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Fibonacci Form and Beyond

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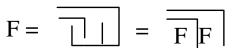
Abstract. This paper develops a context for the well-known Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, ...) in terms of self-referential forms and a basis for mathematics in terms of distinctions that is harmonious with G. Spencer-Brown's Laws of Form and Heinz von Foerster's notion of an eigenform. The paper begins with a new characterization of the infinite decomposition of a rectangle into squares that is characteristic of the golden rectangle. The paper discusses key reentry forms that include the Fibonacci form, and the paper ends with a discussion of the structure of the "Fibonacci anyons" a bit of mathematical physics that relates to the quantum theory of the self-interaction of the marked state of a distinction.

1. Introduction

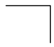
To the extent that this paper is just about the Fibonacci sequence it is about the Fibonacci Form



whose notation we shall explain in due course. The *Fibonacci form* is an infinite abstract form that reenters its own *indicational space*, combined with a shift that produces a Fibonacci number of divisions of the interior space at every depth. This is the infinite description of the well-known recursion that produces the *Fibonacci sequence*.

This paper will provide the background for this point of view. We begin by recalling the relationship of the Fibonacci sequence to the *golden mean* and the *golden rectangle* and we prove a theorem in Sec. 2 showing the uniqueness of the infinite spiraling decomposition of the golden rectangle into distinct squares.

The proof of the theorem shows Fibonacci numbers and the golden mean arise naturally from this classical geometric context. Section 3 provides background about *Laws of Form* and the formalism in which the mark



is seen to represent a *distinction*. With these concepts in place, we are prepared to discuss *infinite reentering forms*, including the Fibonacci form in Sec. 4. Section 5 continues the discussion of infinite forms and *eigenforms* in the sense of VON FOERSTER (1981).

Finally, Sec. 6 returns to the Fibonacci sequence via the patterns of self-interaction of the mark. Here the mark is conceived of as an "elementary particle" that can interact with itself either to produce itself or to annihilate itself. There exist a Fibonacci number of patterns of interaction of a collection of n marks, leading to the unmarked state. This Fibonacci property of the self-interactions of the mark is a link between Laws of Form, topology, and quantum information theory. We give only the beginning of this relationship and refer the reader to the literature for more detailed information.

2. Geometry of the Golden Ratio

In Fig. 1 we have a rectangle of length 377 units and width 233 units. It is paved with squares of sizes,

$$233 \times 233,\ 144 \times 144,\ 89 \times 89,\ 55 \times 55,\ 34 \times 34,\ 21 \times 21,$$
$$13 \times 13,\ 8 \times 8,\ 5 \times 5,\ 3 \times 3,\ 2 \times 2,\ 1 \times 1,\ 1 \times 1.$$

This attests to the fact that for the Fibonacci numbers,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377,$$

the sum of the squares of the Fibonacci numbers up to a given Fibonacci number is equal to the product of that Fibonacci number with its successor. For example,

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 8 \times 13.$$

Letting,

$$f(0) = 1$$
$$f(1) = 1$$
$$f(2) = 2$$
$$f(3) = 3$$
$$f(4) = 5$$

where,

$$f(n + 1) = f(n) + f(n - 1),$$

it follows that a rectangle of size $f(n) \times f(n + 1)$ can be paved with a spiraling pattern of squares of different sizes, ending with a repetition of the 1×1 square. (The fascinating problem of paving a rectangle with squares of all unequal sizes has been studied by BROOKS *et al*. (1940, 1961).

It is well-known that the process of cutting off squares can be continued to infinity if we start with a rectangle that is of the size $\phi \times 1$ where ϕ is the *golden mean* $\phi = (1+\sqrt{5})/2$.

This is not surprising. Such a process will work when the new rectangle is similar to the original one, i.e.,

$$W/(L - W) = L/W.$$

Taking $W = 1$, we find that $1/(L - 1) = L$, whence $L^2 - L - 1 = 0$, whose positive root is the golden mean.

It is also well-known that the golden ratio is the limit of successive ratios of Fibonacci numbers with $1 < 3/2 < 8/5 < 21/13 < \ldots < \phi < \ldots < 13/8 < 5/3 < 2$.

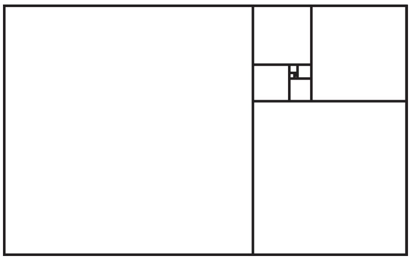


Fig. 1. The Fibonacci rectangles.

We ask:

Is there any other proportion for a rectangle, other than the Golden Proportion, that will allow the process of cutting off successive squares to produce an infinite paving of the original rectangle by squares of different sizes? The answer is: No!

Theorem.

The only proportion that allows the pattern of cutting off successive squares to produce an infinite paving of the original rectangle by squares of different sizes is the golden ratio.

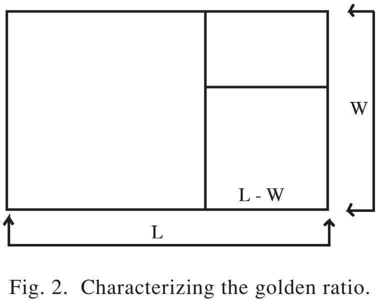


Fig. 2. Characterizing the golden ratio.

Proof. (View Fig. 2)

Suppose the original rectangle has width W and length L. In order for the process of removing a square from the rectangle to produce a new rectangle whose own excised square is smaller than the first square, we need the new rectangle to have width $L - W$ and length W with $L - W < W$. With this inequality, the square removed from the new rectangle will be smaller than the first square cut off from the original rectangle.

In order for this pattern of cutting off squares to be continued, we need an infinite sequence of inequalities, each derived from the previous one. That is, we start with initial length L and width W. We excise a square of size $W \times W$ obtaining a new length $L' = W$ and a new width $W' = L - W$. We require that $W' < L'$ ad infinitum:

$$\begin{array}{l} W < L \\ L - W < W \\ W - (L - W) < L - W \\ L - W - (W - (L - W)) < W - (L - W) \\ \ldots \end{array}$$

Using algebra it follows that,

$$\begin{array}{l} W < L \\ L - W < W \\ 2W - L < L - W \\ 2L - 3W < 2W - L \\ 5W - 3L < 2L - 3W \\ 5L - 8W < 5W - 3L \\ 13W - 8L < 5L - 8W \\ \ldots \end{array}$$

The Fibonacci pattern is now apparent from these inequalities:

$$\begin{array}{l} W < L \\ L < 2W \\ 3W < 2L \\ 3L < 5W \\ 8W < 5L \\ 8L < 13W \\ 21W < 13L \\ \ldots \end{array}$$

or

$$\begin{array}{l} : \\ L/W > 1 \\ L/W < 2 \\ L/W > 3/2 \end{array}$$

$L/W < 5/3$
$L/W > 8/5$
$L/W < 13/8$
$L/W > 21/13$
...

Thus we see from this pattern that L/W is sandwiched between ratios of Fibonacci numbers in the same way as the golden mean,

$$1 < 3/2 < 8/5 < 21/13 < \ldots < L/W < \ldots < 13/8 < 5/3 < 2.$$

This implies that $L/W = \phi = (1+\sqrt{5})/2$. Thus the rectangle is *golden*. QED

The Theorem can be looked at from another point of view. As above, let $L' = W$ and $W' = L - W$ with the assumption that $W < L$ and $L - W < W$. Then,

$$L/W = (W + (L - W))/W = 1 + (L - W)/W = 1 + 1/(W/(L - W)).$$

Thus,

$$L/W = 1 + 1/(L/W)'.$$

Where,

$$(L/W)' = L'/W'.$$

If we define $P = P_0 = L/W$ and $P_{n+1} = ((L/W)_n)'$ then,

$$P = 1 + 1/P_1 = 1 + 1/(1 + 1/P_2)$$
$$= 1 + 1/(1 + 1/(1 + 1/P_3)) = \ldots$$

If the sequence of P_n has a limiting value, then

$$L/W = P = 1 + 1/(1 + 1/(1 + 1/\ldots))$$

and we have proved that $(L/W)' = L/W$. In this way we recapture the well-known formula for the golden mean as a continued fraction (see KAPPRAFF and ADAMSON in this issue, KAPPRAFF in this issue and KAPUSTA in this issue).

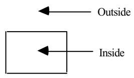
$$= 1 + 1/(1 + 1/(1 + 1/\ldots)).$$

Note again that we did not start with the assumption that $L/W = L'/W'$. We proved that the golden proportion follows from the assumption that one can continue the dissection of a rectangle into squares ad infinitum.

3. Laws of Form

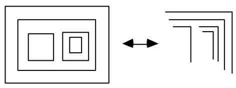
Laws of Form by George Spencer-Brown (SPENCER-BROWN, 1969) is a lucid book with a topological notation based on a single symbol, the mark:

This symbol represents a distinction between its inside and its outside:

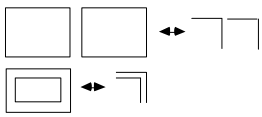


As is evident from the figure above, the mark is regarded as a shorthand for a rectangle drawn in the plane and dividing the plane into the regions inside and outside the rectangle.

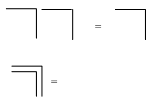
In this notation the idea of a distinction is instantiated in the distinction that the mark makes in the plane. Patterns of non-intersecting marks (that is non-intersecting rectangles) are called *expressions*. For example,



In this example, I have illustrated both the rectangle and the marked version of the expression. In an expression you can say definitively of any two marks whether one is or is not inside the other. The relationship between two marks is either that one is inside the other, or that neither is inside the other. These two conditions correspond to the two elementary expressions shown below.



The mathematics in Laws of Form begins with two laws of transformation pertaining to these two basic expressions. Symbolically, these laws are:



In the first of these equations (*the law of calling*) two adjacent marks condense to a single mark, or a single mark expands to form two adjacent marks. In the second equation (*the law of crossing*) two marks, one inside the other, disappear to form the unmarked state indicated by *nothing at all*. Alternatively, the unmarked state can give birth to two nested marks. A calculus is born of these equations, and the mathematics can begin.

Before introducing this notation, SPENCER-BROWN (1969) begins his book with a discussion of the concept of a *distinction*.

> "We take as given the idea of a distinction and the idea of an indication, and that it is not possible to make an indication without drawing a distinction. We take therefore the form of distinction for the form."

From here he elucidates two laws:

1. The value of a call made again is the value of the call.
2. The value of a crossing made again is not the value of the crossing.

The two symbolic equations above correspond to these laws. The way in which they correspond is worth discussing. First look at the law of calling. It says that the value of a repeated name is the value of the name. In the equation,

$$\urcorner\,\urcorner = \urcorner$$

one can view either mark as the name of the state indicated by the outside of the other mark. In the other equation

$$\overline{\urcorner\,}\big| = $$

the state indicated by the outside of a mark is the state obtained by crossing from the state indicated on the inside of the mark. Since the marked state is indicated on the inside, the outside must indicate the unmarked state. The *Law of Crossing* indicates how opposite forms can fit into one another and vanish into the Void, or how the Void can produce opposite and distinct forms that fit one another, hand in glove.

The same interpretation yields the equation

$$\urcorner = \urcorner$$

where the left-hand side is seen as an instruction to cross from the unmarked state, and the right hand side is seen as an indicator of the marked state. The mark has a double meaning: it can be seen as an operator, transforming the state on its inside to a different state on its outside; and it can be seen as the name of the marked state. That combination of meanings is compatible with this interpretation.

From indications and their calculus, one moves to algebra where it is understood that a variable can indicate either the presence or absence of a mark. Thus,

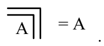
$$\overline{\overline{A\,}\big|\,}\big|$$

stands for the two possibilities

$$\overline{\overline{\urcorner\,}\big|\,}\big| = \urcorner \quad , A = \urcorner$$

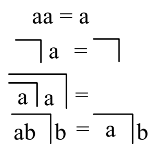
$$\overline{\overline{\ \ }\big|\,}\big| = \qquad , A = $$

In all cases of A we have

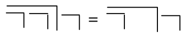


Thus begins the algebra of this non-numerical arithmetic of forms. The primary algebra that emerges is a subtle precursor to Boolean algebra.

Other examples of algebraic rules are the following:



Each of these rules is easy to understand from the point of view of the arithmetic of the mark. Just ask what you will get if you substitute values of a and b into the equation. For example, in the last equation, if a and b are both marked, then the equation becomes



which is certainly true, by the law of calling.

Once we have an algebra we can solve equations. Spencer-Brown pointed out that one should consider equations of degree greater than one in the primary algebra just as one does in elementary algebra. *Such equations can involve self-reference*. First consider ordinary algebra. The expression,

$$x^2 = ax + b$$

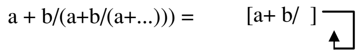
is a quadratic equation with solutions that are sometimes complex numbers of the form $R + Si$ where $i^2 = -1$. One can re-write this equation as,

$$x = a + b/x.$$

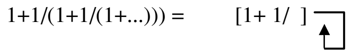
In this form it is indeed self-referential, with x re-entering the expression on the right. We could "solve" it by an infinite reentry or infinite continued fraction:

$$x = a + b/(a + b/(a + b/(a + b/(a + \cdots))))).$$

In this infinite formalism it is literally the case that $x = a + b/x$ and we can write



to indicate how this form reenters its own *indicational space*. This formal solution to the quadratic equation converges to a real solution when the quadratic equation has real roots. For example, if $a = 1 = b$, then



converges to the positive solution of $x^2 = x + 1$, which is the golden mean, $\phi = (1+\sqrt{5})/2$.

On the other hand, the quadratic equation may have imaginary roots. (This happens when $a^2 + 4b$ is negative.) Under these circumstances, the formal solution does not represent a real number. For example, if i denotes the square root of minus one, then we can write

i = -1/(-1/(-1/...)) = [-1/] 

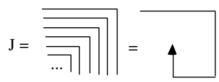
to denote a formal number with the property that

$$i = -1/i.$$

Spencer-Brown makes the point that one can follow the analogy of introducing imaginary numbers in ordinary algebra by introducing *imaginary Boolean values* in the arithmetic of logic. An apparently paradoxical equation such as

$$J = \overline{J\;}\rceil$$

can be regarded as an analog of the quadratic $x = -1/x$, and its solutions will be values that are other that marked or unmarked, other than true or false.

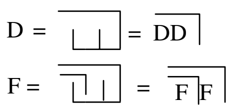


In other words, *J* can be represented by an infinite form that reenters its own indicational space.

4. Infinite Recursive Forms

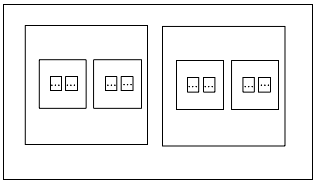
Constructions involving the mark, suggest that all possible expressions be considered, including infinite expressions, with no arithmetic initials other than commutativity of juxtapositions. We shall call such expressions *forms*. We shall discuss some of the phenomenology of infinite forms that are described by reentry.

The simplest example of such a form is the *reentering mark* J, described above. The next two examples by order of simplicity are,



I call *D* the *doubling form*, and F the *Fibonacci form*.

A look at the recursive approximations to *D* shows immediately why we have called it the doubling form (approximations are done in box form):

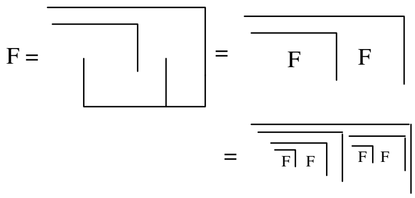
D = 

We see from looking at the approximations, that the number of divisions of D doubles at each successive depth beyond depth zero. Letting D_n denote the number of divisions of D at depth n, we see that $D_n = 2^{n-1}$.

Given any forms G and H, we define G_n to be the number of divisions of G at depth n. We have the basic formulas:

$$\overline{G}\rceil_{n+1} = G_n$$
$$(GH)_n = G_n + H_n$$

In the case of the Fibonacci form, we have



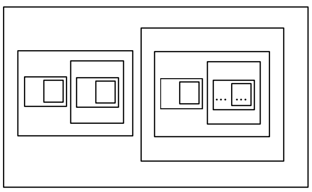
Thus

$$F_{n+1} = \overline{\overline{F}\rceil \; F}\rceil_{n+1}$$
$$= \overline{F}\rceil_n + F_n$$
$$F_{n+1} = F_{n-1} + F_n$$

For the Fibonacci form, $F_{n+1} = F_n + F_{n-1}$ with $F_0 = F_1 = 1$. The depth counts in this form are the Fibonacci numbers,

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

with each integer the sum of the preceding two integers.



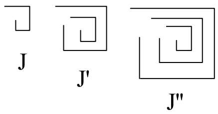
The Fibonacci Form

It is natural to define the *growth rate* $m(G)$ of a form G to be limit of the ratios of successive depth counts as the depth goes to infinity.

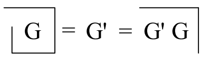
$$m(G) = \lim_{n\to\infty} G_{n+1} / G_n .$$

Then we have $m(D) = 2$, and $m(F) = (1+\sqrt{5})/2$, the golden mean.

Finally, here is a natural hierarchy of recursive forms, obtained each from the previous by enfolding one more reentry.



Given any form G, we define G' by the formula shown below, so that



$$G'_{n+1} = G'_n + G_{n-1}$$

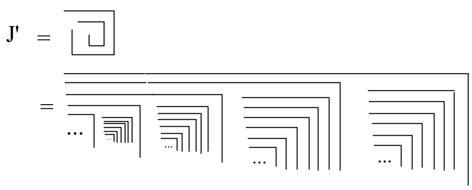
This implies that

$$G'_{n+1} - G'_n = G_{n-1}.$$

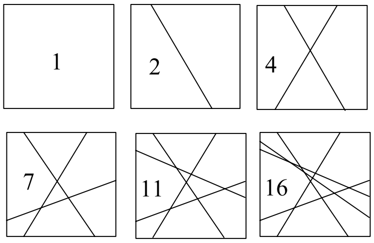
Thus the discrete difference of the depth series for G′ is (with a shift) the depth series for G. The sequence J, J', J'', J''', ... is particularly interesting because:

The depth sequence $(J^{(n)})_k$ is equal to the maximal number of divisions of n-dimensional Euclidean space by $k-1$ hyperspaces of dimension $n-1$.

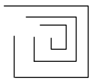
We will not prove this result here, but note that J takes the role of a point (dimension zero) with $J_k = 1$ for all k, while J' satisfies $J'_{k+1} = J'_k + 1$ $(k > 0)$, so that $J'_k = k - 1$ for $k > 1$. This is the formula for the number of divisions of a line by $k - 1$ points.



To think about the divisions of hyperspace, consider an arbitrary collection of mutually non-parallel lines, no two of which intersect at a point. If a new line is added, it will cut several existing regions into two regions. The number of new regions is equal to the number of divisions made in the new line. This is a verbal description of the basic recursion given above.



1 + 1 = 2
2 + 2 = 4
4 + 3 = 7
7 + 4 = 11
11 + 5 = 16
...

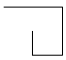


The very simplest recursive forms yield a rich complexity of behaviors that lead directly to the mathematics of imaginary numbers, oscillations, patterns of growth, dimensions, and geometry.

There is an eternity and a spirit at the center of each complex form. That eternity may be an idealization, a “fill-in”, but it is nevertheless real. In the end it is that eternity, that eigenform unfolding the present moment that is all that we have. We know each other through our idealizations of the other. We know ourselves through our idealization of ourselves. We become what we were from the beginning, a Sign of Itself.

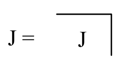
5. Eigenforms

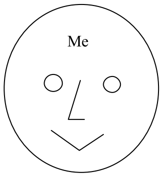
Consider the reentering mark.



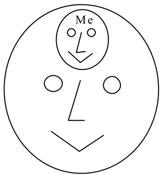
This is an archetypal example of an *eigenform* in the sense of VON FOERSTER (1981). An eigenform is a solution to an equation, a solution that occurs at the level of form, not at the level of number. We live in a world of eigenforms. You thought those forms that you see are actually “out there”? Out where? The very space, the context that you regard as your external world is an eigenform. It is your organism’s solution to the problem of distinguishing itself in a world of actions.

The shifting boundary of Myself/MyWorld is the dynamics of the form that “you” are. The reentering mark is the solution to the equation

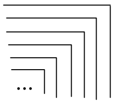


where the right-angle bracket distinguishes a space in the plane. This is not a numerical equation. One does not even need to know any particularities about the behavior of the mark to satisfy this equation. It is akin to solving, Me = 

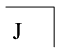
by attempting to create a space where “I” can be both myself and inside myself, as is true of our psychological locus. And this can be solved by an infinite regress of Me’s inside of Me’s.

Me = 

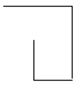
In a similar manner, we may solve the equation for J by an infinite nest of boxes

J = 

Note that in this form of the solution, layered like an onion, the entire infinite form reenters its own indicational space. It is indeed a solution to the equation

J = 

The solution in the form

J = 

is meant to indicate how the form reenters its own indicational space. This reentry notation is due to G. Spencer-Brown. Although he did not write down the reentering mark itself in his book *Laws of Form*, it is implicit in the discussion in chapter 11 of that book.

It is not obvious that we should take infinite regress as a model for the way we are in the world. Everyone has experienced being between two reflecting mirrors and the veritable infinite regress that arises at once in that situation. Physical processes can happen more rapidly than the speed of our discursive thought, and thereby provide ground for an excursion to infinity.

Here is one more example. This is the eigenform of the Koch fractal as shown in KAUFFMAN (1987). In this case one can write the eigenform equation

$$K = K\{KK\}K.$$

The curly brackets in the center of this equation refer to the fact that the two middle copies within the fractal are inclined with respect to one another and with respect to the two outer copies. In the figure below we show the geometric configuration of the reentry.

The Koch fractal reenters its own indicational space four times (that is, it is made up of four copies of itself, each one-third the size of the original. We say that the Koch fractal has *replication rate* four and write $R(K) = 4$. We say it has *length ratio* three and write $F(K) = 3$.

In describing the fractal recursively, one starts with a segment of a given length L. This is replaced by $R(K)$ segments each of length $L' = L/F(K)$. In the equation above we see that $R(K) = 4$ is the number of reentries, and $F(K)$ is the number of groupings in the reentry form.

It is worth mentioning that the fractal dimension D of a fractal such as the Koch curve is given by the formula,

$$D = \ln(R)/\ln(F)$$

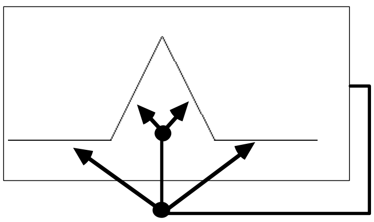
where R is the replication rate of the curve , F is the length ratio and $\ln(x)$ is the natural logarithm of x.

In the case of the Koch curve one has $D = \ln(4)/\ln(3)$. The fractal dimension measures the fuzziness of the limit curve. For curves in the plane, this can vary between 1 and 2, with curves of dimension, two having space-filling properties.

It is worth noting that we have, for the case of an abstract, grouped reentry form such as K = K{KK}K, a corresponding abstract notion of fractal dimension, as described above

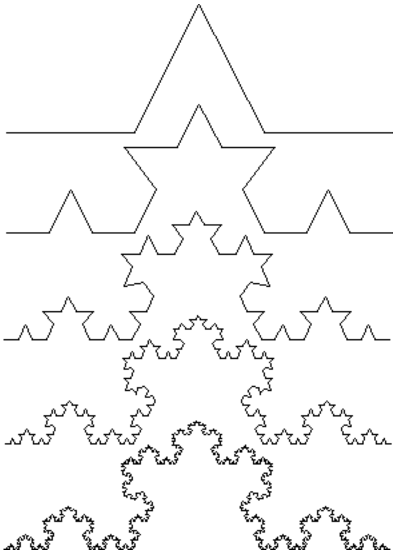
$$D(K) = \ln(\text{Number of Reentries})/\ln(\text{Number of Groupings}).$$

This example shows that the abstract notion of dimension interfaces with the actual geometric fractal dimension in the case of appropriate geometric realizations of the form. There is more to investigate in this interface between reentry form and fractal form.



K = K { K K } K

In the geometric recursion, each line segment at a given stage is replaced by four line segments of one third its length, arranged according to the pattern of reentry as shown in the figure above. The recursion corresponding to the Koch eigenform is illustrated in the next figure. Here we see the sequence of approximations leading to the infinite self-reflecting eigenform that is known as the Koch snowflake fractal.



Five stages of recursion are shown. To the eye, the last stage vividly illustrates how the ideal fractal form contains four copies of itself, each one-third the size of the whole. The abstract schema

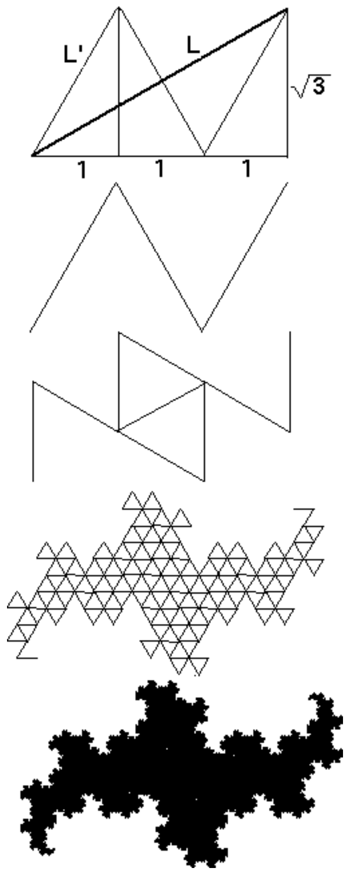
$$K = K\{KK\}K$$

for this fractal can itself be iterated to produce a "skeleton" of the geometric recursion:

$$\begin{aligned} K &= K\{KK\}K \\ &= K\{KK\}K\{K\{KK\}KK\{KK\}K\}K\{KK\}K \\ &= \ldots \end{aligned}$$

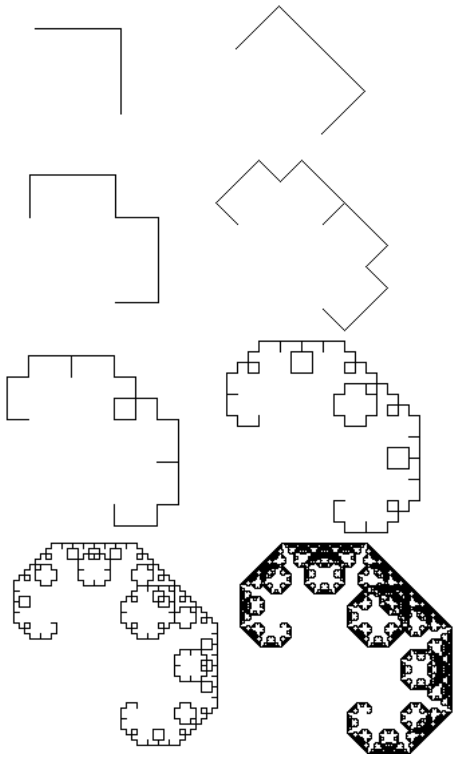
We have only performed one line of this skeletal recursion. There are sixteen K's in this second expression just as there are sixteen line segments in the second stage of the geometric recursion. Comparison with this abstract symbolic recursion shows how geometry aids intuition.

Geometry is much deeper and more surprising than the skeletal forms (see KAPUSTA in this issue). The next example illustrates this very well. Here we have the initial length L being replaced by three copies of L' with L/L' equal to $\sqrt{3}$ (to see that $L/L' = \sqrt{3}$, refer to the illustration below and note that $L' = \sqrt{1+3} = 2$, while $L = \sqrt{9+3} = 2\sqrt{3}$). Thus this fractal curve has dimension $D = \ln(3)/\ln(\sqrt{3}) = 2$. In fact, it is strikingly clear from the illustration that the curve is space-filling. It tiles its interior space with triangles and has another fractal curve as the boundary limit.



The relationship between eigenforms and the geometry of physical, mental, symbolic, and spiritual landscapes is a subject that is in need of deep exploration. Compare with KAUFFMAN (2003).

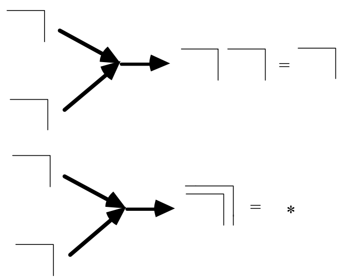
As a final fractal example for this section, consider the beautiful specimen SB generated by the Spencer-Brown mark. That is, the generator for this fractal is a ninety degree bend. Each segment is replaced by two segments at ninety degrees to one another, and the ratio of old segment to new segment is $\sqrt{2}$. Thus we have $D(SB) = \ln(2)/\ln(\sqrt{2}) = 2$, another space-filler. Notice how in the end, we have an infinite form that is a superposition of two smaller copies of itself at ninety degrees to one another.



It is generally thought that the miracle of being able to recognize an object arises in some simple way from the assumed existence of the object and the action of our perceiving systems. What must be understood is that this act of cognition is a fine tuning to the point at which the action of the perceiver, and the perception of the object are indistinguishable. Such tuning requires an intermixing of the perceiver and the perceived that goes beyond description. Yet in terms of mathematical entities such as number or fractal pattern, part of the process is slowed down to the point where we can begin to apprehend it. There is a stability in the comparison, in the one-to-one correspondence that is a process happening at once in the present time. The closed loop of perception occurs in the eternity of present individual time. Each such process depends upon linked and ongoing eigenbehaviors and yet is seen as simple by the perceiving mind.

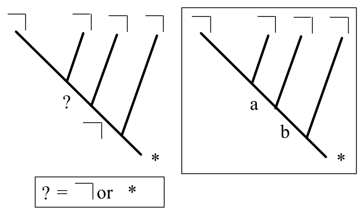
6. Fibonacci Particles

Think of the Spencer-Brown mark as an “elementary particle” that has two modes of interaction. Two marks can interact to produce either one mark or nothing.



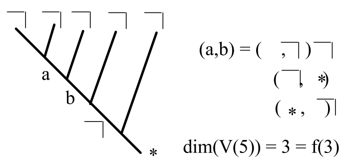
Here we have indicated the interactions, with * denoting the unmarked state. An entity whose only interactions are to produce itself or annihilate itself is surely the simplest idea of an elementary particle! The mark embodies this idea by the choice through which two marks interact either by calling or crossing. The choice at the level of distinctions is the question as to whether one distinction is inside or outside of the other.

By studying the structure of this single-particle theory, Laws of Form and quantum theory can be brought together. The purpose of including this structure in the present paper is because of its relationship to the Fibonacci numbers. We look at the possible successive interactions of this particle. Consider the diagram below.



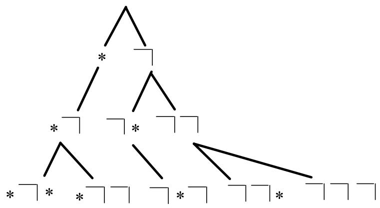
In this diagram we have illustrated four initial particles that are to interact in the pattern shown above. That is, the left two particles interact to produce the question mark (which can be either marked or unmarked). Then the question mark interacts with a mark to produce a mark, and the mark interacts with the fourth mark across the top to produce an unmarked state, shown at the bottom of the diagram. If we want an unmarked state to appear at the bottom of the diagram then the last interaction must be between two marks, since an unmarked state interacting with a marked state can produce only a marked state. The question mark can be either marked or unmarked to accomplish the overall pattern. In the box next to this process diagram we have indicated the form that was filled in. In the form we have unknown states a and b that can be either marked or unmarked. We see that the possibilities for a and b are $(a, b) = (U, M)$ and $(a, b) = (M, M)$ where U stands for the unmarked state, and M stands for the marked state. With two solutions, we say that this space of processes is two-dimensional.

Now consider the processes that will solve the analogous problem with five initial marks.



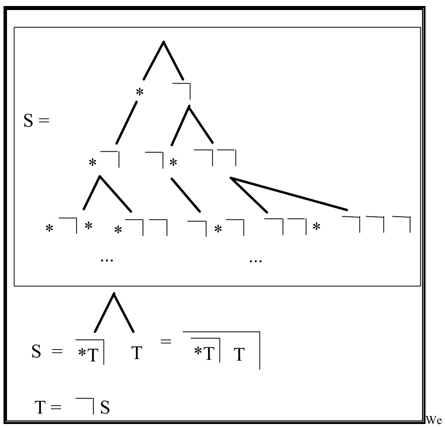
Now we get sequences of the form (a, b, M) and we see that the solutions (a, b) are: (M, M), (M, U), (U, M). Solutions of the form, $(a, b) = (U, U)$ are not valid. We cannot have two consecutive unmarked states in this game, since any given unmarked state will interact with one of the initial marks to produce a marked state. Thus we see that a general process space with $n + 2$ initial marks, ending with the unmarked state, has solutions of the form $(a_1, a_2, ..., a_n, M)$ where a = $(a_1, a_2, ..., a_n)$ is an arbitrary sequence of marked and unmarked states with no two consecutive terms unmarked.

In this way we are led to consider sequences of marked and unmarked states such that no two consecutive elements of the sequence are unmarked. It is easy to see that the number of such sequences of length n is the $n + 1$-th Fibonacci number $f(n + 1)$ where $f(0) = 1, f(1) = 1, f(2) = 2$, etc.

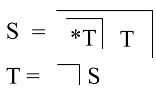


As a result, the dimension of the space $V(n + 2)$ of interactions of $n + 2$ elementary marked particles to produce a single unmarked particle is the Fibonacci number $f(n)$.

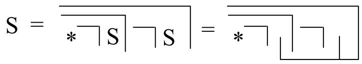
Note that the infinite tree indicated above is divided into two infinite trees (the one below * and the one below the mark) with the left tree obtained from the right tree by shifting it down one level and placing a star on the left of every sequence in the right tree. We shall call the right tree, T, and the whole tree S. Then the right tree is obtained from the whole tree by putting a mark to the left of each sequence on the whole. Thus we have the situation diagrammed below. In writing equations we have used a mark *over* a symbol to denote the down-shifting of the corresponding tree structure.



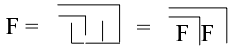
We can combine these equations to obtain



As a result,



The tree and re-entry form S produces the so-called *rabbit sequence* characteristic of symbolic of dynamic systems at the brink of chaos as described in appendix *D* by KAPPRAFF in this issue. With this reentry form for the Fibonacci sequences, we return again to the pattern of the Fibonacci Form.



This natural appearance of the Fibonacci Numbers and the Fibonacci Form, as a result of the self-interactions of the mark, has far-reaching consequences. It turns out that there is an intimate relationship between the properties of this model and certain unitary representations of the braid group, and that these representations can be used to generate a robust set of unitary matrices. These matrices can, in turn be configured as a topological quantum field theory, and this field theory can model the operations of a quantum computer. See KAUFFMAN (1991, 2005) and PRESKILL for more details of this story of the *Fibonacci Anyons*. So we see that the Fibonacci Sequence gets around.

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