imprint on cosmic behavior. And that is what has led myself, as it has all admirers of the golden mean, along our paths to discovery.

I would like to thank Professor Jay Kappraff who, over the years, has shown great interest in my search for beauty, and not only encouraged me to present part of these discoveries in this special issue of FORMA, but helped in its editing.

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