Generating Geometric Patterns Using Periodic Functions

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Quadratic curves are typical examples of geometric forms. This paper looks at selected quadratic curves and describes their transformation into cubic curves for use as axes for generating geometric patterns. Using periodic functions with t as an intermediary variable, we defined these curves as universal cubic curves expressed as $x = f(t)$, $y = g(t)$ and $z = h(t)$. Using universal cubic curves selected as the axes and selected quadratic curves as motifs, we then generated original geometric patterns by applying affine transformations. This paper describes our investigation of mathematical modeling-based generation of 3D geometric patterns with cubic curves as the axes and quadratic curves as motifs, and of the transformation of the patterns into two dimensions to generate geometric pattern variations. Through pattern generation, this paper aims to provide a basic methodology that can be used in fields such as art and design.
Key words: Composition Forms, Theory of Plastic Art, Geometric Pattern, Periodic Function, Affine Transformation

1. Introduction

Patterns can be classified into natural patterns (such as fingerprints and whorls, or formations created by wind or waves such as sand dunes and coastlines) and man-made patterns (such as graphs, shapes, and drawings). Man-made patterns include patterns called motifs created for purposes such as decoration in fields such as art, craft, design and architecture (Takaki, 2003). Typical examples include Celtic motifs from Ireland, arabesque motifs from the Near and the Middle East, Byzantine motifs from ancient Rome and motifs found on Japanese Jomon-era pottery. This paper only discusses man-made patterns, considered to be synonymous with motifs.

Patterns can be considered to have originated as ornamentation on artifacts such as cave paintings from Ardèche in modern-day France or ancient sarcophagi from Japan (Takaki, 2003). Patterns are closely related to ornamentation, a concept clearly defined in the writing of prominent German art historian Wilhelm Worringer (Wilhelm, W.) who said that the essence of ornamentation is the expression of a culture's artistic desires with the utmost purity and clarity. The history of design has been charted in compilations of ornamental design created by writers such as Owen Jones (Owen, O.) (known as the greatest design theorist of the 19th century), Auguste Racinet (M. A. Racinet) (a 19th century French designer) and Stuart Durant (Stuart, D.) (known for his research on the history of 20th-century ornamentation and design). These compilations show the relationships between the patterns created by cultures from around the world and ornamental design. For example, geometric patterns have featured prominently in designs that use motifs from nature (see Fig. 1). The history of these patterns can be considered the history of the fusion between nature and geometry.

The place of patterns today in relation to this history and to ornamentation can be described as follows:

In the art world, new approaches to ornamental designs can be found in works such as the organically patterned wall paintings of Kiyoshi Kuroda and the mathematically modeled patterns of Asao Tokoro (MOT Manual 2010). Ornamental designs conveyed through geometric patterns that reflect a new view of nature can be seen in the field of design (in the printed-fabric fashion designs of world-famous French fashion house Leonard (Shirley, S.), as well as in architecture, interior design and textile design (in the patterns shown on everyday media (TRIBOUILLARD).

New features of geometric patterns can be found in the ornamentation of the 21st century. In the field of fashion design for example, digital media are being used to create a variety of designs with component elements of combined patterns, geometric patterns, organic shapes and elements with ethnic origins (Diana, N. and Christina, U.).

Advances in computer graphics are enabling computer-generated pattern research that examines geometric pattern generation from the perspective of mathematical modeling. Examples include a study by Odaka on patterns generated by original algorithms (Odaka, 1998), one by Yokoyama showing pattern design variation created by unique use of applications (Yokoyama, Y.), and one by Morita on simulations of shapes from the natural world (Morita, K.). The patterns examined by these studies are geometric patterns created in so-called 'virtual spaces'. But these studies could indicate one approach to the design of geometric patterns using digital media. When defining the current role of pattern design for ornamentation, these studies could bring
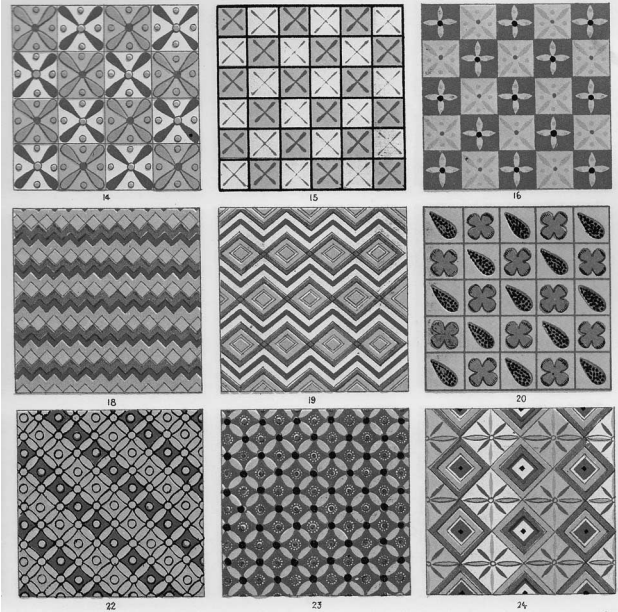
Fig. 1. Ornaments of Egypt/Those that have been taken from murals painted on tombstones in Egypt around/A part (see references GRAMMTIC DER ORNAMENT (Owen, J.)).

Fig. 2. Parallel projection of universal golden ellipse.

(a)

(b)

Fig. 3. Translation of golden ellipse on straight axix.

new value to the production of works with pattern motifs created using various media.

In light of these observations, this paper describes the transformation of quadratic curves (typical geometric forms) into cubic curves, which are used as axes for generating geometric patterns. Using periodic functions with t as an intermediary variable, we defined these curves as universal cubic curves expressed as $x = f(t)$, $y = g(t)$ and $z = h(t)$. We then investigated the use of these universal cubic curves to generate geometric patterns in three dimensions by mathematical modeling, and the transformation of these patterns into two dimensions to generate geometric pattern variations. Through pattern generation, this paper aims to provide a basic methodology that can be used in fields such as art and design.

2. Method for Geometric Pattern Generation

Odaka has described a method of generating geometric patterns that uses a basic figure as a base and applies a periodic function to it to create variations by translation or by a combination of translation and revolution (Odaka, 1998). Transformations of a given base form by translation, revolution or a combination of translation and revolution are known as affine transformations. They are a very effective method of pattern generation.

The method described in this paper is based on this approach. It uses universal cubic curves and quadratic curves to generate geometric patterns from affine transformations. Specifically, it uses selected cubic curves as the axes, uses selected quadratic curves as motifs, and generates geometric patterns from continuous curves by translation, or a combination of translation and revolution.

2.1 Setting of universal cubic curves as axes
2.1.1 Using universal golden ellipse
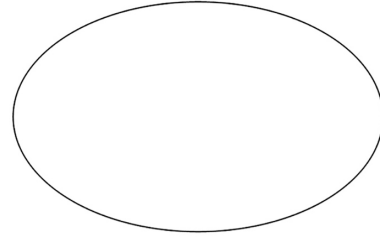An ellipse that can be expressed by a periodic function
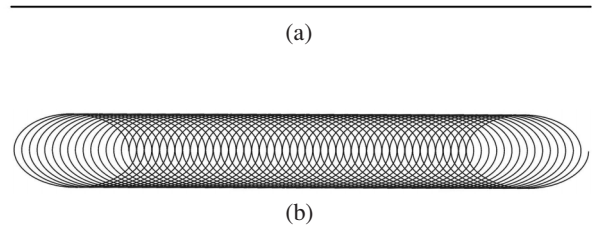
can be expressed as follows using t as an intermediate variable (Japan Society for Graphic Science), (a: major axis of ellipse; b: minor of ellipse; $a > b$):

$$x = f(t) = a \sin t$$
$$y = g(t) = b \cos t. \quad (1)$$

We then provided a formula for the z-axis direction and created Formula 2 to generate a universal ellipse. Formula 2 is for an example ellipse. Formula 2 could be used to generate several different variations of universal cubic curves for quadratic curves expressible by periodic functions. Figure 2 shows an example universal cubic curve generated from Formula 2. The ratio of the major axis (a) and the minor axis (b) of the ellipse in Fig. 2 has been set to the golden ratio, $\tau = (1 + \sqrt{5})/2$. This paper are refers these ellipses as golden ellipses when they are quadratic curves, or as universal golden ellipses when they are cubic curves (a_1: major axis of golden ellipse; b_1: minor axis of golden ellipse; $a_1 = \tau b_1$; c_1: rational number; f: natural number).

$$x = f(t) = a_1 \sin t$$
$$y = g(t) = b_1 \cos t$$
$$z = h(t) = c_1 \sin f t. \quad (2)$$

2.2 Application of affine transformations
To generate geometric patterns, we used golden ellipses as the examples and arbitrary forms as the axes and applied affine transformations. The affine transformation methods used were translations or combinations of translations and revolutions.
2.2.1 Affine transformations with straight-line axis
We tried subjecting a golden ellipse to a simultaneous translation and revolution (i.e., rotation) on straight-line axis.
a. Translation of golden ellipse on straight-line axis
The first step was to try translating the golden ellipse on straight-line axis. To make the golden ellipse into a continuous curve, we created Formula 3 by adding the formula
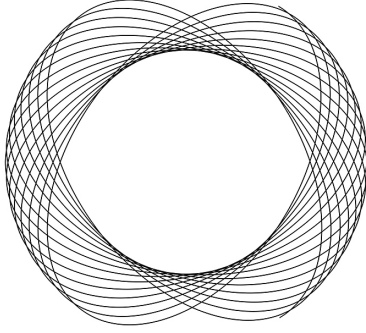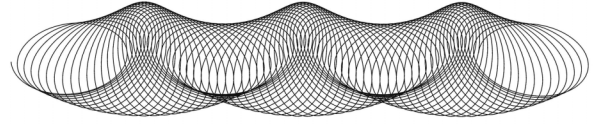
Fig. 4. Rotation of golden ellipse.

Fig. 5. Combining translation and rotation.

(a)

(b)
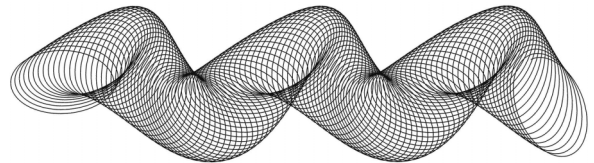
(c)

Fig. 6. Using sine curve at axes.

$s_1 = d_1\pi/j$ to the y-axis direction and adding the formula to the z-axis direction. Figure 3 shows the result. Figure 3b is an example in which the golden ellipse is translated with a period value of 120π, using the axis of Fig. 3a.

$$x_1 = at + x$$
$$y_1 = b + y$$
$$z_1 = c\sin(s_1 t)\pi/h. \qquad (3)$$

b. Rotation of golden ellipse

The second step was to try rotating the golden ellipse. To achieve rotation, we created Formula 4 by adding $s_2 = d_2\pi/j$. Figure 4 shows an example with a rotation period value of 30π.

$$x_2 = x\cos(s_2 t)\pi/h - y\sin(s_2 t)\pi/h$$
$$y_2 = x\sin(s_2 t)\pi/h + y\cos(s_2 t)\pi/h. \qquad (4)$$

c. Combining translation and rotation

The third step was to try translating and rotating the golden ellipse simultaneously. To do so, we created Formula 5 by taking Formula 3 and restructuring it with Formula 4.

$$x_3 = at + x_2$$
$$y_3 = b + y_2$$
$$z_3 = c\sin(s_1 t)\pi/h. \qquad (5)$$

Figure 5 shows an example generated from Formula 5, with a rotation coefficient d_2 of 1.5 and period value of 120π.

2.2.2 Affine transformations using sine curves as axes

We devised a model for generating geometric patterns using golden ellipses as motifs. An example using sine curves is shown below.

First, we created Formula 6 by applying Formula 3 to a sine curve equation (r: radius of rotating circle). Second, we devised the equations $h = m2^n j$ (m: natural number; n: integer) and $d_2 = m2^n d_1$ (m: natural number; n: integer), which are used when determining the values of the h, j, d_1 and d_2 coefficients that determine the pattern shape.

$$x_4 = a_1 t + x$$
$$y_4 = b_1\sin(s_1 t)\pi/h + y$$
$$z_4 = c_1\sin(s_1 t)\pi/h. \qquad (6)$$

We then took Formula 6 and restructured it with Formula 4 to create Formula 7.

$$x_5 = a_1 t + x_2$$
$$y_5 = b_1\sin(s_1 t)\pi/h + y_2$$
$$z_5 = c_1\sin(s_1 t)\pi/h. \qquad (7)$$

Substituting $(s_1 t)\pi/h = g_1$ into this equation enables Formula 6 to be expressed as Formula 8, and Formula 7 as Formula 9.

$$x_6 = a_1 t + x$$
$$y_6 = b_1\sin g_1 + y$$
$$z_6 = c_1\sin g_1. \qquad (8)$$

$$x_7 = a_1 t + x_2$$
$$y_7 = b_1\sin g_1 + y_2$$
$$z_7 = c_1\sin g_1. \qquad (9)$$

Figure 6 shows the translation of a golden ellipse using sine curves as the axes, and the result when a rotation is added to the translation.

Figure 6a shows the cycloid axes. Figure 6b shows an example of a translation of a golden ellipse using the axis of Fig. 6a. Figure 6c shows an example of a rotation added to a translation of a universal ellipse using the axis of Fig. 6a.

Using the models above as a base, this paper describes the application of Formulas 8 and 9 as methods of generating geometric patterns. Patterns generated using Formula

Table 1.   Universal circle used as axes and golden ellipse used as motifs ($r_1 = 2\tau$, $a_1 = 12\tau$, $b_1 = 1$, $c_1 = 1$, $f_1 : -10.2\pi \sim 306.6\pi$, $f_2 : -16.8\pi \sim 360\pi$).

	h	d_1	s_1	j	d_2	s_2
(c)	30	1	0.017π	60	—	—
(d)	30	1	0.017π	60	0.5	0.008π

(a) A universal circle as an axis

(b) An ellipse as a motif

(c) A Type A geometric pattern

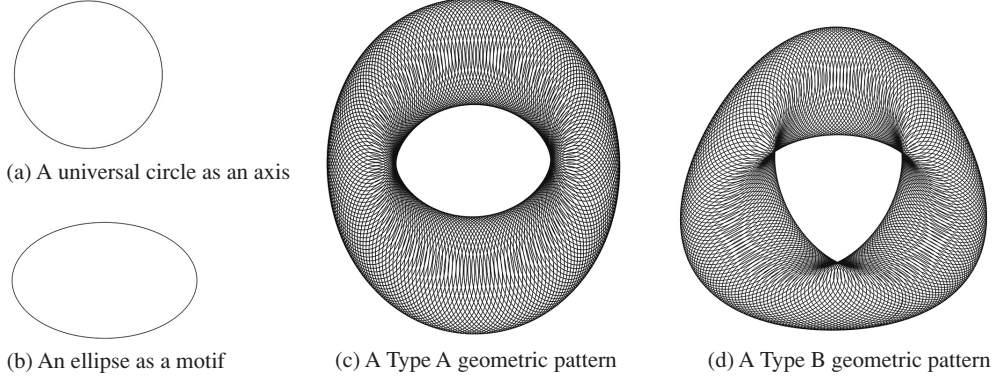(d) A Type B geometric pattern

Fig. 7. Geometric patterns using combinations of universal circle used as axes and golden ellipse used as motifs.

8 are referred to as Type A patterns (translation type), and patterns generated using Formula 9 as Type B patterns (type combining translation and rotation). The only figures used for transformation are figures that can be expressed by periodic functions.

The programs used were created by Mathematica.

3.   Geometric Pattern Generation

We investigated the generation of Type A and B geometric patterns by combining arbitrary axes and arbitrary motifs nd using Formulas 8 and 9. This paper describes the transformation of the geometric curves (quadratic curves) used as the axes into closed curves in three dimensions. In other words, these curves correspond to the knots of knot theory in topology. The f value in the z formula is handled as the knot generation coefficient.

Tables 1 to 3 and 5 to 13 show the important coefficient values for determining Type A or Type B mode in the diagrams.

To determine the relationship between h and j and d_1 and d_2, we have devised the equations $h = m2^n j$ (m: natural number; n: integer) and $d_2 = m2^n d_1$ (m: natural number; n: integer) for the method described in this paper. In each curve, f_1 indicates the initial/final value of the Type A period, and f_2 the initial/final value of the Type B period. The values of s_1 and s_2 are rounded to four decimal places.

The diagrams are all shown in parallel projection. Figure 7(a) is the universal cubic curve used as the axes, Fig. 7(b) is the quadratic curve used as the motif, Fig. 7(c) is a Type A geometric pattern, and Fig. 7(d) is a Type B geometric pattern.

The first step is to generate geometric patterns by transforming a circle to a cubic curve and using the curve as the axes and arbitrary curves as the motifs.

3.1   Geometric pattern generation with universal circle as axes

We used a circle (the simplest closed curve) as the quadratic curve. Using it as the axes, we transformed the quadratic curve to a cubic curve, then used the cubic curve to generate a Type A geometric pattern corresponding to Formula 8, and a Type B geometric pattern corresponding to Formula 9. This paper refers to this cubic curve as a universal circle. We selected typical quadratic curves as the motifs: ellipses, epicycloids, and hypocycloids.

Formula 8 can be used to express Type A by Formula 10, and Formula 9 can be used to express Type B by Formula 11. In these formulas, x_{mi}, y_{mi} and z_{mi} are the variables of the equation generating the pattern of Type A and x_{vi}, y_{vi} and z_{vi} are the variables of the equation generating the pattern of Type B. x_{ni} and y_{ni} are the motif equations. x_{nj} and y_{nj} are the equations that rotate the motifs (r_1: radius of circle; $r_1 > 0$).

$$x_{mi} = r_1 \sin g_1 + x_{ni}$$
$$y_{mi} = r_1 \cos g_1 + y_{ni}$$
$$z_{mi} = c_1 \sin g_1. \tag{10}$$

$$x_{vj} = r_1 \sin g_1 + x_{nj}$$
$$y_{vj} = r_1 \cos g_1 + y_{nj}$$
$$z_{vj} = c_1 \sin g_1. \tag{11}$$

The geometric patterns generated as described above are shown in Figs. 7 to 9.

3.1.1 Using golden ellipse as motifs

Restructuring Formulas 1 and 10 enables Type A geometric patterns to be expressed by Formula 12.

$$x_{m1} = r_1 \sin g_1 + x$$
$$y_{m1} = r_1 \cos g_1 + y$$
$$z_{m1} = c_1 \sin g_1. \tag{12}$$

Table 2. Universal circle used as axes and epicycloid used as motifs ($r_1 = 10$, $a_1 = 2$, $b_1 = 1$, $c_1 = 1$, $f_1 : -\pi \sim 252.5\pi$, $f_2 : -0.5 \sim 252.5\pi$).

	H	d_1	s_1	j	d_2	s_2
(c)	25	1	0.02π	50	—	—
(d)	25	1	0.02π	50	0.5	0.01π

(a) A universal circle as an axis

(b) An epicycloid as a motif

(c) A Type A geometric pattern
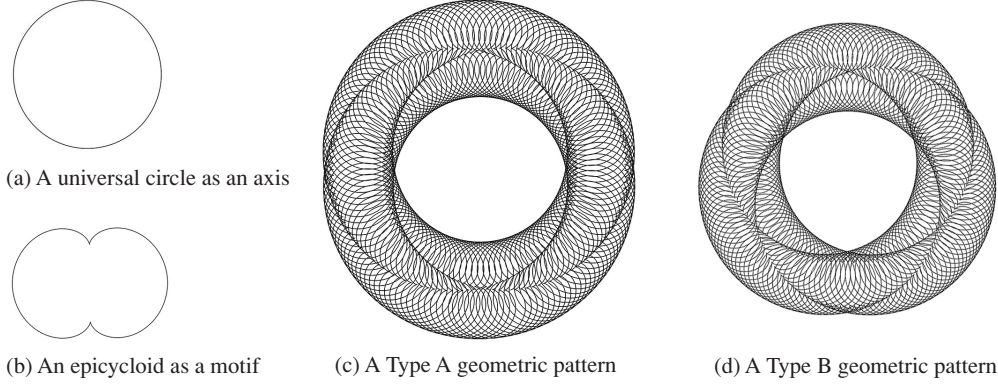
(d) A Type B geometric pattern

Fig. 8. Geometric patterns using combinations of universal circle used as axes and epicycloid used as motifs.

Restructuring Formulas 4 and 11 enable Type B geometric patterns to be expressed by Formula 13.

$$x_{v1} = r_1 \sin g_1 + x_2$$
$$y_{v1} = r_1 \cos g_1 + y_2$$
$$z_{v1} = c_1 \sin g_1. \tag{13}$$

Figure 7c shows a Type A geometric pattern generated from Formula 12. Figure 7d shows a Type B geometric pattern generated from Formula 13.

3.1.2 Using epicycloid as motifs

An epicycloid that can be expressed by a periodic function can be expressed as follows using t as an intermediate variable (a_2: radius of fixed circle; b_2: radius of rotating circle; $a_2 > b_2$):

$$x_3 = (a_2 + b_2) \cos t - b_2 \cos(a_2 + b_2)t/b_2$$
$$y_3 = (a_2 + b_2) \sin t - b_2 \sin(a_2 + b_2)t/b_2. \tag{14}$$

Restructuring Formulas 10 and 14 enables Type A geometric patterns to be expressed by Formula 15.

$$x_{m2} = r_1 \sin g_1 + x_3$$
$$y_{m2} = r_1 \cos g_1 + y_3$$
$$z_{m2} = c_1 \sin g_1. \tag{15}$$

We then created Formula 16 to achieve rotation of an epicycloid.

$$x_4 = x_3 \cos(s_2t)\pi/h - y_3 \sin(s_2t)\pi/h$$
$$y_4 = x_3 \sin(s_2t)\pi/h + y_3 \cos(s_2t)\pi/h. \tag{16}$$

Substituting $(s_2t)\pi/h = g_2$ into this equation enables Formula 16 to be expressed as Formula 17.

$$x_4 = x_3 \cos g_2 - y_3 \sin g_2$$
$$y_4 = x_3 \sin g_2 + y_3 \cos g_2. \tag{17}$$

Restructuring Formulas 11 and 17 enable Type B geometric patterns to be expressed by Formula 18.

$$x_{v2} = r_1 \sin g_1 + x_4$$
$$y_{v2} = r_1 \cos g_1 + y_4$$
$$z_{v1} = c_1 \sin g_1. \tag{18}$$

Figure 8c shows a Type A geometric pattern generated from Formula 15. Figure 8d shows a Type B geometric pattern generated from Formula 18.

3.1.3 Using hypocycloid as motifs

A hypocycloid that can be expressed by a periodic function can be expressed as follows using t as an intermediate variable (a_2: radius of fixed circle; b_2: radius of rotating circle; $a_2 > b_2$):

$$x_5 = (a_2 - b_2) \cos t + b_2 \cos(a_2 - b_2)t/b_2$$
$$y_5 = (a_2 - b_2) \sin t - b_2 \sin(a_2 - b_2)t/b_2. \tag{19}$$

Restructuring Formulas 10 and 19 enables Type A geometric patterns to be expressed by Formula 20.

$$x_{m3} = r_1 \sin g_1 + x_5$$
$$y_{m3} = r_1 \cos g_1 + y_5$$
$$z_{m3} = c_1 \sin g_1. \tag{20}$$

We then created Formula 21 to achieve rotation of a hypocycloid.

$$x_6 = x_5 \cos g_2 - y_5 \sin g_2$$
$$y_6 = x_5 \sin g_2 + y_5 \cos g_2. \tag{21}$$

Restructuring Formulas 11 and 21 enables Type B geometric patterns to be expressed by Formula 22.

$$x_{v3} = r_1 \sin g_1 + x_6$$
$$y_{v3} = r_1 \cos g_1 + y_6$$
$$z_{v3} = c_1 \sin g_1. \tag{22}$$

Table 3. Universal circle used as axes and hypocycloid used as motifs ($r_1 = 10$, $a_2 = 4.5$, $b_2 = 1.5$, $c_1 = 1$, $f_1 : 0.0 \sim 324.0\pi$, $f_2 : -0.0 \sim 324.0\pi$).

	h	d_1	s_1	j	d_2	s_2
(c)	20	0.5	0.013π	40	—	—
(d)	20	0.5	0.013π	40	1	0.025π

(a) A universal circle as an axis

(b) A hypocycloid as a motif   (c) A Type A geometric pattern   (d) A Type B geometric pattern
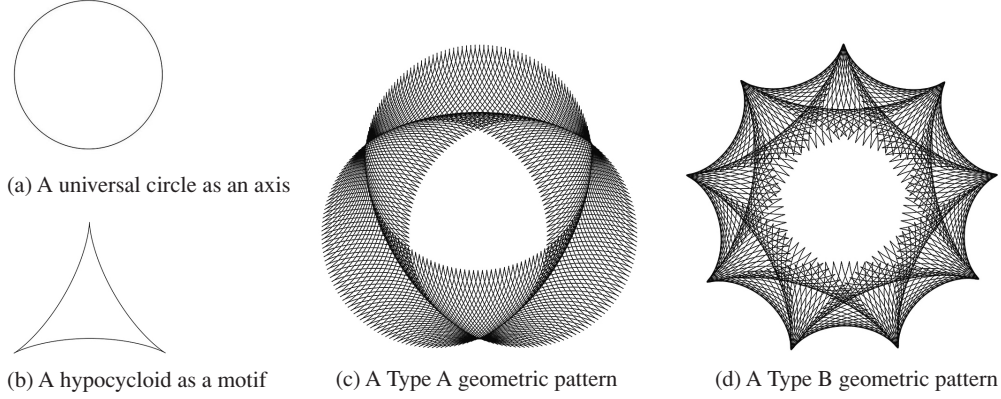
Fig. 9. Geometric patterns using combinations of universal circle used as axes and hypocycloid used as motifs.

Figure 9c shows a Type A geometric pattern generated from Formula 20. Figure 9d shows a Type B geometric pattern generated from Formula 22.

3.2 Setting for generating geometric patterns using combinations of curves as axes and curves as motifs

The second step is to generate geometric patterns using selected universal cubic curves as the axes and selected quadratic curves as the motifs.

3.2.1 Curves used as axes

In the method described in this paper, the curves selected for use as motifs are ellipses, epicycloids, and hypocycloids (which are quadratic curves). These curves are transformed into cubic curves and used as the axes to generate geometric patterns. Each of these transformed cubic curves is named according to the quadratic curve it was created from. For example, a cubic curve transformed from an epicycloid is referred to as a universal epicycloid. This paper treats these curves as the equivalent of knots in topology. Three types of knots were selected unknots. The relationship between a curve transformed into a cubic curve for use as the axes and its knot is as follows:

a. Universal golden ellipse: unknots
b. Universal epicycloid: unknots
c. Universal hypocycloid: unknots

3.2.2 Curves sed as motifs

Curves A_2 to C_2 shown below were selected as the curves used as motifs. In other words, the quadratic curves that are the motifs are the same as the parallel projections of the cubic curves used as the axes.

a. golden ellipse
b. epicycloid
c. hypocycloid

3.2.3 Combination of curves used as axes and curves used as motifs

We investigated the generation of geometric patterns by

Table 4. Curves used as axes and curves used as motifs.

	A_2	B_2	C_2
A_1	A_1-A_2	A_1-B_2	A_1-C_2
B_1	B_1-A_2	B_1-B_2	B_1-C_2
C_1	C_1-A_2	C_1-B_2	C_1-C_2

combining the curves used as the axes and the curves used as the motifs. Table 1 shows the combinations (A_1 to C_1 indicate the figures used as the axes, and A_2 to C_2 indicate the figures used as the motifs).

3.3 Generating geometric patterns using combinations of curves used as axes and curves used as motifs

Using the combinations shown in Table 4, we investigated the generation of geometric patterns by combining the curves used as the axes and the curves used as the motifs. Type A geometric patterns were created by restructuring the curve's equation with Formula 8, and Type B geometric patterns by restructuring the curve's equation with Formula 9. Figures 10 to 18 show the generated geometric patterns. For each diagram, Fig. 10(a) is the curve used as the axes, Fig. 10(b) is the curve used as the motifs, Fig. 10(c) is a Type A patterns, and Fig. 10(d) is a Type B patterns.

3.3.1 Using universal golden ellipses as axes

a. Using golden ellipse as motifs (A_1-A_2)

Formula 1 is restructured with Formula 8 to express the Type A geometric pattern by Formula 23 (a_3: major axis of the golden ellipse as an axis; b_3: minor axis of the golden ellipse as an axis; $a_3 = \tau b_3$).

$$x_{m6} = a_3 \sin g_1 + x$$
$$y_{m6} = b_3 \cos g_1 + y$$
$$z_{m6} = c_3 \sin g_1. \tag{23}$$

Formula 4 is restructured with Formula 9 to express the

Table 5. Universal golden ellipse used as axes and golden ellipse used as motifs ($a_3 = 10\tau$, $b_3 = 10$, $c_3 = 1$, $a_1 = 5\tau$, $b_1 = 5$, $f_1 : -1.5\pi \sim 363\pi$, $f_2 : -30\pi \sim 363\pi$).

	H	d_1	s_1	j	d_2	s_2
(c)	30	1	0.033π	30	—	—
(d)	30	1	0.003π	30	0.5	0.017π

(a) A universal golden ellipse as an axis

(b) A golden ellipse as a motif   (c) A Type A geometric pattern   (d) A Type B geometric pattern
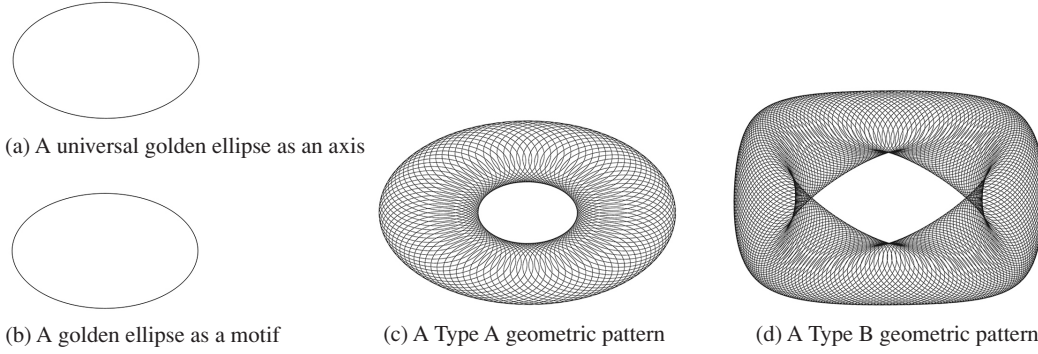
Fig. 10. Geometric patterns using combinations of universal golden ellipse used as axes and golden ellipse used as motifs.

Table 6. Universal golden ellipse used as axes and epicycloid used as motifs ($a_3 = 10\tau$, $b_3 = 10$, $c_3 = 1$, $a_2 = 2.0$, $b_2 = 1$, $f_1 : -0.5\pi \sim 181.5\pi$, $f_2 : -0.5\pi \sim 363\pi$).

	H	d_1	s_1	j	d_2	s_2
(s)	30	1	0.033π	30	—	—
(d)	30	0.5	0.017π	30	1	0.033π

(a) A universal golden ellipse as an axis

(b) An epicycloid as a motif   (c) A Type A geometric pattern   (d) A Type B geometric pattern
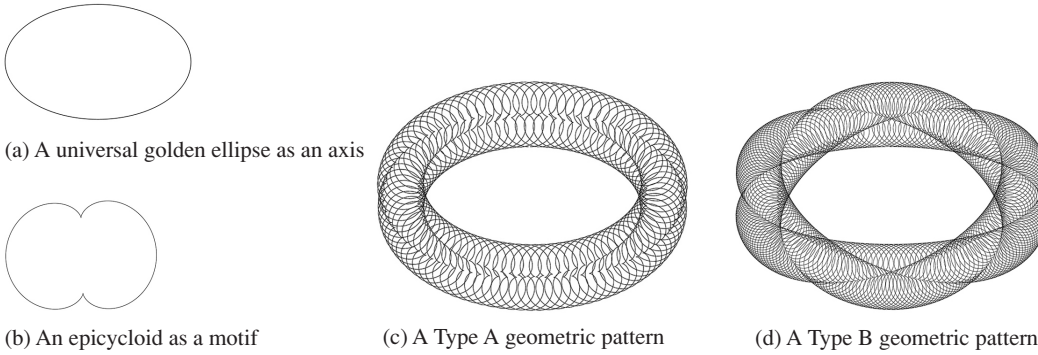
Fig. 11. Geometric patterns using combinations of universal golden ellipse used as axes and epicycloid used as motifs.

Type B geometric pattern by Formula 24.

$$x_{v6} = a_3 \sin g_1 + x_2$$
$$y_{v6} = b_3 \cos g_1 + y_2$$
$$z_{v6} = c_3 \sin g_1. \tag{24}$$

Figure 10c shows a Type A geometric pattern generated from Formula 23. Figure 10d shows a Type B geometric pattern generated from Formula 24.
b. Using epicycloid as motifs (A_1-B_2)
Restructuring Formulas 8 and 14 enable Type A geometric patterns to be expressed by Formula 25.

$$x_{m7} = a_3 \sin g_1 + x_3$$
$$y_{m7} = b_3 \cos g_1 + y_3$$
$$z_{m7} = c_3 \sin g_1. \tag{25}$$

Restructuring Formulas 9 and 16 enable Type B geometric patterns to be expressed by Formula 26.

$$x_{v7} = a_3 \sin g_1 + x_4$$
$$y_{v7} = b_3 \cos g_1 + y_4$$
$$z_{v7} = c_3 \sin g_1. \tag{26}$$

Figure 11c shows a Type A geometric pattern generated from Formula 25. Figure 11d shows a Type B geometric pattern generated from Formula 26.
c. Using hypocycloid as motifs (A_1-C_2)
Restructuring Formulas 8 and 19 enable Type A geometric patterns to be expressed by Formula 27.

$$x_{m8} = a_3 \sin g_1 + x_5$$
$$y_{m8} = b_3 \cos g_1 + y_5$$

Table 7. Universal golden ellipse used as axes and hypoycloid used as motifs ($a_3 = 10\tau$, $b_3 = 10$, $c_3 = 1$, $a_2 = 3.6$, $b_2 = 1.2$, $f_1 : -20\pi \sim 324\pi$, $f_2 : -15\pi \sim 364.5\pi$).

	h	d_1	s_1	j	d_2	s_2
(c)	40	1	0.025π	40	—	—
(d)	30	0.5	0.017π	30	0.5	0.017π

(a) A universal ellipse as an axis

(b) A hypocycloid as a motif

(c) A Type A geometric pattern
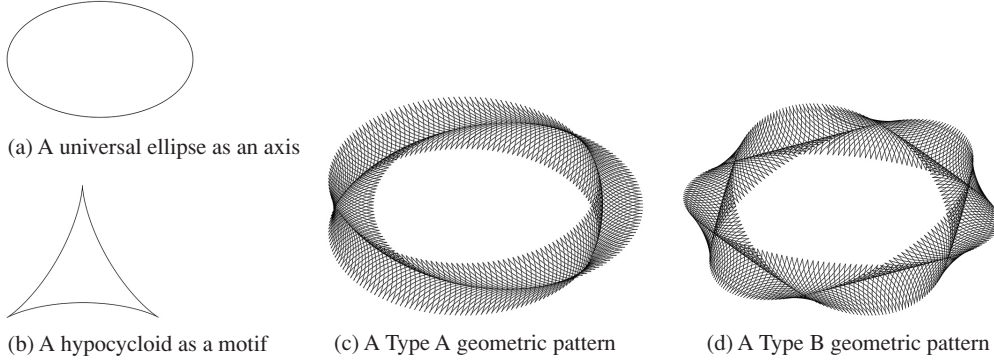
(d) A Type B geometric pattern

Fig. 12. Geometric patterns using combinations of universal golden ellipse used as axes and hypocycloid used as motifs.

Table 8. Universal epicycloid used as axes and golden ellipse used as motifs ($a_3 = 2$, $b_3 = 1$, $c_3 = 1$, $a_1 = 0.6\tau$, $b_1 = 0.6$, $f_1 : 0 \sim 363\pi$, $f_2 : 0 \sim 546\pi$).

	h	d_1	s_1	j	d_2	s_2
(c)	30	0.5	0.017π	30	—	—
(d)	30	0.5	0.017π	30	2	0.067π

(a) A universal epicycloid as an axis

(b) A golden ellipse as a motif

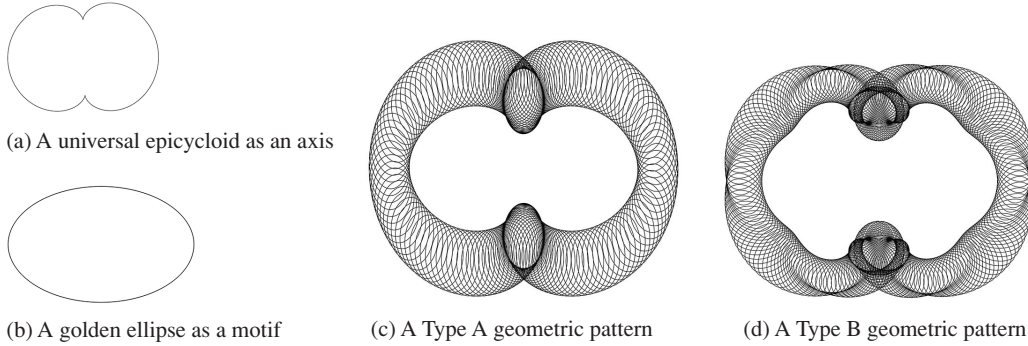(c) A Type A geometric pattern

(d) A Type B geometric pattern

Fig. 13. Geometric patterns using combinations of universal epicycloid used as axes and golden ellipse used as motifs.

$$z_{m8} = c_3 \sin g_1. \tag{27}$$

Restructuring Formulas 9 and 21 enable Type B geometric patterns to be expressed by Formula 28.

$$x_{v8} = a_3 \sin g_1 + x_6$$
$$y_{v8} = b_3 \cos g_1 + y_6$$
$$z_{v8} = c_3 \sin g_1. \tag{28}$$

Figure 12c shows a Type A geometric pattern generated from Formula 27. Figure 12d shows a Type B geometric pattern generated from Formula 28.

3.3.2 Using universal epicycloid as axes

a. Using golden ellipse as motifs (B_1-A_2)

Formula 1 is restructured with Formula 8 to express the Type A geometric pattern by Formula 14 (a_3: radius of fixed circle of epicycloid as an axis; b_3: radius of rotating circle of epicycloid as an axis).

$$x_{m11} = (a_3 + b_3) \cos g_1 - b_3 \cos(a_3 + b_3)g_1/b_3 + x$$
$$y_{m11} = (a_3 + b_3) \sin g_1 - b_3 \sin(a_3 + b_3)g_1/b_3 + y$$
$$z_{m11} = c_3 \sin g_1. \tag{29}$$

Formula 4 is restructured with Formula 29 to express the Type B geometric pattern by Formula 30.

$$x_{v11} = (a_3 + b_3) \cos g_1 - b_3 \cos(a_3 + b_3)g_1/b_3 + x_2$$
$$y_{v11} = (a_3 + b_3) \sin g_1 - b_3 \sin(a_3 + b_3)g_1/b_3 + y_2$$
$$z_{v11} = c_3 \sin g_1. \tag{30}$$

Figure 13c shows a Type A geometric pattern generated

Table 9. Universal epicycloid used as axes and epicycloid used as motifs ($a_3 = 12$, $b_3 = 1$, $c_3 = 1$, $a_2 = 3$, $b_2 = 1$, $f_1 : 0 \sim 366\pi$, $f_2 : 0 \sim 366\pi$).

	h	d_1	s_1	j	d_2	s_2
(c)	30	0.5	0.017π	30	—	—
(d)	30	0.5	0.017π	30	0.5	0.017π

(a) A universal epicycloid as an axis

(b) An epcycloid as a motif

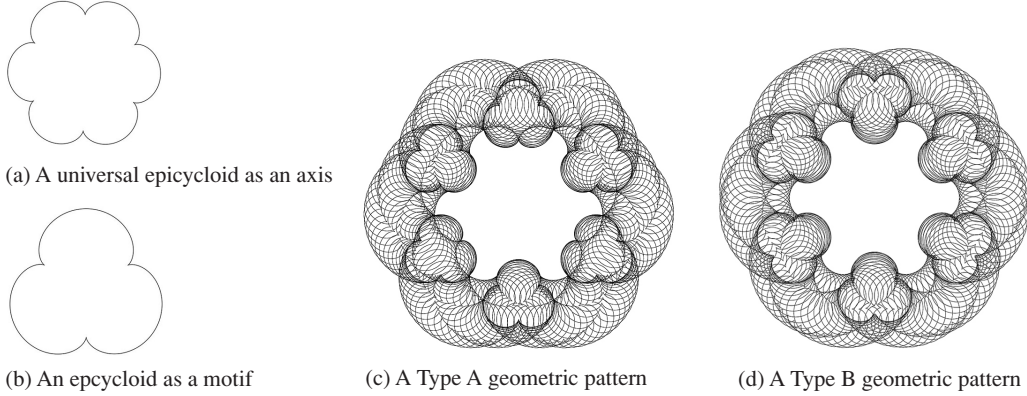(c) A Type A geometric pattern

(d) A Type B geometric pattern

Fig. 14. Geometric patterns using combinations of universal epicycloid used as axes and epicycloid used as motifs.

Table 10. Universal epicycloid used as axes and hypocycloid used as motifs ($a_3 = 21$, $b_3 = 7$, $c_3 = 1$, $a_2 = 12$, $b_2 = 4$, $f_1 : 0 \sim 360\pi$, $f_2 : 0 \sim 488\pi$).

	H	d_1	s_1	j	d_2	s_2
(c)	30	0.5	0.017π	30	—	—
(d)	20	1/6	0.008π	20	1/6	0.008π

(a) A universal epicycloid as an axis

(b) A hypocycloid as a motif

(c) A Type A geometric pattern
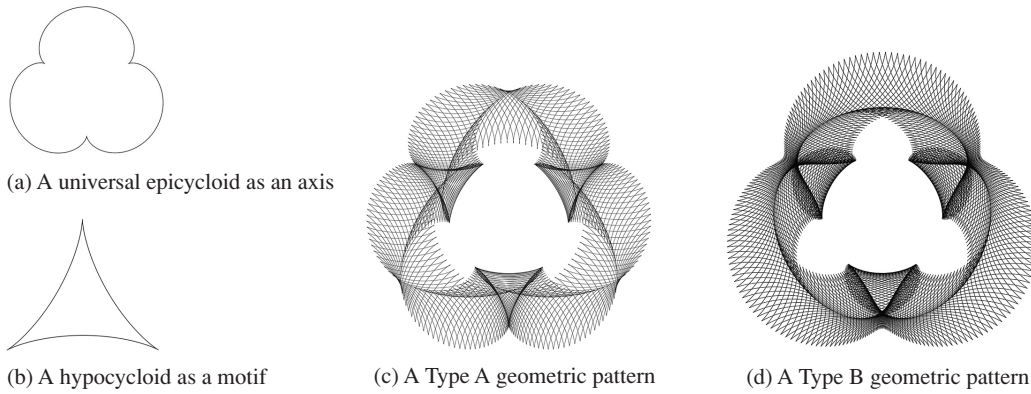
(d) A Type B geometric pattern

Fig. 15. Geometric patterns using combinations of universal epicycloid used as axes and hypocycloid used as motifs.

from Formula 29. Figure 13d shows a Type B geometric pattern generated from Formula 30.

b. Using epicycloid as motifs (B_1-B_2)

Type A patterns were created by restructuring Formula 14 with Formula 29, and Type B patterns by restructuring Formula 17 with Formula 30.

Figure 14c shows a Type A geometric pattern using epicycloid as a motif. Figure 14d shows a Type B geometric pattern using epicycloid as a motif.

c. Using hypocycloid as motifs (B_1-C_2)

Type A geometric patterns were created by restructuring Formula 19 with Formula 29, and Type B geometric patterns by restructuring Formula 22 with Formula 30.

Figure 15c shows a Type A geometric pattern using

hypocycloid as a motif Fig. 15d shows a Type B geometric pattern using hypocycloid as a motif.

3.3.3 Using universal hypocycloid as axes

a. Using golden ellipses as motifs (C_1-A_2)

Formula 1 is restructured with Formula 8 to express the Type A geometric pattern by Formula 19 (a_3: radius of fixed circle of hypocycloid as an axis; b_3: radius of the rotating circle of hypocycloid as an axis).

$$x_{m12} = (a_3 - b_3) \cos g_1 + b_3 \cos(a_3 - b_3)g_1/b_3 + x$$
$$y_{m12} = (a_3 - b_3) \sin g_1 - b_3 \sin(a_3 - b_3)g_1/b_3 + y$$
$$z_{m12} = c_3 \sin g_1. \tag{31}$$

Formula 4 is restructured with Formula 9 to express the

Table 11.   Universal hypocycloid used as axes and hypocycloid used as motifs ($a_3 = 24$, $b_3 = 3$, $c_3 = 1$, $a_1 = 5\tau$, $b_1 = 5.0$, $f_1 : 0 \sim 660\pi$, $f_2 : 0 \sim 490\pi$).

	H	d_1	s_1	j	d_2	s_2
(c)	20	0.125	0.013π	10	—	—
(d)	30	0.125	0.013π	10	0.125	0.013π

(a) A universal hypoycloid as an axis

(b) A golden ellipse as a motif

(c) A Type A geometric pattern
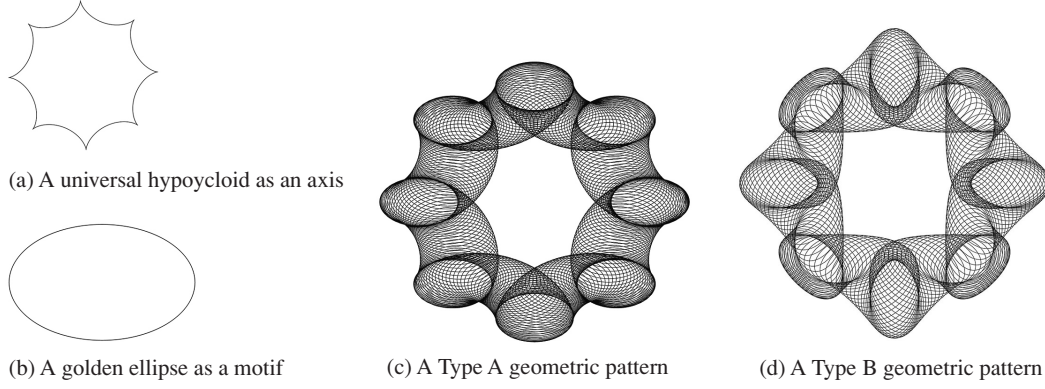
(d) A Type B geometric pattern

Fig. 16.   Geometric patterns using combinations of universal hypocycloid used as axes and golden ellipse used as motifs.

Table 12.   Universal hypocycloid used as axes and epicycloid used as motifs ($a_3 = 32$, $b_3 = 8$, $c_3 = 1$, $a_2 = 6\tau$, $b_2 = 1.0$, $f_1 : 0 \sim 360\pi$, $f_2 : 0 \sim 500\pi$).

	h	d_1	s_1	j	d_2	s_2
(c)	30	0.5	0.017π	30	—	—
(d)	25	0.25	0.01π	25	0.25	0.01π

(a) A universal hypocycloid as an axis

(b) An epcycloid as a motif

(c) A Type A geometric pattern
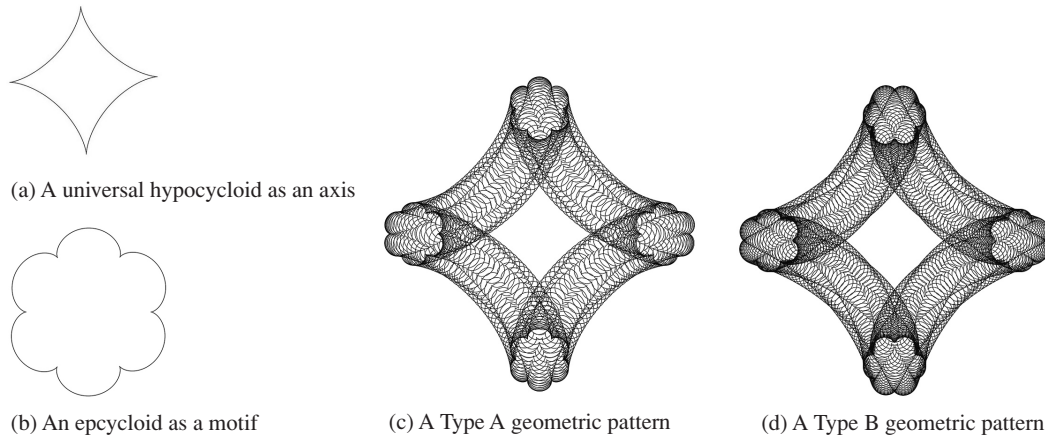
(d) A Type B geometric pattern

Fig. 17.   Geometric patterns using combinations of universal hypocycloid used as axes and epicycloid used as motifs.

Type B geometric pattern by Formula 32.

$$x_{v12} = (a_3 - b_3) \cos g_1 - b_2 \cos(a_3 - b_3)g_1/b_3 + x_2$$
$$y_{v12} = (a_3 - b_3) \sin g_1 - b_2 \sin(a_3 - b_3)g_1/b_3 + y_2$$
$$z_{v12} = c_3 \sin g_1. \tag{32}$$

Type A geometric patterns were created by restructuring Formula 1 with Formula 31, and a Type B geometric pattern by restructuring Formula 4 with Formula 32. Figure 16c shows a Type A geometric pattern using golden ellipse as a motif. Figure 16d shows a Type B geometric pattern using epicycloid as a motif.

b. Using epicycloid as motifs (B_1-B_2)

Type A geometric patterns were created by restructuring Formula 14 with Formula 31, and Type B geometric patterns by restructuring Formula 17 with Formula 32.

Figure 17c shows a Type A geometric pattern using epicycloid as a motif. Figure 16d shows a Type B geometric pattern using epicycloid as a motif.

c. Using hypocycloid as motifs (B_1-C_2)

Type A geometric patterns were created by restructuring Formula 19 with Formula 31, and Type B geometric patterns by restructuring Formula 21 with Formula 32.

Table 13. Universal hypocycloid used as axes and hypocycloid used as motifs ($a_3 = 12$, $b_3 = 2$, $c_3 = 1$, $a_2 = 3\tau$, $b_2 = 1$, $f_1 : 0 \sim 640\pi$, $f_2 : 0 \sim 730\pi$).

	h	d_1	s_1	j	d_2	s_2
(c)	40	0.5	0.013π	40	—	—
(d)	30	0.333	0.017π	10	0.333	0.033π

(a) A universal hypocycloid as an axis

(b) A hypocycloid as a motif

(c) A Type A geometric pattern
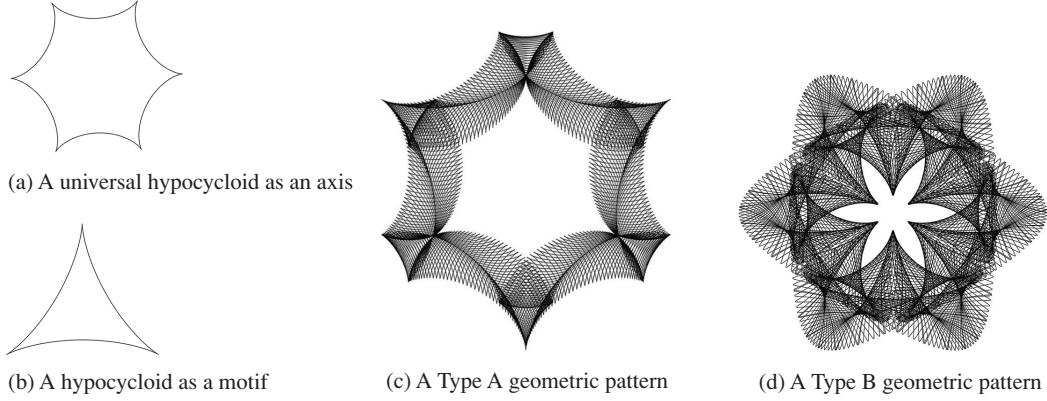
(d) A Type B geometric pattern

Fig. 18. Geometric patterns using combinations of universal hypocycloid used as axes and hypocycloid used as motifs.

Figure 18c shows a Type A geometric pattern using hypocycloid as a motif. Figure 18d shows a Type B geometric pattern using hypocycloid as a motif.

4. Discussion

We generated geometric patterns using four types of geometric curves as the axes and three types of geometric curves as motifs. The key points of the method described in this paper are as follows: 1) We devised equations corresponding to the Type A and Type B geometric patterns devised to determine geometric shapes. 2) We devised the coefficient equation corresponding to Type A geometric patterns ($s_1 = d_1\pi/J$), and the coefficient equation corresponding to Type B geometric patterns ($s_2 = d_2\pi/J$). 3) We devised the equations $h = m2^n j$ (m: natural number; n: integer) and $d_2 = m2^n d_1$, which are used when determining the values of the h, j, d_1 and d_2 coefficients that determine the pattern shape. We used these key points to devise a methodology for generating geometric patterns. We investigated the effectiveness of this methodology. Our findings can be summarized as follows:

4.1 Determining factors for pattern generation and pattern features

The shape of the generated pattern is determined by the values of the coefficients h, j, d_1, and d_2. We generated several patterns using a wide range of different values for each of the coefficients and determined that the values below are appropriate for our method. Type A and Type B geometric pattern shape differences are determined by the combinations of values used for the h and j, and d_1 and d_2 coefficients. To set the d_1 and d_2 coefficients, we set d_1 for Type A geometric patterns and d_1 for Type B geometric patterns to the same value or to different values.

a. Using universal circle as axes

When using a universal circle as the axes: To set the h and j coefficients, we determined the value of h by setting

$j = 40, 50$ or 60; $m = 1$ and $n = -1$ for Type A geometric patterns. To set the d_1 and d_2 coefficients, we determined value of d_2 by setting $d_1 = 0.5$ or 1; $m = 1$; and $n = -1$ or 1 for Type B geometric patterns.

With golden ellipse as motif: Figure 7c shows an example of a generated Type A geometric pattern for which $d_1 = 1$. The pattern in Fig. 7c is a convex curve in the shape of the pattern opening. Figure 7d shows an example of a generated Type B geometric pattern for which $d_1 = 1$ and $d_2 = 0.5$. The pattern in Fig. 7d is a convex curve with three cusps in the shape of the pattern opening. Varying the value of m to 3, 4, and 5 at inflection point d_2 enables generation of variations in the shape of the pattern opening.

With epicycloid as a motif: The curve shown in Fig. 8b is a closed curve with two inflection points. The pattern's internal shape is therefore determined by the number of inflection points, in contrast to a golden ellipse. Figure 8c shows an example of a Type A geometric pattern generated with $d_1 = 1$. Figure 8d shows an example of a Type B geometric pattern generated with $d_1 = 1$ and $d_2 = 0.5$. It is considered that the shape of the pattern opening of Fig. 8c is a convex curve with two inflection points and the shape of the pattern opening Fig. 8d is a convex curve with three inflection points.

With hypocycloid as a motif: The curve shown in Fig. 9b is a closed curve with three inflection points. Figure 9c shows an example of a Type A geometric pattern generated with $d_1 = 0.5$. Figure 9d shows an example of a Type B geometric pattern generated with $d_1 = 0.5$ and $d_2 = 1$. The visual outline of the pattern of Fig. 9d is a closed curve with nine inflection points.

b. Using universal golden ellipse as axes

When using a universal golden ellipse as the axes: To set the h and j coefficients, we determined the value of h by setting $j = 30$, or 40; $m = 1$ and $n = 0$. To set the

d_1 and d_2 coefficients, we determined the value of d_2 by setting $d_1 = 0.5$ or 1; $m = 1$; and $n = -1$, 0 or 1. In contrast to a universal circle, the use of a universal golden ellipse as the axes exhibits features in the major axis and the minor axis directions when the motif is subjected to affine transformations.

With golden ellipse as motif: Figure 10c shows an example of a generated Type A geometric pattern for which $d_1 = 1$. The pattern in Fig. 10c is an ellipse in the shape of the pattern opening. Figure 10d shows an example of a generated Type B geometric pattern for which $d_1 = 1$ and $d_2 = 0.5$. The pattern in Fig. 10d is a convex curve with three inflection points in the shape of the pattern opening. Just as when using a universal circle as the axes, varying the value of m to 3, 4 and 5 enables generation of variations in the shape of the pattern opening.

With epicycloid as a motif: The motif is the same as curve shown in Fig. 8b, so the shape of the axes is seen in pattern shape differences. Figure 11c shows an example of a Type A pattern generated with $d_1 = 1$. The pattern in Fig. 11c is a convex curve with two inflection points in the shape of the pattern opening. Figure 11d shows an example of a Type B geometric pattern generated with $d_1 = 0.5$ and $d_2 = 1$. The pattern in Fig. 11d is a convex curve with seven inflection points in the shape of the pattern opening.

With hypocycloid as a motif: The motif is the same as curve shown in Fig. 9b. Figure 12c shows an example of a Type A geometric pattern generated with $d_1 = 1$. Figure 12d shows an example of a Type B geometric pattern generated with $d_1 = 0.5$ and $d_2 = 0.5$. The value of coefficient d_2 is more effective at generating a complex shape in Type B geometric pattern than in Type A geometric pattern.

c. Using universal epicycloid as axes

When using a universal epicycloid as the axes: To set the h and j coefficients, we determined the value of h by setting $j = 40$, 50 or 60; $m = 1$ and $n = 0$. To set the d_1 and d_2 coefficients, we determined the value of d_2 by setting $d_1 = 0.5$ or 1; $m = 1$; and $n = -1$, 0 or 1. When using a universal epicycloid as the axes, the axes are unknots, while also being handled as closed curves. Curves are quantified according to the values of their inflection points. Geometric patterns are given features according to how the motif placement is set at the inflection points of each axis.

With golden ellipse as a motif: The curve used as the axes shown in Fig. 13a is a convex curve with two inflection points. Figure 13c shows an example of a Type A geometric pattern generated with $d_1 = 0.5$. Figure 13d shows an example of a Type B geometric pattern generated with $d_1 = 0.5$ and $d_2 = 2$. The value of rotation coefficient d_2 is more effective at generating a complex shape in Type B geometric pattern than in Type A geometric pattern.

With epicycloid as a motif: The curve used as the axes shown in Fig. 14a is a closed curve with six inflection points. The curve used as the motif shown in Fig. 14b is a closed curve with three inflection points. Figure 14c shows an example of a Type A geometric pattern generated with $d_1 = 0.5$. Figure 14d shows an example of a Type B geometric pattern generated with $d_1 = 0.5$ and $d_2 = 0.5$. For the Type A and Type B geometric pattern motifs, we varied the value of d_1 and d_2 and set rotation angles of 60° and 180° for each.

With hypocycloid as a motif: The curve used as the axes shown in Fig. 15a is a closed curve with three inflection points. The curve used as the motif shown in Fig. 15b is a closed curve with three inflection points. Figure 15c shows an example of a Type A geometric pattern generated with $d_1 = 0.5$. Figure 15d shows an example of a Type B geometric pattern generated with $d_1 = 1/6$ and $d_2 = 1/6$. The value of coefficient d_2 is more effective at generating a complex shape in Type B geometric pattern than in Type A geometric pattern.

d. Using universal hypocycloid as axes

When using a universal hypocycloid as the axes: To set the h and j coefficients, we determined the value of h by setting $j = 40$, 50 or 60; $m = 1$, 2 or 3; and $n = -1$ or 0. To set the d_1 and d_2 coefficients, we determined the value of d_2 by setting $d_1 = 0.5$ or 1; $m = 1$; and $n = -1$, 0 or 1. When using a universal hypocycloid as the axes, the axes are unknots, while also being handled as closed curves. Curves are quantified according to the values of their inflection points. Geometric patterns are given features according to how the motif placement is set at the inflection points of each curve as an axis.

With golden ellipse as a motif: The curve used as the axes shown in Fig. 16a is a closed curve with eight inflection points. Figure 16c shows an example of a Type A geometric pattern generated with $d_1 = 0.125$. The pattern in Fig. 16c is a closed curve with eight inflection points in the shape of the pattern opening. Figure 16d shows an example of a Type B geometric pattern generated with $d_1 = 0.125$ and $d_2 = 0.125$. The pattern in Figs. 16c and 16d are closed curves with eight inflection points in the shape of the pattern opening. The rotation angle of the motif when placed at inflection points is 180° in Type A geometric patterns and 45° in Type B geometric patterns.

With epicycloid as a motif: The curve used as the axes shown in Fig. 17a is a closed curve with four inflection points. The curve used as the motif shown in Fig. 17b is a closed curve with eight inflection points. Figure 17c shows an example of a Type A geometric pattern generated with $d_1 = 0.5$. Figure 17d shows an example of a Type B geometric pattern generated with $d_1 = 0.25$ and $d_2 = 0.25$. The pattern in Figs. 17c and 17d are closed curves with four inflection points in the shape of the pattern opening.

With hypocycloid as a motif: The curve used as the axes shown in Fig. 18a is a closed curve with six inflection points. The curve used as the motif shown in Fig. 18b is a closed curve with three inflection points. Figure 18c shows an example of a Type A geometric pattern generated with $d_1 = 0.5$. The pattern in Fig. 18c is a closed curve with six inflection points in the shape of the pattern opening. Figure 18d shows an example of a Type B geometric pattern generated with $d_1 = 1/6$ and $d_2 = 1/3$. The pattern in Fig. 18d is a closed curve with twelve inflection points in the shape of the pattern opening.

The value of coefficient d_2 is more effective at generating a complex shape in Type B geometric pattern than in Type A geometric pattern.

4.2 Pattern generation evaluation

Using arbitrary cubic curves as the axes and arbitrary quadratic curves as the motif, we generated geometric patterns by means of the original method we devised. To enable an evaluation of the geometric patterns reported on in this paper, we invite candid feedback from those who work in the fields of art and design. We would like to investigate our method further, using the feedback to gain an objective assessment of whether it is useful for design work.

5. Conclusion

This paper has described our investigation of a method of generating geometric patterns. We transformed quadratic curves into cubic curves and defined them as universal cubic curves. Quadratic curves for use as motifs were selected from among golden ellipses, epicycloids, and hypocycloids. They were transformed into universal cubic curves for use as the axes, and subjected to translation or a combination of translation and rotation to generate each pattern. The results of our investigation of the use of universal cubic curves described in this paper indicate some potential application to geometry-based art and design works. Future work in this area could result in the generation of variations by combining types of forms used as axes and types of forms used as motifs.

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