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Some Graph-Theoretical Aspects of the Golden Ratio: Topological Index, Isomatching Graphs, and Golden Family Graphs

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Abstract. By defining the non-adjacent number, $p(\mathrm{G}, k)$, and topological index, Z_{G}, for a graph G, several sequences of graphs are shown to be closely related to the golden ratio, τ. Namely, the Z-values of the path and cycle graphs are Fibonacci, and Lucas numbers, respectively, and thus the ratio of consecutive terms of Z converges to τ. Several new sequences of graphs (golden family graphs) were found whose Z-values are either Fibonacci or Lucas numbers, or their multiples. Interesting mathematical relations among them are introduced and discussed.

1. Introduction

It has already been shown by LUCAS (1876) that the Fibonacci numbers can be obtained from the Pascal's triangle by rotation and addition in a certain way. The golden ratio can then be asymptotically obtained by taking the ratio of consecutive Fibonacci numbers, whose graph-theoretical aspects have been pointed out by the present author (HOSOYA, 1971, 1973) in terms of the topological index, Z, which is the sum of the non-adjacent numbers for a given graph. Similar properties of the Lucas numbers related to the golden ratio have also been demonstrated.

Recently several new sequences of graphs were found whose Z-values are either Fibonacci or Lucas numbers, or their multiples. They are called *golden family graphs*, and their interesting mathematical structure will be presented in this paper. The methodology developed here can be applied to the problems of general recursive sequences, widening their field of algebraic analysis to geometrical or graph theoretical realms.

2. Graph, Topological Index, and Fibonacci Numbers

In graph theory (HARARY, 1969) a graph, G, is a set of vertices and edges. We are concerned only with non-directed and connected graphs. Except for a few cases, multiple edges are excluded. *Path graphs*, S_n, and *cycle graphs*, C_n, are the most fundamental

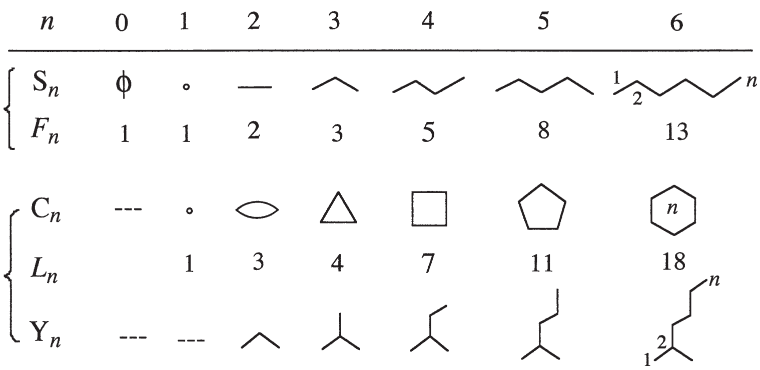

Fig. 1. Path graphs (S_n), cycle graphs (C_n), and Y_n graphs. The topological indices of S_n are Fibonacci numbers, F_n, while those of the isomatching pair of C_n and Y_n are Lucas numbers, C_n.

sequences of tree and non-tree graphs, respectively, where n denotes the number of vertices (Fig. 1).

The present author (HOSOYA, 1971, 1973) has defined the *topological index*, Z, for characterizing the topological nature of graphs representing the carbon atom skeleton of hydrocarbon molecules (CVETKOVIC *et al.*, 1995; BALABAN and IVANCIUC, 1999; KOSHY, 2001). In order to obtain the value of Z for G one first defines the *non-adjacent number*, $p(\mathrm{G}, k)$, or *k-matching number*, as the number of ways to choose k-disjoint edges (bonds) from G. The largest value of k for G with n vertices is denoted by $m = [n/2]$. The non-adjacent number, $p(\mathrm{G}, m)$, for G with even n is equal to the perfect matching number, or the Kekulé number, $K(\mathrm{G})$, in chemistry. The topological index Z for G is defined as the sum of all $p(\mathrm{G}, k)$ numbers.

The $p(\mathrm{G}, k)$ numbers have a special algebraic meaning for graphs, especially for trees. Namely, the characteristic polynomial, $P_{\mathrm{G}}(x)$,

$$P_{\mathrm{G}}(x) = (-1)^n \det(\boldsymbol{A} - x\boldsymbol{E}), \tag{1}$$

of a given tree G is expressed in terms of the $p(\mathrm{G}, k)$ numbers as,

$$P_{\mathrm{G}}(x) = \sum_{k=0}^{m} (-1)^k p(\mathrm{G}, k) x^{n-2k} \quad (\mathrm{G} \in \text{tree}). \tag{2}$$

Here $\boldsymbol{A}$ and $\boldsymbol{E}$ are the adjacency and identity matrices of order n. For general graphs with cycles, $P_{\mathrm{G}}(x)$ can also be expressed in terms of the $p(\mathrm{G}, k)$ numbers for the set of subgraphs of G but in a more complicated fashion (HOSOYA, 1972).

Table 1 shows the $p(\mathrm{G}, k)$ numbers and Z-values for smaller members of S_n and C_n. By joining a pair of vertices with a pair of edges one gets *digon*, C_2, given in Fig. 1, which has the same set of $p(\mathrm{G}, k)$ numbers and thus has the same Z-value as S_3. We refer to a set of

Table 1. Non-adjacent numbers $p(\mathrm{G}, k)$ and topological indices Z of path (a) and cycle graphs (b).

	(a)						(b)					
	Path graph S_n						Cycle graph C_n					
n	$p(\mathrm{G},k)$					$Z=F_n$	$p(\mathrm{G},k)$					$Z=L_n$
	$k=$ 0	1	2	3	4		$k=$ 0	1	2	3	4	
0	1					1	2					2
1	1					1	1					1
2	1	1				2	1	2				3
3	1	2				3	1	3				4
4	1	3	1			5	1	4	2			7
5	1	4	3			8	1	5	5			11
6	1	5	6	1		13	1	6	9	2		18
7	1	6	10	4		21	1	7	14	7		29
8	1	7	15	10	1	34	1	8	20	16	2	47
	FT						LT					

graphs with this property as *isomatching graphs*. However, S_3 and C_2 are not *isospectral graphs*, since they have different $P_{\mathrm{G}}(x)$s. Of course, among tree graphs isomatching graphs are always isospectral with respect to each other.

Most important for Table 1 is that the topological indices of the sequences of path and cycle graphs are the Fibonacci (F_n) and Lucas (L_n) numbers, respectively. Note, however, that the initial conditions for F_n,

$$F_0 = F_1 = 1, \tag{3}$$

in Table 1 are shifted by one step from the conventional definition (VOROBIEV, 1961; HOGGATT, 1969) while those for L_n,

$$L_0 = 2,\ L_1 = 1, \tag{4}$$

are the same.

The ratio of consecutive terms from these two sequences are well known to converge to the golden mean, $\tau = (1+\sqrt{5})/2$,

$$F_n/F_{n-1},\ L_n/L_{n-1} \to \tau, \tag{5}$$

as a result of their common recursive relation,

$$f_n = f_{n-1} + f_{n-2}. \tag{6}$$

Numerous sequences of graphs can be generated whose Z-value ratios converge to τ. Let us therefore define *golden family graphs* as those sequences of graphs whose Z-values form

Table 2. Pascal's and asymmetrical Pascal's triangles.

a) Pascal's triangle (PT)

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

b) Asymmetrical Pascal's triangle (APT)

						1						
					1		2					
				1		3		2				
			1		4		5		2			
		1		5		9		7		2		
	1		6		14		16		9		2	
1		7		20		30		25		11		2

c) PT + PT' = APT

						1						
					1		1+1					
				1		2+1		1+1				
			1		3+1		3+2		1+1			
		1		4+1		6+3		4+3		1+1		
	1		5+1		10+4		10+6		5+4		1+1	
1		6+1		15+5		20+10		15+10		6+5		1+1

either F_n or L_n sequences, or their multiples. It is already known that the sequence of graphs, Y_n, given in Fig. 1 are isomatching with C_n and belong to the golden family. It can be proved that there exist no isomatching sequence of graphs with S_n, but there are known to be a number of graphs whose Z-values are terms from F_n or L_n but with different $p(G, k)$ distributions. The main purpose of the present paper is to introduce new members of the golden family.

For later discussion, let us call the two right-angled arrays of numbers in Table 1 the *Fibonacci triangle**, FT, and the *Lucas triangle***, LT, both of which are closely related to the *Pascal's triangle*. Both FT and LT can be generated from the pair of numbers in the first and second rows by successive application of the "rook sum." First, select a given number; move to the southeast neighbor; add both numbers; and assign their sum as the southern neighbor of the latter number. This recurrence relation turns out to be a variation of the "Y-sum" in Pascal's triangle.

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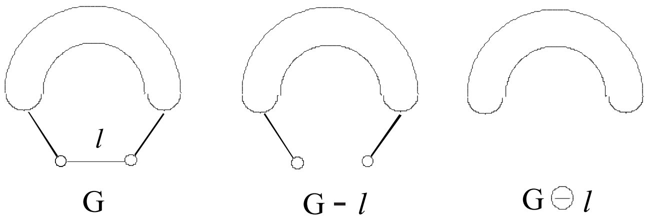
*This is different from the "Fibonacci triangle" proposed earlier by the present author (HOSOYA, 1976).

**This is a little different from the "Lucas triangle" proposed by FEINBERG (1967).



$$p(\mathrm{G},k) = p(\mathrm{G}-l,k) + p(\mathrm{G}\ominus l, k-1)$$

$$\mathrm{Q}(\mathrm{G},x) = \mathrm{Q}(\mathrm{G}-l,x) + x\,\mathrm{Q}(\mathrm{G}\ominus l, x)$$

$$\mathrm{Z}(\mathrm{G}) = \mathrm{Z}(\mathrm{G}-l) + \mathrm{Z}(\mathrm{G}\ominus l)$$

Fig. 2. Recursion formulas of $p(\mathrm{G}, k)$, $\mathrm{Q}_{\mathrm{G}}(x)$, and Z_{G}, using a pair of subgraphs, G–l and G$\ominus l$ which are obtained by deleting an arbitrary chosen edge l from G.

3. Pascal's Triangle and Asymmetrical Pascal's Triangle

Pascal's triangle is a triangular array of the *binomial coefficients*,

$$B(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{7}$$

(BONDARENKO, 1993). The triangle shown in Table 2a is the standard form and denoted here as PT. In principle, all integers in PT can be generated from only the top 1 by applying the *Y-sum*.

The Fibonacci numbers from this arrangement can be obtained by adding the numbers along the northeast diagonals (HOGGATT, 1969; KOSHY, 2001). By rotating PT of Table 2a by about 30 degree clockwise, a skewed form of FT can be recognized. This means that all the integers of PT replicate the whole family of $p(\mathrm{G}, k)$ numbers for the sequence of path graphs, S_n, and *vice versa*. This graph-theoretical connection of PT has already been pointed out by the present author (HOSOYA, 1971, 1973).

From this discussion one obtains the following important combinatorial relation,

$$p(\mathrm{S}_n, k) = \binom{n-k}{k}. \tag{8}$$

Similar graph-theoretical features of Lucas numbers can be obtained by adding a pair of PTs by shifting one row with respect to the other to give the *asymmetrical Pascal's triangle* (*APT*), as shown in Table 2(b) and (c) (HOSOYA, 1995, 1998). This relation can be expressed symbolically by

$$\text{PT} + \text{PT}' \rightarrow \text{APT}. \tag{9}$$

If an element of APT is denoted by A(n, k), the above relation corresponds to the following identity,

$$\text{A}(n, k) = \text{B}(n, k) + \text{B}(n-1, k-1). \tag{10}$$

FEINBERG (1967) arrived at this triangle by expanding the polynomial $(x+y)^{n-1}(x+2y)$, and called it the Lucas triangle. However, our LT in Table 1 can be obtained from it (APT of Table 2b) by rotation of about 30 degree clockwise in the way that FT, was derived from PT. KAPPRAFF (2002) calls APT the *generalized Pascal's triangle*.

If a PT is subtracted from another PT after shifting downwards by two steps along the right slant roof, one also gets an APT. However, in this case an extra pair of 1's is attached on top of the right slant diagonal of the APT (HOSOYA, 1998). Then one may denote this capped triangle as APT′, and this relation can be expressed symbolically by,

$$\text{PT} - \text{PT}'' \rightarrow \text{APT}'. \tag{11}$$

This APT′ gives the degree of degeneracy of the angular part of the atomic wave-function of hypothetical n-dimensional hydrogen-like atoms.

4. Recursion Formula

In what follows, it is useful to consider the *Z-counting polynomial*, $\text{Q}_\text{G}(x)$, defined by,

$$\text{Q}_\text{G}(x) = \sum_{k=0}^{m} p(\text{G}, k) x^k. \tag{12}$$

With this polynomial the topological index is defined as follows:

$$Z_\text{G} = \sum_{k=0}^{m} p(\text{G}, k) \tag{13}$$

$$= \text{Q}_\text{G}(1). \tag{14}$$

By using Eq. (8), $\text{Q}_\text{G}(x)$ for S_n is written as

$$\text{Q}_{\text{S}_n}(x) = \sum_{k=0}^{m} \binom{n-k}{k} x^k. \tag{15}$$

We then have,

$$Z_{\text{S}_n} = \text{Q}_{\text{S}_n}(1) = F_n. \tag{16}$$

As the size of the graph increases, the number of steps for enumerating $p(\mathrm{G}, k)$ numbers and Z-values increases quite rapidly. For this purpose several efficient recursion formulas have been proposed using the inclusion-exclusion principle (HOSOYA, 1971, 1973; HOSOYA and MOTOYAMA, 1985). The simplest and most fundamental one will be explained here.

Consider graph G in Fig. 2, where l is an arbitrarily chosen edge. Define subgraph G–l as the graph obtained from G by deleting l, and G$\ominus l$ as the graph obtained by deleting l together with all the edges incident to l. The recursion relations for $p(\mathrm{G}, k)$ and $\mathrm{Q_G}(x)$ are expressed by,

$$p(\mathrm{G}, k) = p(\mathrm{G}-l, k) + p(\mathrm{G}\ominus l, k-1) \tag{17}$$

and

$$\mathrm{Q_G}(x) = \mathrm{Q}_{\mathrm{G}-l}(x) + x\mathrm{Q}_{\mathrm{G}\ominus l}(x). \tag{18}$$

As a result, one gets the following useful recursion formula for Z_G:

$$Z_\mathrm{G} = Z_{\mathrm{G}-l} + Z_{\mathrm{G}\ominus l}. \tag{19}$$

The first term on the rhs of Eq. (17) is the number of l-exclusive possibilities from G, while the second term gives the number of l-inclusive possibilities. Note that the argument in the second term is k–1, since l has already been included as a member of the k edges. The Recursion Formula (18) of the Z-counting polynomial can be obtained by summing Eq. (17) after multiplying by x^k. The x in the second term of Eq. (18) is automatically factored out, since it accounts for the selection of edge l for l-inclusive enumeration.

Examples of the use of the recursion formulas, Eqs. (17)–(19), are given in Fig. 3. Applying them to C_n, as in Fig. 3a one gets,

$$p(\mathrm{C}_n, k) = p(\mathrm{S}_n, k) + p(\mathrm{S}_{n-2}, k-1), \tag{20}$$

$$\mathrm{Q}_{\mathrm{C}_n}(x) = \mathrm{Q}_{\mathrm{S}_n}(x) + x\mathrm{Q}_{\mathrm{S}_{n-2}}(x), \tag{21}$$

and

$$\begin{aligned} Z_{\mathrm{C}_n} &= Z_{\mathrm{S}_n} + Z_{\mathrm{S}_{n-2}} \\ &= \mathrm{Q}_{\mathrm{C}_n}(1) = L_n. \end{aligned} \tag{22}$$

The explicit form of $\mathrm{Q}_{\mathrm{C}n}(x)$ is derived by combining Eqs. (15) and (21) as,

$$\mathrm{Q}_{\mathrm{C}_n}(x) = \sum_{k=0}^{m} \frac{n}{n-k} \binom{n-k}{k} x^k. \tag{23}$$

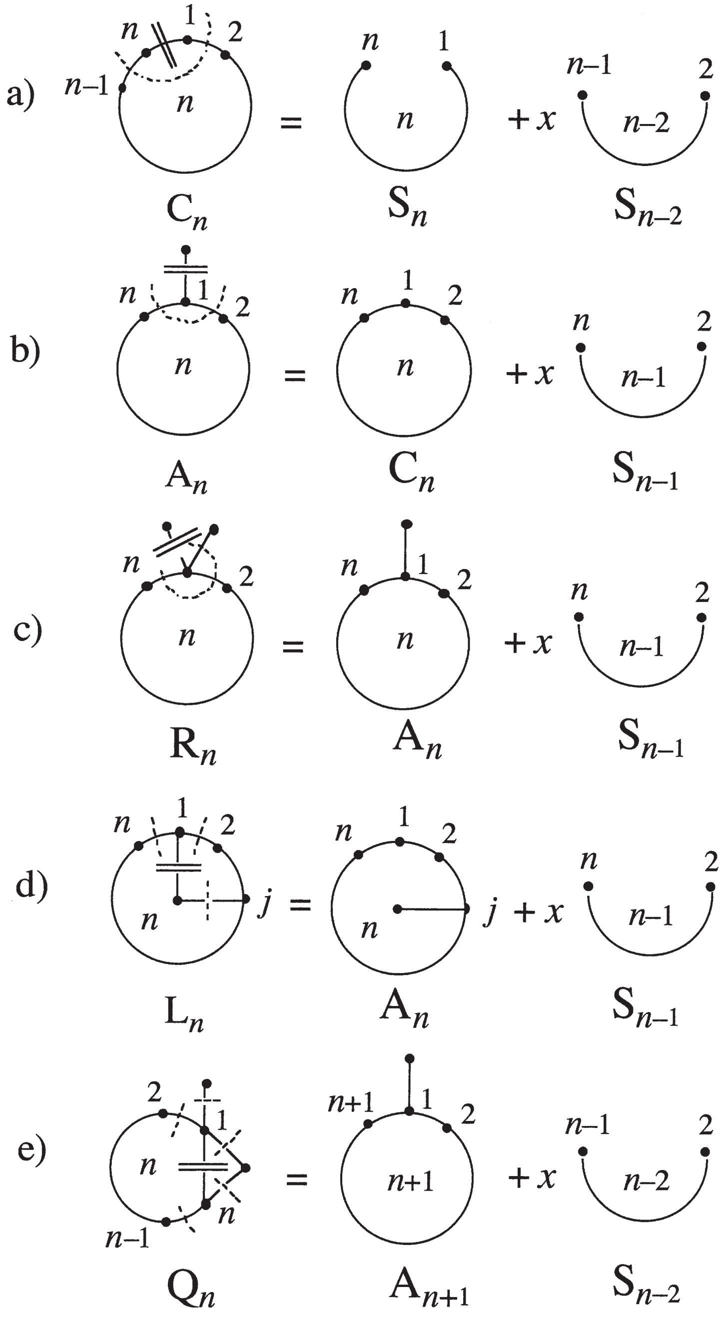


Fig. 3. Examples for the use of the recursion formula of Fig. 2 for deriving the characteristic quantities of various golden family graphs. G–*l* is obtained by deleting the double-crossed edge *l*, and further deletion of the edges crossed by dotted lines yields G⊖*l*.

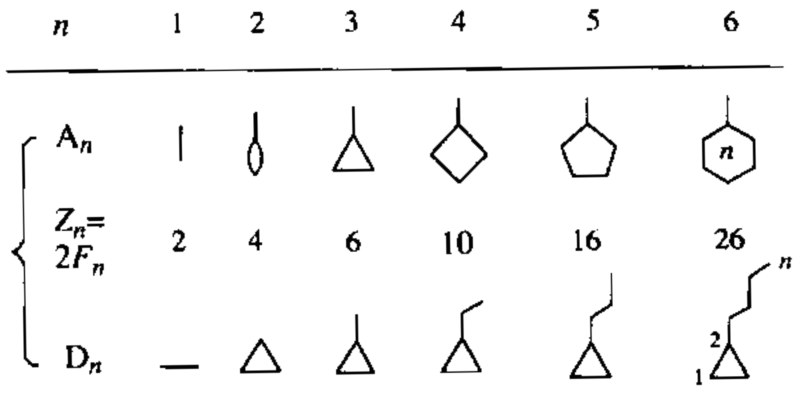


Fig. 4. A pair of isomatching golden family graphs, apple (A_n) and worm (D_n) graphs, whose Z-values are $2F_n$.

The first equality of Eq. (22) represents the well known relationship,

$$L_n = F_n + F_{n-2}. \tag{24}$$

5. Discovery of New Golden Family Graphs

The $p(G, k)$ numbers and Z-values for smaller tree and non-tree graphs have been extensively tabulated by the group of the present author (MIZUTANI *et al.*, 1971; KAWASAKI *et al.*, 1971). Several new members of the golden family graphs were discovered by scrutinizing these tables.

5.1. Apple and worm graphs

We refer to the two sequences of graphs in Fig. 4 as *apple graphs* (A_n) and *worm graphs* (W_n), whose Z-values are both equal to $2F_n$. They are found to be isomatching with respect to each other, and their $p(G, k)$ numbers are given in Table 3a. Graph A_n is constructed from C_n and a branch of unit length. By cutting the terminal edge and applying the recursion formula as in Fig. 3b, the following expressions can be derived:

$$Q_{An}(x) = Q_{Cn}(x) + xQ_{Sn-1}(x) \tag{25}$$

and

$$\begin{aligned} Z_{An} &= Z_{Cn} + Z_{Sn-1} \\ &= (F_n + F_{n-2}) + F_{n-1} = 2F_n. \end{aligned} \tag{26}$$

The general expression for the $p(G, k)$ numbers of these two sequences of graphs are obtained as follows by using Eqs. (15) and (23):

Table 3. Non-adjacent numbers $p(\mathrm{G}, k)$ and topological indices Z of A_n and D_n graphs (a) as derived from a pair of FT′s (b)

	(a)						(b)				
	A_n, D_n $p(\mathrm{G},k)$					Z=2F_n	FT + FT'''				
n	k = 0	1	2	3	4						
1	1	1				2	1	1			
2	1	3				4	1	2+1			
3	1	4	1			6	1	3+1	1		
4	1	5	4			10	1	4+1	3+1		
5	1	6	8	1		16	1	5+1	6+2	1	
6	1	7	13	5		26	1	6+1	10+3	4+1	
7	1	8	19	13	1	42	1	7+1	15+4	10+3	1
8	1	9	26	26	6	68	1	8+1	21+5	20+6	5+1

$$p(\mathrm{A}_n,k)=p(\mathrm{C}_n,k)+p(\mathrm{S}_{n-1},k-1)$$
$$=\frac{\left(n^2+n-nk-k^2\right)(n-1-k)!}{k!\,(n+1-2k)!}\quad (n\geq 2). \tag{27}$$

In the case of D_n one needs to open the triangle by cutting one of its edges to get S_{n+1} (see Fig. 4), and follow the standard procedure. Then a result similar to A_n can be obtained,

$$Z_{\mathrm{D}n} = F_{n+1} + F_{n-2}$$
$$= (F_n + F_{n-1}) + F_{n-2} = 2F_n. \tag{28}$$

Equation (27) also holds for D_n.

It is interesting to note that the right-angled triangle of Table 3a can be generated from a pair of FT by shifting three steps as demonstrated in Table 3b. This relation can be expressed symbolically by,

$$\mathrm{FT} + \mathrm{FT}''' \rightarrow \mathrm{A}_n,\ \mathrm{D}_n, \tag{29}$$

which can be justified by considering the first equality of Eq. (27) and with the aid of Eq. (20) as,

$$p(\mathrm{A}_n, k) = p(\mathrm{S}_n, k) + p(\mathrm{S}_{n-2}, k-1) + p(\mathrm{S}_{n-1}, k-1)$$
$$= p(\mathrm{S}_{n+1}, k) + p(\mathrm{S}_{n-2}, k-1). \tag{30}$$

5.2. Rabbit, clock, and cocktail glass graphs

Figure 5 illlustrates a quartet of isomatching sequences of graphs, R_n, L_n, M_n, and O_n, whose Z-values are Fibonacci numbers, but with different $p(\mathrm{G}, k)$ distributions as shown in Table 4a. Note that R_1 is S_3, and the lowest three members of L_n and M_n (n = 1–3)

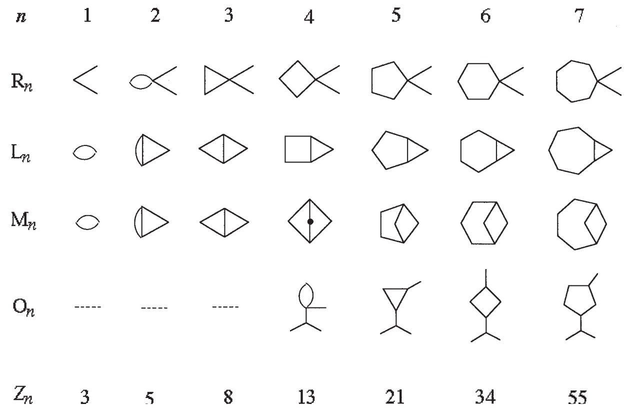


Fig. 5. The quartet isomatching golden family graphs, rabbit (R_n), clocks, (L_n and M_n), and cocktail glass (O_n) graphs, whose Z-values are F_n.

Table 4. Non-adjacent numbers $p(G, k)$ and topological indices Z of R_n, L_n (L-1), M_n (L-2), and O_n graphs (a) as derived from LT and FT (b).

	(a)						(b)				
	R_n, L_n, M_n, O_n $p(G,k)$					$Z = F_{n+2}$	LT + FT"'				
n	$k = 0$	1	2	3	4						
1	1	2				3	1	2			
2	1	4				5	1	3+1			
3	1	5	2			8	1	4+1	2		
4	1	6	6			13	1	5+1	5+1		
5	1	7	11	2		21	1	6+1	9+2	2	
6	1	8	17	8		34	1	7+1	14+3	7+1	
7	1	9	24	19	2	55	1	8+1	20+4	16+3	2
8	1	10	32	36	10	89	1	9+1	27+5	30+6	9+1

beginning with C_2 are identical. Only the sequence O_n begins from $n = 4$. Because of their shape, we nickname them: *rabbit*, *clock-1*, *clock–2*, and *cocktail graphs*, respectively. The reason for the naming of the clock graph will be explained later.

It is evident from Fig. 3c that the recursion formulas for R_n can be derived as

$$Q_{Rn}(x) = Q_{An}(x) + xQ_{Sn-1}(x) \tag{31}$$

and

$$\begin{aligned} Z_{Rn} &= Z_{An} + Z_{Sn-1} \\ &= 2F_n + F_{n-1} = F_{n+2}. \end{aligned} \tag{32}$$

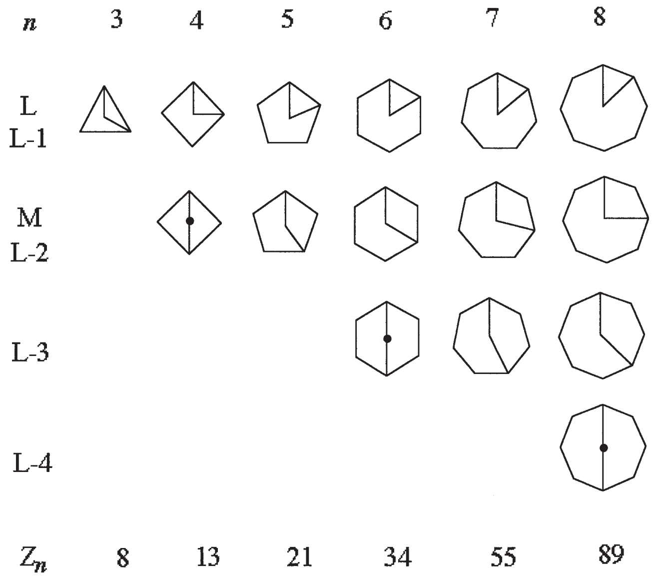


Fig. 6. Four isomatching clock graphs.

As shown in Fig. 3d, the recursion relations for L_n (j = 2) and M_n (j = 3) are,

$$Q_{L_n}(x) = Q_{An}(x) + xQ_{Sn-1}(x) \tag{33}$$

and

$$\begin{aligned} Z_{L_n} &= Z_{An} + Z_{Sn-1} \\ &= 2F_n + F_{n-1} = F_{n+2}. \end{aligned} \tag{34}$$

The recursion relations for O_n are not as simple to express but nevertheless exist. The general form of the $p(G_n, k)$ numbers can be derived as,

$$\begin{aligned} p(G_n, k) &= p(A_n, k) + p(S_{n-1}, k-1) \\ &= \frac{(n^2 + n - 2k^2)(n-1-k)!}{k!\,(n+1-2k)!} \quad (n \geq 2) \quad (G = R,\ L,\ M,\ O), \end{aligned} \tag{35}$$

In this case, it is observed that the $p(G, k)$ numbers of this quartet of graphs can be obtained by adding a pair of FT and LT as shown in Table 4b. This relation is expressed symbolically as,

$$\mathrm{LT} + \mathrm{FT}''' \rightarrow \mathrm{R}_n, L_n, \mathrm{M}_n, \mathrm{O}_n. \tag{36}$$

This relation can be explained beginning with the first equality of Eq. (35) as

$$\begin{aligned} p(\mathrm{G}_n, k) &= p(\mathrm{C}_n, k) + 2p(\mathrm{S}_{n-1}, k-1) \\ &= p(\mathrm{C}_n, k) + p(\mathrm{S}_{n-1}, k-1) + p(\mathrm{S}_{n-2}, k-1) + p(\mathrm{S}_{n-3}, k-2) \\ &= p(\mathrm{C}_n, k) + p(\mathrm{C}_{n-1}, k-1) + p(\mathrm{S}_{n-2}, k-1) \\ &= p(\mathrm{C}_{n+1}, k) + p(\mathrm{S}_{n-2}, k-1) \quad (\mathrm{G} = \mathrm{R, L, M, O}). \end{aligned} \tag{37}$$

All members of the sequence of isomeric clock graphs, L-n, up to $n = 4$ are shown in Fig. 6. The reason why they are isomatching has already been explained above. Note, however, that neither of them are isospectral with respect to each other.

We have seen that the Z-values of all sequences of graphs shown in Figs. 4–6 are either F_n or its multiple, and thus they belong to the golden family.

5.3. Q-shaped graphs

A pair of isomatching sequences of graphs, Q_n, and O_n, were found whose Z-values are Lucas numbers but with different $p(\mathrm{G}, k)$ distributions from C_n (See Table 1). As shown in Fig. 3e the $Q_\mathrm{G}(x)$ of Q_n can be decomposed into fragments, as follows, by using Eqs. (21) and (25):

$$\begin{aligned} Q_{\mathrm{Q}n}(x) &= Q_{\mathrm{A}n+1}(x) + xQ_{\mathrm{S}n-2}(x) \\ &= Q_{\mathrm{C}n+1}(x) + xQ_{\mathrm{S}n}(x) + xQ_{\mathrm{S}n-2}(x) \\ &= Q_{\mathrm{S}n+1}(x) + x\{Q_{\mathrm{S}n}(x) + Q_{\mathrm{S}n-1}(x) + Q_{\mathrm{S}n-2}(x)\}. \end{aligned} \tag{38}$$

The $Q_\mathrm{G}(x)$ of Ω_n are the same. Since the general expressions for their $p(\mathrm{G}, k)$ and $Q_\mathrm{G}(x)$ are so complex, there is no point in describing them here.

6. Silver Family Graphs

The *silver mean* is the positive root, $\theta = 1 + \sqrt{2}$, of $x^2 - 2x - 1 = 0$ (DUPUIS and LAWLOR, 1979; KAPPRAFF, 2002, and KAPPRAFF and ADAMSON, and KAPUSTA in this issue), which is the limit of the ratio of consecutive terms of both the *Pell sequences* (ALEXANDERSON, 1966), P_n, and *Pell*-Lucas sequence (HORADAM and MAHON, 1985), Q_n. defined by,

$$P_n = 2\,P_{n-1} + P_{n-2}, \text{ with } P_0 = 1, P_1 = 2^* \tag{39}$$

and

$$\mathrm{Q}_n = 2\mathrm{Q}_{n-1} + \mathrm{Q}_{n-2}, \text{ with } \mathrm{Q}_0 = \mathrm{Q}_1 = 2 \tag{40}$$

---

*Conventional initial conditions are $P_0 = 0$ and $P_1 = 1$.

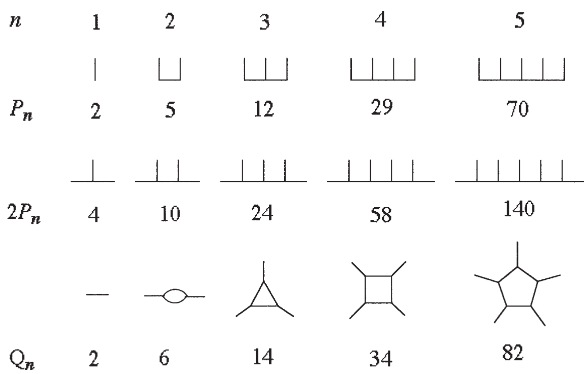


Fig. 7. Three series of graphs, U_n, T_n, and CU_n, which belong to the silver family graphs. Their Z-values are either Pell numbers, P_n or Pell-Lucas numbers, Q_n, or their multiples.

Since there are several sequences of graphs whose Z-value ratios converge to $\theta = 1 + \sqrt{2}$, we shall call them *silver family graphs*. Three sequences of graphs, U_n, T_n, and CU_n, are shown in Fig. 7.

One can extend this type of graph-theoretical analysis to a variety of number sequences. In this way mathematical relations satisfied by a set of recursive number sequences that obey the same recursion formula can be transformed into another set of sequences of graphs. In this way, a fundamental understanding of their whole mathematical structure can be obtained both visually and systematically.

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