Original Paper *Forma*, **19**, 367–387, 2004

Golden Fields, Generalized Fibonacci Sequences, and Chaotic Matrices

Jay KAPPRAFF[1]*, Slavik JABLAN[2], Gary W. ADAMSON[3] and Radmila SAZDANOVICH[2]

[1]*New Jersey Institute of Technology, Newark, NJ 07102, U.S.A.*
[2]*The Mathematical Institute, Knez Mihailova 35, P.O. Box 367, 11001, Belgrade, Serbia and Montenegro*
[3]*P.O. Box 124571, San Diego, CA 92112-4571, U.S.A.*
**E-mail address: kappraff@verizon.net*

(Received March 11, 2005; Accepted March 20, 2005)

Keywords: Fibonacci Sequence, Golden Mean, Regular Polygons, Diagonals, Lucas Polynomials, Chebyshev Polynomials, Mandelbrot Set

Abstract. The diagonals of regular n-gons for odd n are shown to form algebraic fields with the diagonals serving as the basis vectors. The diagonals are determined as the ratio of successive terms of generalized Fibonacci sequences. The sequences are determined from a family of triangular matrices with elements either 0 or 1. The eigenvalues of these matrices are ratios of the diagonals of the n-gons, and the matrices are part of a larger family of matrices that form periodic trajectories when operated on by a matrix form of the Mandelbrot operator at a point of full-blown chaos. Generalized Mandelbrot matrix operators related to Lucas polynomials have similar periodic properties.

1. Introduction

It is well known that the ratio of successive terms of the Fibonacci sequence approaches the golden mean, $\tau = (1+\sqrt{5})/2$, in the limit and that the diagonal of a regular pentagon with unit edge has length τ. We show that the Fibonacci sequence can be generalized to characterizing all of the diagonals of regular n-gons for n an odd integer. Furthermore, a geometric sequence in τ is also a Fibonacci sequence and shares all of the algebraic properties inherent in the integer Fibonacci sequence. Similar sequences involving the diagonals of higher order n-gons also have algebraic properties. In fact we shall show that the diagonals form the bias vectors of a field. We shall call these, as STEINBACH (1997) did, “golden fields”. Products and quotients of the diagonals of an n-gon can be expressed as a linear combination of the diagonals.

The results depend strongly on a set of polynomials related to the Fibonacci numbers, and the Lucas polynomials, both of which are related to the Chebyshev polynomials. All of the roots of the Fibonacci polynomials are of the form $x = 2\cos(k\pi/n)$ while the Lucas polynomials map $2\cos A \mapsto 2\cos mA$. As a result, we show that a family of matrices with 0, 1, –1 elements form periodic trajectories when operated on by matrix forms of the Lucas polynomials. We refer to these as Mandelbrot Matrix Operators since the Lucas polynomial $L_2(x)$ corresponds to the Mandelbrot operator at the extreme left hand point on the real axis,

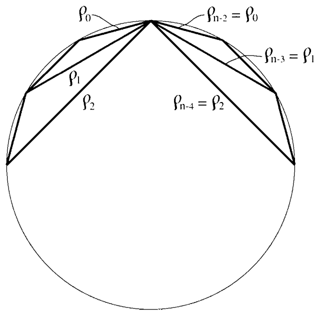

Fig. 1. The diagonals, ρ_k, of an n-gon are shown where ρ_0 denotes the edge.

a point of full-blown chaos. KAPPRAFF and ADAMSON (2005) have shown in a previous paper that the higher order Lucas equations lead to generalized Mandelbrot sets.

2. Preliminaries

Our work is based on the Diagonal Product Formula (DPF) of STEINBACH (1997).

Proposition: Diagonal Product Formula:

Consider a regular n-gon (Fig. 1) for odd n and let ρ_0 be the length of a side and ρ_k the length of the k-th diagonal with $k = (n–3)/2$. Then

$$\begin{aligned} \rho_0\rho_k &= \rho_k \\ \rho_1\rho_k &= \rho_{k-1} + \rho_{k+1} \\ \rho_2\rho_k &= \rho_{k-2} + \rho_k + \rho_{k+2} \\ \rho_3\rho_k &= \rho_{k-3} + \rho_{k-1} + \rho_{k+1} + \rho_{k+3} \\ &\vdots \\ \rho_h\rho_k &= \sum_{i=0}^{h} \rho_{k-h+2i}. \end{aligned} \tag{1}$$

In what follows we shall let $\rho_0 = 1$.

Using a chain of substitutions in the DPF, STEINBACH (1997) derived for the regular n-gon, the following formula basic to the combinatorics of polygons,

$$C(k,0)x^k - C(k-1,1)x^{k-2} + C(k-2,2)x^{k-4} - \cdots = C(k-1,0)x^{k-1} - (k-2,1)x^{k-3} + \cdots \tag{2}$$

where $k = (n–1)/2$ and $C(i, j) = i!/j!(i–j)!$.

If we write Eq. (2) as $P_k(x) = 0$, $P_k(x)$ has the recurrence relation, $P_{k+1}(x) = xP_k(x) – P_{k–1}(x)$ where $P_{–1} = 1$ and $P_0 = 1$. $P_k(x)$ is referred to as the DPF polynomials.

Consider the following identity, the proof of which is given in Theorem 1 of Appendix A:

$$\sin 2nA / \sin 2A = K_n(x) = (-1)^k P_k(x) P_k(-x). \tag{3a}$$

where $x = 2\cos 2A$, $k = (n-1)/2$ and $K_n(x)$ is the sequence of polynomials called *Fibonacci polynomials of the second kind* (see "Generalized Binet Formulas" by KAPPRAFF and ADAMSON in another article in this issue) since the absolute values of the coefficients of K_n sum to the n-th Fibonacci number as shown in KOSHY (2001), KAPPRAFF (2002) and HOSOYA in this issue. They are generated by the recursion,

$$K_{k+1}(x) = xK_k(x) - K_{k-1}(x) \text{ where } K_1 = 1 \text{ and } K_2 = x.$$

The first seven Fibonacci polynomials are,

$$\begin{aligned}
K_1 &= 1 \\
K_2 &= x \\
K_3 &= x^2 - 1 = (x-1)(x+1) \\
K_4 &= x^3 - 2x = \left(x^2 - 2\right)x \\
K_5 &= x^4 - 3x^2 + 1 = \left(x^2 - x - 1\right)\left(x^2 + x - 1\right) \\
K_6 &= x^5 - 4x^3 + 3x = \left(x^2 - 3\right)\left(x^2 - 1\right)x \\
K_7 &= x^6 - 5x^4 + 6x^2 - 1 = \left(x^3 - x^2 - 2x + 1\right)\left(x^3 + x^2 - 2x - 1\right).
\end{aligned} \tag{3b}$$

We prove in Theorem 2 of Appendix A that the Fibonacci polynomials of the second kind are related to the derivatives of the Chebyshev polynomials of the first kind. They are also generated from Pascal's triangle as shown in "Generalized Binet Formulas" by KAPPRAFF and ADAMSON in this issue.

Note that the sum of the absolute values of the coefficients of K_n is the n-th Fibonacci number. If $A = j\pi/2n$, it follows from Eqs. (3a) and (3b) that $K_n(x) = 0$ and that $\pm 2\cos j\pi/7$ are roots of $P_3(x)$ and $P_3(-x)$. For example,

$$\sin 14A / \sin 2A = K_7(x) = -P_3(x)P_3(x)$$

where $x = 2\cos 2A$ and, $P(x) = \cos 7A/\cos A$ and $P(-x) = -\sin 7A/\sin A$.

Note that the sum of the absolute values of the coefficients of K_7 is 13, the 7-th Fibonacci number. If $A = j\pi/14$, it follows that $K_7(x) = 0$ and that $\pm 2\cos j\pi/7$ are roots of $P_3(x)$ and $P_3(-x)$.

A general formula for the j-th diagonal of an n-gon with unit edge from KAPPRAFF (2002) is,

$$\rho_j = \frac{\sin\frac{(j+1)\pi}{n}}{\sin\frac{\pi}{n}} \text{ for } 0 \le j \le \frac{n-3}{2} \tag{4}$$

where ρ_0 is the edge of the n-gon.

3. The Pentagon

We begin with a statement of the case for $n = 5$, the pentagon. The standard Fibonacci sequence, $F_5^{(1)}$ is,

$$a_1\ a_2\ a_3\ \ldots\ a_k\ a_{k+1}\ \ldots = 1\quad 1\quad 2\quad 3\quad 5\quad 8\quad 13\quad 21\quad \ldots \tag{5}$$

where $\lim(a_{k+1}/a_k) = \tau$ where $\tau = (1 + \sqrt{5})/2$, the golden mean.

The following τ-sequence has identical algebraic properties as the integer sequence,

$$1\quad \tau\quad \tau^2\quad \tau^3\quad \tau^4\quad \ldots\quad \tau^k\quad \ldots \tag{6}$$

i.e., it is also a Fibonacci sequence where,

$$1 + \tau = \tau^2. \tag{6a}$$

Since the diagonal of the pentagon with unit edge has length τ, we shall refer to this as a ρ_1-sequence, where $\rho_1 = \tau$.

Equation (6a) satisfies the DPF for $n = 5$. We present this in Table 1 as a multiplication table expressed as left $\times$ top.

Table 1.

$\times$	1	ρ_1
1	1	ρ_1
ρ_1	ρ_1	$1 + \rho_1$

From this relation we can derive a generating matrix for the ρ_1-sequence by considering successive pairs of elements from the sequence to be a vector, i.e.,

$$\vec{v_1} = (\rho_1, 1)^T,\quad \vec{v_2} = (\rho_1^2, \rho_1)^T,\quad \vec{v_3} = (\rho_1^3, \rho_1^2)^T,\ \ldots$$

Consider the matrices,

$$M_5^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{7a}$$

and

$$M_5^{(1)^{-1}} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \tag{7b}$$

where $M_5^{(1)}\overrightarrow{v_n} = \overrightarrow{v_{n+1}}$. Therefore, $M_5^{(1)}(\rho_1, 1)^T = (\rho_1^{\,2}, \rho_1)^T = (1+\rho_1, \rho_1)^T$. The notation $M_5^{(1)}$ refers to the fact that the matrix generates the ρ_1-sequence for the 5-gon.

The same matrix also generates the Fibonacci sequence where, $M_5^{(1)}\overrightarrow{u_n} = \overrightarrow{u_{n+1}}$ where $\overrightarrow{u_1} = (1, 1)^T$, $\overrightarrow{u_2} = (2, 1)^T$, $\overrightarrow{u_3} = (3, 2)^T$, ...

The eigenvalues of the inverse matrix $M_5^{(1)^{-1}}$ in order of decreasing absolute values are

$$\lambda_1 = -2\cos\frac{\pi}{5}, \ \ \lambda_2 = 2\cos\frac{2\pi}{5}, \tag{8}$$

obtained as the zeros of the irreducible characteristic polynomial,

$$P_2(-x) = x^2 + x - 1 \tag{9}$$

where $P_2(x)$ is the generating polynomial of Eq. (2) for $n = 5$. That the eigenvalues of Eq. (8) are the zeros of Polynomial (9) follows from Eq. (3a). The eigenvalues can also be written as the ratio of diagonals,

$$\lambda_1 = -\frac{\rho_1}{\rho_0}, \ \ \lambda_2 = -\frac{\rho_0}{\rho_1}. \tag{10}$$

Furthermore, it follows from the DPF that, in general, when n is prime, quotients of the diagonals can be written as a linear combination of diagonals (including edge 1) with coefficients 0, 1, –1. For $n = 5$, Table 2 presents the ratio of diagonals, expressed in terms of left ÷ top.

Table 2.

÷	1	ρ_1
1	1	$\rho_1 - 1$
ρ_1	ρ_1	1

Thus the diagonals of a pentagon form a golden field with basis vectors: 1, ρ_1.

4. The Heptagon

Denote the two diagonals of a heptagon by ρ_1 and ρ_2 ($\rho_0 = 1$). From Eq. (4),

$$\rho_1 = 1.801 \ldots \text{ and } \rho_2 = 2.24 \ldots$$

From Eq. (1), the DPF, the product of diagonals are given by Table 3 expressed as left × top.

Table 3.

$\times$	1	ρ_1	ρ_2
1	1	ρ_1	ρ_2
ρ_1	ρ_1	$1 + \rho_2$	$\rho_1 + \rho_2$
ρ_2	ρ_2	$\rho_1 + \rho_2$	$1 + \rho_1 + \rho_2$

Consider the ρ_2 -sequence,

$$1 \quad \rho_1 \quad \rho_2 \quad \rho_1\rho_2 \quad \rho_2^2 \quad \rho_1\rho_2^2 \quad \rho_2^3 \quad \rho_1\rho_2^3 \quad \ldots \tag{11}$$

and the vectors,

$$\vec{v_1} = (\rho_2, \rho_1, 1)^T, \ \vec{v_2} = (\rho_2^2, \rho_1\rho_2, \rho_2)^T, \ \vec{v_2} = (\rho_2^3, \rho_1\rho_2^2, \rho_2^2)^T, \ \ldots$$

Using the relationships in Table 3, we define the matrix,

$$M_7^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{12a}$$

and

$$M_7^{(2)-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \tag{12b}$$

where, $M_7^{(2)}(\rho_2, \rho_1, 1)^T = (\rho_2^2, \rho_1\rho_2, \rho_2)^T = (1 + \rho_1 + \rho_2, \rho_1 + \rho_2, \rho_2)^T$. Matrix $M_7^{(2)}$ generates the ρ_2-sequence for the 7-gon and will be referred to as the principal matix.

Likewise, $M_7^{(2)} \overrightarrow{u_n} = \overrightarrow{u_{n+1}}$ where,

$$\overrightarrow{u_1} = (1,1,1)^T,\ \overrightarrow{u_2}(3,2,1)^T,\ \overrightarrow{u_3} = (6,5,3)^T,\ \ldots$$

results in the generalized Fibonacci sequence, $F_7^{(2)}$,

$$a_1\ a_2\ a_3 \ldots a_k\ a_{k+1} \ldots = 1 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 6 \quad 11 \quad 14 \quad 25 \quad 31 \quad \ldots \tag{13}$$

where

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{11}{6}, \frac{25}{14},\ \ldots \to \lim \frac{a_{2k}}{a_{2k-1}} = \rho_1,$$

and

$$\frac{1}{1}, \frac{3}{1}, \frac{6}{3}, \frac{14}{6}, \frac{31}{14},\ \ldots \to \lim \frac{a_{2k+1}}{a_{2k-1}} = \rho_2.$$

The irreducible characteristic polynomial of the inverse matrix $M_7^{(2)^{-1}}$ is,

$$P_3(x) = x^3 - x^2 - 2x + 1 \tag{14}$$

which can be derived from Eq. (2) for $n = 7$. As a result of Eq. (3a), its roots are the eigenvalues,

$$\lambda_1 = 2\cos\frac{\pi}{7},\ \lambda_2 = -2\cos\frac{2\pi}{7},\ \text{and}\ \lambda_3 = 2\cos\frac{3\pi}{7} \tag{15}$$

where,

$$\lambda_1 = \frac{\rho_1}{\rho_0},\ \lambda_2 = -\frac{\rho_2}{\rho_1},\ \text{and}\ \lambda_3 = \frac{\rho_0}{\rho_2}. \tag{16}$$

Table 4 lists the quotients of the diagonals represented as sums of diagonals expressed as left ÷ top.

Table 4.

$\div$	1	ρ_1	ρ_2
1	1	$1 + \rho_1 - \rho_2$	$\rho_2 - \rho_1$
ρ_1	ρ_1	1	$\rho_1 - 1$
ρ_2	ρ_2	$\rho_2 - 1$	1

Therefore the diagonals of a 7-gon form a golden field with basis vectors 1, ρ_1, ρ_2 and coefficients 0, 1, –1.

In deriving Matrix (12a) only two of the three DPF relations expressed by Table 3 are used. The third relation is expressed by another, ρ_1-sequence,

$$1 \quad \rho_2 \quad \rho_1 \quad \rho_2\rho_1 \quad \rho_1^2 \quad \rho_2\rho_1^2 \quad \rho_1^3 \quad \rho_2\rho_1^3 \quad \ldots \tag{17}$$

and the vectors,

$$\overrightarrow{v_1} = \left(\rho_2, \rho_1, 1\right)^T, \ \overrightarrow{v_2} = \left(\rho_1^2, \rho_1\rho_2, \rho_1\right)^T, \ \overrightarrow{v_3} = \left(\rho_1^3, \rho_2\rho_1^2, \rho_1^2\right)^T, \ \ldots$$

From Table 3 we define matrix $M_7^{(1)}$ as,

$$M_7^{(1)} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{18}$$

where,

$$M_7^{(1)}\overrightarrow{v_1} = M_7^{(1)}\left(\rho_1, \rho_2, 1\right)^T = \left(\rho_1^2, \rho_1\rho_2, \rho_1\right)^T = \left(1 + \rho_1, \rho_1 + \rho_2, \rho_1\right)^T$$

and, $M_7^{(1)}\overrightarrow{v_n} = \overrightarrow{v_{n+1}}$.

The corresponding generalized Fibonacci sequence, $F_7^{(1)}$ is,

$$1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 4 \quad 3 \quad 7 \quad 6 \quad 13 \quad 10 \quad 23 \quad 19 \quad \ldots \tag{19}$$

where, $\overrightarrow{u_1} = (1, 1, 1)^T$, $\overrightarrow{u_2} = (2, 2, 1)^T$, $\overrightarrow{u_3} = (3, 4, 2)^T$, ... and $M_7^{(1)}\overrightarrow{u_n} = \overrightarrow{u_{n+1}}$ and,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{6}{3}, \frac{10}{6}, \ \ldots \to \lim \frac{a_{2k+1}}{a_{2k-1}} = \rho_2$$

$$\frac{1}{1}, \frac{2}{1}, \frac{4}{2}, \frac{7}{3}, \frac{13}{6}, \ \ldots \rightarrow \lim \frac{a_{2k}}{a_{2k-1}} = \rho_1.$$

5. The Nonagon

A similar analysis can be carried out for the 9-gon. We state the results. From Eq. (4),

$$\rho_1 = 1.879 \ldots, \ \rho_2 = 2.532 \ldots, \text{ and } \rho_3 = 2.879 \ldots$$

From Eq. (1) (DPF), Table 5 expresses the multiplication table for $n = 9$ as left ÷ top.

Table 5.

÷	1	ρ_1	ρ_2	ρ_3
1	1	ρ_1	ρ_2	ρ_3
ρ_1	ρ_1	$1 + \rho_2$	$\rho_1 + \rho_3$	$\rho_2 + \rho_3$
ρ_2	ρ_2	$\rho_1 + \rho_3$	$1 + \rho_2 + \rho_3$	$\rho_1 + \rho_2 + \rho_3$
ρ_3	ρ_3	$\rho_2 + \rho_3$	$\rho_1 + \rho_2 + \rho_3$	$1 + \rho_1 + \rho_2 + \rho_3$

Consider the ρ_3-sequence,

$$1 \quad \rho_1 \quad \rho_2 \quad \rho_3 \quad \rho_1\rho_3 \quad \rho_2\rho_3 \quad \rho_3^2 \quad \rho_1\rho_3^2 \quad \rho_2\rho_3^2 \quad \rho_3^3 \quad \rho_1\rho_3^3 \quad \cdots \tag{20}$$

From Table 5 we derive the principal matrix,

$$M_9^{(3)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{21a}$$

and

$$M_9^{(3)^{-1}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \tag{21b}$$

and the generalized Fibonacci sequence, $F_9^{(3)}$,

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 3 \quad 4 \quad 7 \quad 9 \quad 10 \quad 19 \quad 26 \quad 30 \quad 56 \quad \ldots \tag{22}$$

where,

$$\frac{1}{1}, \frac{2}{1}, \frac{7}{4}, \frac{19}{20}, \frac{56}{30}, \ \ldots \to \lim \frac{a_{3k-1}}{a_{3k-2}} = \rho_1,$$

$$\frac{1}{1}, \frac{3}{1}, \frac{9}{4}, \frac{26}{10}, \frac{75}{30}, \ \ldots \to \lim \frac{a_{3k}}{a_{3k-2}} = \rho_2,$$

$$\frac{1}{1}, \frac{4}{1}, \frac{10}{4}, \frac{30}{10}, \frac{85}{30}, \ \ldots \to \lim \frac{a_{3k+1}}{a_{3k-2}} = \rho_3.$$

The eigenvalues of $M_9^{(3)^{-1}}$ are the zeros of the characteristic equation,

$$P_4(-x) = x^4 + x^3 - 3x^2 - 2x + 1 = (x + 1)(x^3 - 3x + 1) \tag{23}$$

where Polynomial (23) is derived from Eq. (2) for $n = 9$.

Note that since $n = 9$ is not prime, this equation is reducible and the factor $(x + 1)$ is the characteristic polynomial of the triangle inscribed within the 9-gon.

The eigenvalues are,

$$\lambda_1 = -2\cos\frac{\pi}{9}, \ \lambda_2 = 2\cos\frac{\pi}{9}, \ \lambda_3 = -2\cos\frac{3\pi}{9} = -1, \ \text{and } \lambda_4 = 2\cos\frac{4\pi}{9} \tag{24}$$

where,

$$\lambda_1 = -\frac{\rho_1}{\rho_0}, \ \lambda_2 = \frac{\rho_3}{\rho_1}, \ \lambda_3 = -\frac{\rho_2}{\rho_2} - 1, \ \text{and } \lambda_4 = \frac{\rho_0}{\rho_3}. \tag{25}$$

Table 6 lists the quotients of the diagonals as sums where the ratios are expressed as left ÷ top:

Table 6.

÷	1	ρ_1	ρ_2	ρ_3
1	1	$\rho_2 - 2$	$\frac{2\rho_2 - \rho_1 - 2}{3}$	$1 + \rho_1 - \rho_2$
ρ_1	ρ_1	1	$\frac{2\rho_2 - \rho_1 + 1}{3}$	$\rho_2 - \rho_1$
ρ_2	ρ_2	$2 + \rho_1 - \rho_2$	1	$\rho_1 - 1$
ρ_3	$\rho_1 + 1$	$\rho_2 - 1$	$\frac{\rho_1 + \rho_2 - 1}{3}$	1

In Table 6 we have eliminated ρ_3 by recognizing that $\rho_3 = \rho_1 + 1$. Note that the coefficients are now rational numbers reflecting that $n = 9$ is not prime. That the ratios with fractional coefficients all have denominator ρ_2 signifies that ρ_2 is the edge of the triangle inscribed in the 9-gon. The basis vectors of the golden field associated with $n = 9$ are now $1, \rho_1, \rho_2$.

The following are the ρ_1 and ρ_2-sequences for $n = 9$:

$$1 \quad \rho_2 \quad \rho_3 \quad \rho_1 \quad \rho_2\rho_1 \quad \rho_3\rho_1 \quad \rho_1^2 \quad \rho_2\rho_1^2 \quad \rho_3\rho_1^2 \quad \rho_1^3 \quad \rho_2\rho_1^3 \quad \ldots \tag{26a}$$

$$1 \quad \rho_1 \quad \rho_3 \quad \rho_2 \quad \rho_1\rho_2 \quad \rho_3\rho_2 \quad \rho_2^2 \quad \rho_1\rho_2^2 \quad \rho_3\rho_2^2 \quad \rho_2^3 \quad \rho_1\rho_2^3 \quad \ldots \tag{26b}$$

Matrices corresponding to the ρ_1 and ρ_2-sequences are,

$$M_9^{(1)} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{27a}$$

and

$$M_9^{(2)} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{27b}$$

The generalized Fibonacci sequence, $F_9^{(1)}$ is,

$$1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 4 \quad 4 \quad 3 \quad 7 \quad 8 \quad 6 \quad 14 \quad 15 \quad 10 \quad \ldots \tag{28}$$

where,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{6}{3}, \frac{10}{6}, \ \ldots \to \lim \frac{a_{3k+1}}{a_{3k-2}} = \rho_1,$$

$$\frac{1}{1}, \frac{2}{1}, \frac{4}{2}, \frac{7}{3}, \frac{14}{6}, \ \ldots \to \lim \frac{a_{3k-1}}{a_{3k-2}} = \rho_2,$$

$$\frac{1}{1}, \frac{2}{1}, \frac{4}{2}, \frac{8}{3}, \frac{15}{6}, \ \ldots \to \lim \frac{a_{3k}}{a_{3k-2}} = \rho_3.$$

The generalized Fibonacci sequence, $F_9^{(2)}$, is,

$$1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 3 \quad 3 \quad 5 \quad 8 \quad 7 \quad 13 \quad 20 \quad 18 \quad 33 \quad 41 \quad 45 \quad \ldots \tag{29}$$

where,

$$\frac{1}{1},\frac{2}{1},\frac{5}{3},\frac{13}{7},\frac{33}{18},\ \ldots \to \lim \frac{a_{3k-1}}{a_{3k-2}} = \rho_1,$$

$$\frac{1}{1},\frac{3}{1},\frac{7}{3},\frac{18}{7},\frac{45}{18},\ \ldots \to \lim \frac{a_{3k+1}}{a_{3k-2}} = \rho_2,$$

$$\frac{1}{1},\frac{3}{1},\frac{8}{3},\frac{20}{7},\frac{51}{18},\ \ldots \to \lim \frac{a_{3k}}{a_{3k-2}} = \rho_3.$$

6. The General Case

An n-gon for odd has $(n–3)/2$ diagonals denoted by,

$$\rho_1, \rho_1,\ \ldots,\ \rho_m \text{ where } m = \frac{n-3}{2}.$$

The ρ_m-sequence is,

$$1 \quad \rho_1 \quad \rho_2 \quad \cdots \quad \rho_m \quad \rho_1\rho_m \quad \rho_2\rho_m \quad \cdots \quad \rho_{m-1}\rho_m \quad \rho_m^2 \quad \rho_1\rho_m^2 \quad \rho_2\rho_m^2 \quad \cdots \quad \rho_{m-1}\rho_m^2 \quad \rho_m^3 \quad \rho_1\rho_m^3 \quad \cdots \tag{30}$$

and the corresponding principal matrix is,

$$M_n^{(m)} = \begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 \\ 1 & 1 & 1 & . & . & 1 & 0 \\ 1 & 1 & . & . & 1 & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & 1 & 0 & . & . & 0 & 0 \\ 1 & 0 & 0 & . & . & 0 & 0 \end{bmatrix} \tag{31a}$$

and

$$M_n^{(m)^{-1}} = \begin{bmatrix} 0 & 0 & 0 & . & . & 0 & 1 \\ 0 & 0 & 0 & . & . & 1 & -1 \\ 0 & 0 & . & . & 1 & -1 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 1 & -1 & . & . & 0 & 0 \\ 1 & -1 & 0 & . & . & 0 & 0 \end{bmatrix}. \tag{31b}$$

From Eq. (2), the characteristic polynomial, where $k = (n–1)/2$ is,

$P_k(x)$ for k odd, and $P_k(–x)$ for k even.

The characteristic polynomials are irreducible when n is prime. If n_1 is a factor of n then the characteristic polynomial is factorable, and either $P_{k_1}(x)$ or $P_{k_1}(–x)$, corresponding to the inscribed n_1-gon is a factor of $P_k(x)$ or $P_k(–x)$. For example, for $n = 9$, Eq. (23) shows that $P_1(–x)$ is a factor of $P_4(–x)$ and shares the root of the inscribed triangle. Likewise for $n = 15$,

$$P_7(x) = x^7 - x^6 - 6x^5 + 5x^4 + 10x^3 - 6x^2 - 4x + 1 = (x-1)\left(x^2 - x - 1\right)\left(x^4 + x^3 - 4x^2 - 4x + 1\right) \tag{32}$$

so that $P_7(x)$ shares the roots of $P_1(x)$, the characteristic polynomial of the inscribed triangle, and $P_2(x)$ is the characteristic polynomial related to the inscribed pentagon. It also follows that any n-gon with n divisible by 3 has $\lambda = \pm 1$ as an eigenvalue.

The eigenvalues can be expressed as,

$$\lambda_j = 2\cos\left(2(k-j)+1\right)k\frac{\pi}{n} \tag{33}$$

where,

$$\left|\lambda_j\right| = \frac{\rho_{2(k-j)}}{\rho_{j-1}} \tag{34}$$

for $j = 1, 2, ..., k$ and $k = (n–1)/2$. Note that in Eq. (34), $\rho_{2i} = \rho_{2(k–i)+1}$ for $(n–1)/2 \le i \le n – 3$.

In addition, there are m generalized Fibonacci and ρ_j-sequences corresponding to matrices, $M_n^{(j)}$ for $j = 1, 2, ..., m$.

Each n-gon has a golden field associated with it in which both products and quotients can be expressed as linear combinations of the diagonals. If n is prime, the coefficients of the quotients are 0, 1, –1, and the basis vectors of the golden field are 1, ρ_1, ρ_2, ..., ρ_m. If n is not prime, the coefficients are rational numbers, and the basis vectors are a subset of: 1, ρ_1, ρ_2, ..., ρ_n.

It can also be shown that the many combinatoric relations involving the numbers of the Fibonacci sequence continue to hold for the generalized Fibonacci sequences. These relationships will be explored in a future paper.

7. Polygons and Chaos

Consider the sequence of Lucas polynomials, L_m, of the second kind as described in “Generalized Binet Formulas” by KAPPRAFF and ADAMSON in this issue. The first six Lucas polynomials are,

$$
\begin{aligned}
L_0(x) &= 2 \\
L_1(x) &= x \\
L_2(x) &= x^2 - 2 \\
L_3(x) &= x^3 - 3x \\
L_4(x) &= x^4 - 4x^2 + 2 \\
L_5(x) &= x^5 - 5x^3 + 5.
\end{aligned}
\tag{35}
$$

They are generated by the recursion,

$$L_{k+1}(x) = xL_1(x) - L(x) \text{ where } L_0 = 2 \text{ and } L_2 = x.$$

The Lucas polynomials are related to the Chebyshev polynomials of the second kind and have the defining property described by KAPPRAFF and ADAMSON (2005), and KAPPRAFF (2002),

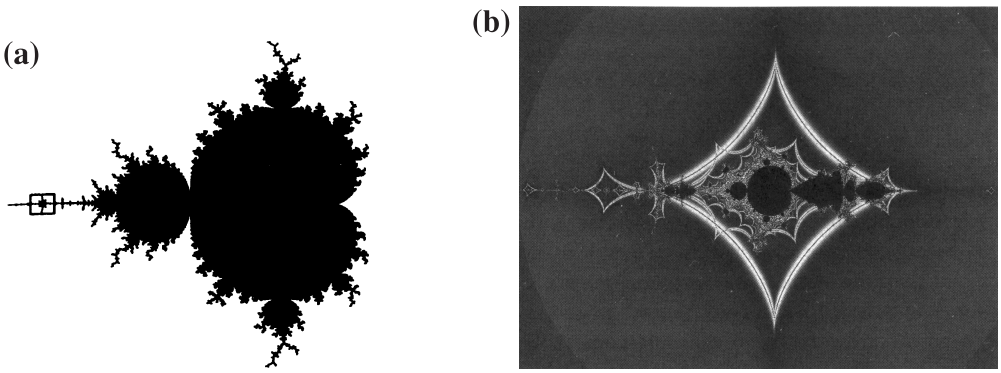

Fig. 2. a) The Mandelbrot set; b) A generalized Mandelbrot set corresponding to $L_6(x)$ as derived by KAPPRAFF and ADAMSON (2005). Computer image created by J. Barrallo.

$$L_m(2\cos\theta) = 2\cos m\theta. \tag{36}$$

In particular, $L_2 = x^2 - 2$ is a special case of the operator that generates the Mandelbrot set,

$$z \mapsto z^2 + c$$

for $c = -2$, the leftmost point on the real axis of the Mandelbrot set shown in Fig. 2a. KAPPRAFF and ADAMSON (2005) have shown that the other Lucas polynomials lead to somewhat more complex Mandelbrot sets such as the one shown in Fig. 2b for $L_6(x)$. Beginning with $x = x_0$, the recursion,

$$x \mapsto x^2 - 2 \tag{37a}$$

generates the trajectory: $x_0, x_1, x_2, \ldots, x_k$ where $x_0 \to x_1 \to x_2 \to \ldots \to x_k \ldots$ If $x_p = x_0$ the trajectory is periodic with period p.

Next consider x to be the $n \times n$ diagonalizable matrix X, and rewrite Eq. (37a) as,

$$X \mapsto X^2 - 2I \tag{37b}$$

where I is the $n \times n$ identity matrix. We refer to Eq. (37b) as the *Mandelbrot Matrix Operator* (*MMO*). We claim that for each n-gon for odd n, setting either $X_0 = -M_n^{(m)^{-1}}$ or $X_0 = -M_n^{(m)^{-1}}$ (see Eq. (31b)) results in a periodic trajectory of period p depending only on the value of n, with the same values of p as described by KAPPRAFF and ADAMSON (2005), i.e., p is the smallest positive integer such that,

$$2^p \equiv \pm 1(\mathrm{mod}\, n). \tag{38}$$

For example, for the pentagon, $n = 5$, using Eq. (7b),

$$X_0 = M_5^{(1)^{-1}} \to X_1 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \to X_2 = M_5^{(1)^{-1}}$$

so that $M_5^{(1)^{-1}}$ repeats with period 2. For the hexagon, $n = 7$, using Eq. (12b),

$$X_0 = -M_7^{(2)^{-1}} \to X_1 = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \to X_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \to X_3 = -M_7^{(2)^{-1}}$$

so that $-M_7^{(2)^{-1}}$ has period $p = 3$.

We state this result as a Theorem.

Theorem: If X is an $n \times n$ diagonalizable matrix and either $X_0 = -M_n^{(m)^{-1}}$ or $X_0 = -M_n^{(m)^{-1}}$, depending on n, the Mandelbrot matrix operator, $X \mapsto X^2 - 2I$ has a periodic trajectory with period p.

Proof: We shall demonstrate this for the case n = 5 and n = 7. The proof for general n follows in a similar manner.

Since X is diagonalizable, there exists a matrix of eigenvectors P such that,

$$X = P^{-1}\Lambda P \tag{39}$$

where, Λ is the matrix of eigenvalues, $\Lambda = \lambda_i\delta_{ij}$ (no summation on i) and δ_{ij} is the Kronecker delta. Replacing X into the MMO (37b) yields,

$$X^2 - 2I = P^{-1}((\lambda_i^2 - 2)\delta_{ij})P. \tag{40}$$

If λ_i or $-\lambda_i$ is given by Eq. (32) then $X = M_n^{(m)^{-1}}$ or $X = -M_n^{(m)^{-1}}$ and the result follows by replacing λ_i or $-\lambda_i$ with its value given by Eq. (33) into Eq. (40) and using Eq. (36) for m = 2. We shall demonstrate this for n = 5 and n = 7.

If n = 5, using Eq. (34),

$$\lambda_1 = -2\cos\frac{\pi}{5} \to 2\cos\frac{2\pi}{5} \to 2\cos\frac{4\pi}{5} = 2\cos\frac{(5-1)\pi}{5} = -2\cos\frac{\pi}{5}.$$

We abbreviate this sequence by considering the coefficients of the numerator of the arguments, i.e.,

$$\lambda_1 = -2\cos\frac{\pi}{5} \equiv 1 \to 2 \to 4 = 2\cos\frac{4\pi}{5} \equiv -2\cos\frac{\pi}{5}.$$

In a similar manner,

$$\lambda_2 = 2\cos\frac{2\pi}{5} \equiv 2 \to 4 \to 8 \equiv 2\cos\frac{8\pi}{5} = 2\cos\frac{2\pi}{5}.$$

Thus we have demonstrated that $M_5^{(1)^{-1}}$ has period 2.

If n = 7, using Eq. (37),

$$-\lambda_1 = -2\cos\frac{\pi}{7} \equiv 1 \to 2 \to 4 \to 8 \equiv 2\cos\frac{8\pi}{7} = -2\cos\frac{\pi}{7}$$

$$-\lambda_2 = -2\cos\frac{2\pi}{7} \equiv 2 \to 4 \to 8 \to 16 \equiv 2\cos\frac{16\pi}{7} = -2\cos\frac{2\pi}{7}$$

$$-\lambda_3 = -2\cos\frac{3\pi}{7} \equiv 3 \to 6 \to 12 \to 24 \equiv 2\cos\frac{24\pi}{7} = -2\cos\frac{3\pi}{5}.$$

Thus we have demonstrated that $-M_7^{(2)^{-1}}$ has period 3.

In a similar manner, as demonstrated by KAPPRAFF and ADAMSON (2005), using Eq. (36), this result continues to hold for the generalized Mandelbrot matrix operators (GMMO), $X \mapsto L_m(X)$ with periods given by the smallest positive integer, p, such that,

$$m^p \equiv \pm 1(\mathrm{mod}\, n)$$

where $L_m(X)$ for $m = 2$, 3, 4, and 5 is given by Eq. (35) with X replacing x.

Slavik Jablan and Radmila Sazdanovich have done an exhaustive computer study of matrices with 0, 1, –1 elements to determine their periodic behavior under the Mandelbrot Matrix Operators. Their results for the case of 3×3 matrices under the Matrix Operator (36b) are summarized:

Of all the 3^9 matrices with elements 0, 1, –1, 384 have period 3. Among these, 120 have the characteristic equation

$$P_3(x) = x^3 - x^2 - 2x + 1 \tag{41a}$$

while 120 matrices have the characteristic equation,

$$P_3(-x) = x^3 + x^2 - 2x - 1. \tag{41b}$$

The first 120 have period {3, –1}, and the other {3, 1} where $\{p, \pm 1\}$ refers to a trajectory with period p and either the matrix M or $-M$ as the initial matrix.

Among these 240 matrices, only matrices are selected with 0, 1, –1 as elements of their inverses. There are 96 such matrices. They are,

a) 48 with {3, 1} and 48 with {3, –1};

b) the first 48 have characteristic polynomial $P_3(-x)$, the other 48 have characteristic polynomial $P_3(x)$.

The characteristic equation for each of the 48 matrices with {3, 1} is invariant under the Matrix Mandelbrot Operator. The characteristic equation of the 48 matrices with {3, –1} transform to matrices with the characteristic equation of {3, 1} under the Mandelbrot operator.

From each set of 48 matrices there are 8 matrices (16 total) that have the form of a DPF inverse in that either upper or lower triangular elements are either 1 or –1 as in Eq. (12a).

Among these 16 matrices with DPF inverses there are exactly 8 matrices that have diagonals all of whose elements are either 1 or all –1 as in Eq. (12b). Four of these matrices

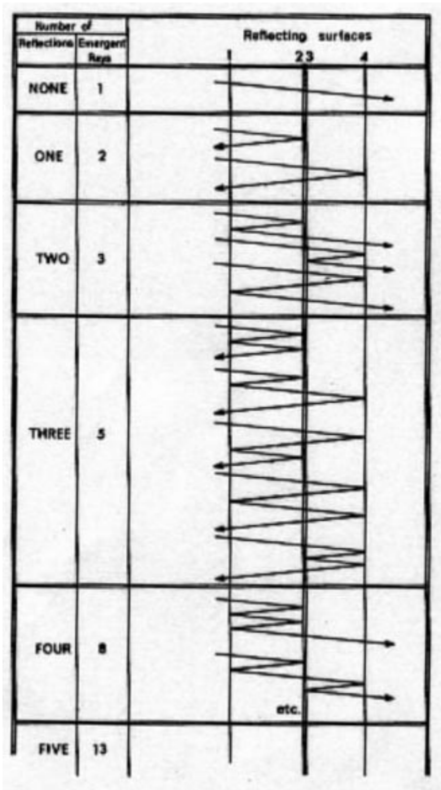

Fig. 3. Multiple reflections in two sheets of glass.

are {3, 1} while the other 4 are {3, –1}.

Similar results can be found for all $k \times k$ matrices and for the other L_m operators.

We can make the following general results for the L_2 MMO:

1. Corresponding to characteristic equation $P_k(x)$ or $P_k(-x)$ there are 2^{k+1} matrices whose inverses are DPF. They can be described by matrices whose main diagonal and its neighbor are filled by 1 or –1;

2. Among these matrices, 2^k will have characteristic equation $P_k(x)$ and the other 2^k will have $P_k(-x)$. One of these will have $\{p, 1\}$ trajectories and the other $\{p, -1\}$;

3. For every k, there are exactly four matrices with only 1 or –1 on the diagonal with trajectory $\{p, 1\}$ and four with $\{p, -1\}$.

8. Reflected Waves

Consider light rays incident to two slabs of glass as shown in Fig. 3. There is one wave with no reflections, 2 waves with 1 reflection, and 3 waves with 2 reflections. In fact for the number of waves, N_k, with k reflections, $N_k = a_{k+1}$ from the $F_5^{(1)}$-sequence (the standard Fibonacci sequence): 1, 2, 3, 5, 8, ..., as demonstrated by HUNTLEY (1970).

Next consider three slabs of glass. It has been shown by MOSER and WYMAN (1973) and HOGGATT and BICKNELL-JOHNSON (1979) that $N_k = a_{2k+1}$, a subsequence: 1, 3, 6, 14, 31, ... of $F_7^{(2)}$ (see Sequence (13)), the generalized Fibonacci sequence associated with the heptagon.

Likewise, for m planes of glass, $N_k = a_{(m-1)k+1}$, a subsequence of the generalized Fibonacci sequence $F_{(2m+1)}^{(m-1)}$.

Appendix A: Fibonacci Polynomials of the Second Kind

We state the following theorem about the Fibonacci polynomials of the second kind:

Theorem 1: For n odd and $k = (n-1)/2$,

$$K_n(x) = \frac{\sin 2nA}{\sin 2A} = 1 + 2\cos 4A + 2\cos 8A + 2\cos 12A + \cdots + 2\cos 4kA \tag{A1a}$$

where $x = 2\cos 2A$.

For n even and $k = (n-1)/2$,

$$K_n(x) = \frac{\sin 2nA}{\sin 2A} = 2\cos 2A + 2\cos 6A + 2\cos 10A + \cdots + 2\cos 2(n-1)A. \tag{A1b}$$

For n odd, $K_n(x) = P_k(x)P_k(-x)$ where $P_k(x)$ are the DPF polynomials (see Eq. (2)).

Proof: Consider the elementary trigonometric identity,

$$\cos jA \sin A = \frac{1}{2}\left(\sin(j+1)A - \sin((j-1)A\right). \tag{A2}$$

Summing this equation for $j = 2, 4, 6, ..., 2k$ yields the collapsing sum,

$$\frac{\sin nA}{\sin A} = 1 + 2\cos 2A + 2\cos 4A + 2\cos 6A + \cdots + 2\cos 2kA.$$

Replacing $A \to 2A$ yields,

$$\frac{\sin 2nA}{\sin 2A} = 1 + 2\cos 4A + 2\cos 8A + 2\cos 12A + \cdots + 2\cos 2(n-1).$$

For n even, and $k = |(n-1)/2|$, setting $j = 1, 3, 5, ..., 2k-1$, yields after replacing $A \to 2A$,

$$\frac{\sin 2nA}{\sin 2A} = 2\cos 2A + 2\cos 6A + 2\cos 10A + \cdots + 2\cos 2(n-1)A.$$

Next we show that K_n satisfies the Recursion Relation (A1). This follows from the elementary trigonometry identity,

$$\sin 2nA = 2\sin(n-1)2A\cos 2A - \sin(n-1)2A. \tag{A3}$$

Let $x = 2\cos 2A$ and $K_n = (\sin 2nA)/(\sin 2A)$. Dividing Eq. (A3) by $\sin 2A$ immediately yields the recursion relation,

$$K_n = xK_{n-1} - K_{n-2}.$$

But since,

$$K_1 = \frac{\sin 2A}{\sin 2A} = 1 \text{ and } K_2 = \frac{\sin 4A}{\sin 2A} = \frac{2\sin 2A\cos 2A}{\sin 2A} = x$$

it follows that $K_n(x)$ must be the Fibonacci-Pascal polynomials with alternating signs also shown in Eq. (3b). It is easy to see that K_n are even functions when n is odd. As a result, if x is a root then so is $-x$ and K_n factors into $P_k(x)P_k(-x)$ where $P_k(x)$ are the DPF polynomials. That $(\sin nA)/(\sin A) = P(-x)$ and $(\cos nA)/(\cos A) = P_k(x)$ where $x = 2\cos 2A$ and $k = (n-1)/2$ can be proved in a similar manner. If n is odd, K_n is an odd function of x.

Theorem 2: $K_n(x) = \dfrac{2}{n}\dfrac{dT_n\left(\frac{x}{2}\right)}{dx}$

The Chebyshev polynomials of the first kind have the property (MASON and HANDSCOMB, 2003),

$$T_n(y) = \cos n\theta \text{ for } y = \cos\theta.$$

Therefore,

$$\frac{dT_n(y)}{dy} = \frac{d\cos n\theta / d\theta}{d\cos\theta / d\theta} = n\frac{\sin n\theta}{\sin\theta}.$$

Let, $\theta = 2A$ and $x = 2\cos 2A = 2y$.

Then it follows that,

$$\frac{1}{n}\frac{dT_n(y)}{dy} = \frac{\sin 2nA}{\sin 2A} = K_n(x).$$

But,

$$\frac{1}{n}\frac{dT_n(y)}{dy} = \frac{2}{n}\frac{dT_n\left(\frac{x}{2}\right)}{dx}.$$

And the result follows. QED.

REFERENCES

HOGGATT, V. E., Jr. and BICKNELL-JOHNSON, M. (1979) Reflections across two and three glass plates, *Fibonacci Quarterly*, **17**, 118–142.

HUNTLEY, H. E. (1970) *The Divine Proportion: A Study in Mathematical Beauty*, Dover, New York.

KAPPRAFF, J. (2002) *Beyond Measure: A Guided Tour through Nature, Myth, and Number*, World Scientific Publ., Singapore.

KAPPRAFF, J. and ADAMSON, G. W. (2005) Polygons and chaos, *J. Dynam. Syst. and Geom. Theories*, **2**, 65–80.

KOSHY, T. (2001) *Fibonacci and Lucas Numbers with Applications*, Wiley, New York.

MASON, J. C. and HANDSCOMB, D. C. (2003) *Chebyshev Polynomials*, Chapman and Hall/CRC, New York.

MOSER, L. and WYMAN, M (1973) Multiple reflections, *Fib. Quart.*, **11**.

STEINBACH, P. (1997) Golden fields: A case for the heptagon, *Math. Mag.* **70**(1).