Research of Pattern Formation, edited by R. Takaki, pp. 23–42.

Generalization of Golden Ratio: Any Regular Polygon Contains Self-Similarity

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Abstract. Aperiodic ordered structure of materials, quasicrystal, was discovered in aluminum manganese alloy in 1984 (Shechtman et al., 1984). It showed icosahedral symmetry which was forbidden in traditional crystallography. The Penrose tiling (Penrose, 1974, 1977) is the key-concept to understand such type of order. The author succeeded in extending the Penrose tiling to three dimension in 1985 (Ogawa, 1985). The present work is an unpublished part of the investigation of quasicrystals and Penrose tiling to understand the origin of self-similarity. The essential parts were carried out in early 1986 and reported at a symposium of the present project in January, 1988.

1. Introduction

Five-fold symmetry seems to locate at a special position among rotational symmetries. Only five-fold one was skipped in crystalline symmetry to which (one-, two-,) three-, four-, and six-fold symmetry belong. It is well known that golden ratio and Fibonacci sequence are associated with the self-similarity contained in a pentagon. It is also well known that the golden ratio is the simplest continued fraction. The generalization of the golden ratio τ to other integer k is straight forward.

$$\tau_k = k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \ldots\ldots}}}, \qquad \tau = \tau_1. \tag{1-1}$$

In any case, an irrational number τ_k

$$\tau_k = \left[k + \sqrt{k^2 + 4} \right] / 2 \tag{1-2}$$

and its inverse number with negative sign

$$\tau_k' = -\tau_k^{-1} = -\left[\sqrt{k^2 + 4} - k \right] / 2 \tag{1-3}$$

are the solutions of quadratic equation

$$\tau_k^2 - k\tau_k - 1 = 0. \tag{1-4}$$

It has a systematic rational approximation

$$k,\ \frac{k^2+1}{k},\ \frac{k^3+2k}{k^2+1},\ \frac{k^4+3k^2+1}{k^3+2k},\ \frac{k^5+4k^3+3k}{k^4+3k^2+1}, \ldots \ \to \tau_k. \tag{1-5}$$

Generally an infinite sequence

$$\{f_n(k)\} = \{1,\ k,\ k^2+1,\ k^3+2k,\ k^4+3k^2+1,\ k^5+4k^3+3k, \ldots\}$$
$$= \sum_{m=0}^{[n/2]} \frac{(n-m-1)!}{m!(n-2m-1)!} k^{n-2m-1} \tag{1-6}$$

which can be defined with a recursion formula of second order

$$f_{n+1} = kf_n + f_{n-1} \tag{1-7}$$

associates with it. The case of $k = 1$ corresponds to the Fibonacci sequence.

However, the generalization in this paper is toward another direction. It is based on the relation between the length of diagonals of regular G-gons with primitive number G. (Though G is restricted to primitive numbers for simplicity in this paper, the arguments go almost in parallel for any integer G. Something troublesome lies in the fact that star polygons are not always single-stroke figures.) Nevertheless, it is turned out later that there are some connections between two directions of generalization. The length d_n of the nth diagonal as a function of the shortest diagonal d_1 is very similar to the general form of the sequence in Eq. (1-6). The difference between them lies only in the sign of the term. It is not clear yet what is the essential fact connecting them.

The important relations for regular pentagons are summarized in Section 2. The similar relations are derived for regular heptagons in Section 3. The relations are generalized to regular G-gons, where G is an arbitrary primitive integer in Section 4. There, a more general viewpoint was introduced.

2. The Golden Ratio and Fibonacci Sequence Associated with Pentagon

In a regular pentagon P ($P_1P_2P_3P_4P_5$) of side length 1, the diagonals are only one kind and of length τ. Being the positive root of the quadratic equation

$$\tau^2 - \tau - 1 = 0, \qquad (2-1)$$

an irrational number

$$\tau = \left(\sqrt{5}+1\right)/2 = 1.6180339887\ldots \qquad (2-2)$$

is known as golden ratio. A regular pentagon is drawn together with its all diagonals in Fig. 1. A pentagram P' ($P_1P_3P_5P_2P_4$), consisting of five diagonals, is a star polygon which is sometimes referred to as a (5/2)-gon. Therein, a side of length τ is divided into three parts as

$$\tau = \frac{1}{\tau} + \frac{1}{\tau^2} + \frac{1}{\tau}. \qquad (2-3)$$

Two diagonals P_1P_3 and P_1P_4 divide P into three isosceles triangles $\Delta P_1P_2P_3$, $\Delta P_1P_3P_4$ and $\Delta P_1P_4P_5$. The side lengths and angles of $\Delta P_1P_2P_3$ and $\Delta P_1P_4P_5$ are $[1, 1, \tau]$ and $\langle\phi, \phi, 3\phi\rangle$ and those of $\Delta P_1P_3P_4$ are $[\tau, \tau, 1]$ and $\langle 2\phi, 2\phi, \phi\rangle$, where $\phi = \cos^{-1}(\tau/2) = \pi/5 = 36°$. The isosceles triangles which are associated with a regular pentagon are only these two kinds,

$$\Delta_\phi(1) = [1, 1, \tau]\langle\phi, \phi, 3\phi\rangle \qquad (2-4)$$

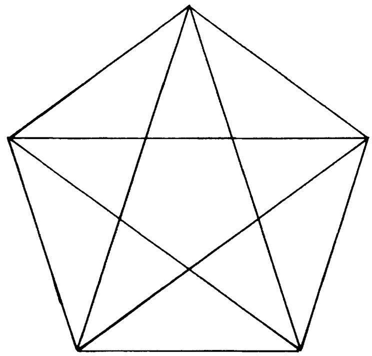

Fig. 1. A regular pentagon together with all of its diagonals. The diagonal length is the golden ratio $\tau \approx 1.6180$ when the edge length in the unit of the side length 1. See that there are two kinds of isosceles triangles listed in Eq. (2-4) and their divisions (Eq. (2-5)) therein.

$$\Delta_{2\phi}(\tau) = [\tau, \tau, 1]\langle 2\phi, 2\phi, \phi\rangle$$

where the notation $\Delta_\phi(1)$ and $\Delta_{2\phi}(\tau)$ are introduced so that the suffix denotes the magnitude of a base angle and the variable in the parenthesis denotes a side length. Any of these isosceles triangles of side length 1 was constructed with a set of similar elements of side length τ^{-1} as easily seen from Fig. 1.

$$\begin{aligned} \Delta_\phi(1) &= 2\Delta_\phi(1/\tau) + \Delta_{2\phi}(1/\tau) \\ \Delta_{2\phi}(1) &= \Delta_\phi(1/\tau) + \Delta_{2\phi}(1/\tau). \end{aligned} \qquad (2-5)$$

These relations between a set of the isosceles triangles and another set of the similar isosceles triangles express a kind of self-similarity. It is the prototype of the self-similarity seen in the Penrose tiling, which is the key concept of the ideal quasicrystal.

Fibonacci sequence $\{f_n\}$ is a semi-infinite sequence of integers which is closely connected with the self-similarity associated with a regular pentagon and with the golden ratio. It is defined by a recursion formula

$$f_0 = 0, \ f_1 = 1, \ f_{n+1} = f_n + f_{n-1}. \qquad (2-6)$$

The first 16 terms are as follows

$$\{f_n;\ n \ge 1\} = \{1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,\ldots\}. \quad (2-7)$$

The recursion formula in Eq. (2-6) is expressed in a matrix form

$$\mathrm{T} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ F_n = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix},\ F_0 = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F_{n+1} = \mathrm{T}F_n = \mathrm{T}^n F_1. \quad (2-8)$$

The matrix T may be referred to as the forward matrix. The eigenvalue equation of T is the same quadratic equation as Eq. (2-1) and the eigenvalues are τ and $1-\tau = -\tau^{-1}$. The corresponding eigenvectors are given by

$$v_1 \equiv \begin{bmatrix} 1 \\ \tau \end{bmatrix},\ v_2 \equiv \begin{bmatrix} -\tau \\ 1 \end{bmatrix}, \quad (2-9)$$

$$\mathrm{T}v_1 = \tau v_1,\ \mathrm{T}v_2 = -\tau^{-1} v_2. \quad (2-10)$$

The sequence can be extended to an infinite sequence with the terms of negative index

$$\{f_{-1}, f_{-2}, f_{-3}, f_{-4}, \ldots\} = \{1,-1,2,-3,5,-8,13,-21,\ldots\}. \quad f_{-n} = (-1)^{n+1} f_n \quad (2-11)$$

by the help of the backward matrix defined as the inverse matrix of T

$$\mathrm{T}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad F_{n-1} = \mathrm{T}^{-1} F_n. \quad (2-12)$$

The ratio f_{n+1}/f_n of the successive terms of the Fibonacci sequence tends in the limit of infinitely large n to the bigger eigenvalue τ

$$f_{n+1} / f_n \to \tau \quad (n \to \infty). \quad (2-13)$$

Apart from Eq. (2-8), T^n for an arbitrary integer n is directly related with Fibonacci sequence

$$\mathrm{T}^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix} \ \mathrm{T}^{-n} = \begin{bmatrix} f_{-n-1} & f_{-n} \\ f_{-n} & f_{-n+1} \end{bmatrix} = (-1)^n \begin{bmatrix} f_{n+1} & -f_n \\ -f_n & f_{n-1} \end{bmatrix} \quad (2-14)$$

$$\frac{\tau T^n}{f_n} \to \begin{bmatrix} 1 & \tau \\ \tau & \tau^2 \end{bmatrix} \quad (n \to \infty) \qquad (2-15)$$

and τ^n and f_n are mutually connected by

$$\tau^n = f_n \tau + f_{n-1}, \quad \tau^{-n} = (-1)^n [f_{n+1} - f_n \tau], \qquad (2-16)$$

$$f_n = \frac{\tau^n - (-\tau)^{-n}}{\tau - \tau^{-1}}. \qquad (2-17)$$

3. A Regular Heptagon

Let H $(H_1 H_2 H_3 H_4 H_5 H_6 H_7)$ be a regular heptagon of side length 1. It is shown together with its all diagonals in Fig. 2. The magnitude of the interior angle at a vertex is 5θ where $\theta = \pi/7$. There are only two kinds of diagonals α and $\beta(>\alpha)$. It is easy to see that

$$\alpha = 2\cos\theta \approx 1\ 801937736 \qquad (3-1)$$

and

$$\beta = \alpha\cos\theta + \cos 2\theta = 2\cos 2\theta + 1 = 4\cos^2\theta - 1 \approx 2\ 246979604 \qquad (3-2)$$

The relations

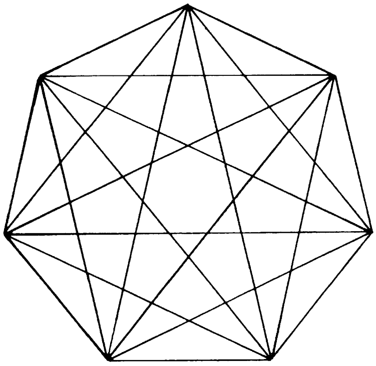

Fig. 2. A regular heptagon together with all of its diagonals. There are two kinds of diagonals $d_1 = \alpha \approx 1.8019$ and $d_2 = \beta \approx 2.2470$. See that there are three kinds of isosceles triangles listed in Eq. (3-8).

$$\alpha^2 = 1+\beta,\ \alpha\beta = \alpha+\beta,\ \beta^2 = 1+\alpha+\beta, \tag{3-3}$$

$$\alpha^{-1} = 1+\alpha-\beta,\ \beta^{-1} = \beta-\alpha,\ \beta/\alpha = \beta-1,\ \alpha/\beta = \alpha-1,\ \alpha^{-1}+\beta^{-1} = 1.$$

More generally, any rational power of α and β is expressed as linear combination of α, β and 1 with only integral coefficients.

There are two star polygons associated with a heptagon H; the one is a (7/2)-gon H' ($H_1H_3H_5H_7H_2H_4H_6$) of side length α and the other is a (7/3)-gon H'' ($H_1H_4H_7H_3H_6H_2H_5$) of side length β. In H', a side α is divided into three parts as

$$\alpha = \frac{1}{\alpha}+\frac{\beta}{\alpha^2}+\frac{1}{\alpha}, \tag{3-4}$$

cf.(2 – 3)

and in H'' a side β into five parts as

$$\beta = \frac{\alpha}{\beta}+\frac{1}{\beta^2}+\frac{1}{\alpha\beta}+\frac{1}{\beta^2}+\frac{\alpha}{\beta}. \tag{3-5}$$

cf.(2 – 3)

If all of fourteen diagonals of H are drawn, a diagonal of length a is divided into five parts

$$\alpha = \frac{1}{\alpha}+\frac{1}{\alpha\beta}+\frac{1}{\beta^2}+\frac{1}{\alpha\beta}+\frac{1}{\alpha}, \tag{3-6}$$

cf.(2 – 3)

and a diagonal of length β into seven parts

$$\beta = \frac{1}{\beta}+\frac{\alpha}{\beta^2}+\frac{1}{\beta^2}+\frac{1}{\alpha\beta}+\frac{1}{\beta^2}+\frac{\alpha}{\beta^2}+\frac{1}{\beta}. \tag{3-7}$$

cf.(2 – 3)

There are three isosceles triangles associated with a heptagon H of side length 1, with the same notation as in the case of a pentagon,

$$\Delta_\theta(1) = [1,1,\alpha]\langle\theta,\theta,5\theta\rangle,$$

$$\Delta_{2\theta}(\alpha) = [\alpha, \alpha, \beta]\langle 2\theta, 2\theta, 3\theta \rangle, \qquad (3-8)$$

$$\text{cf.}(2-4)$$

$$\Delta_{3\theta}(\beta) = [\beta, \beta, 1]\langle 3\theta, 3\theta, \theta \rangle.$$

These triangles are divided into the miniatures of these three triangles as shown in Fig. 3. These self-similar relations, are expressed as,

$$\Delta_{\theta}(1) = 2\Delta_{\theta}(1/\beta) + 2\Delta_{2\theta}(1/\beta) + \Delta_{3\theta}(1/\beta),$$

$$\Delta_{2\theta}(1) = 2\Delta_{\theta}(1/\beta) + 3\Delta_{2\theta}(1/\beta) + \Delta_{3\theta}(1/\beta), \qquad (3-9)$$

$$\text{cf.}(2-5)$$

$$\Delta_{3\theta}(1) = \Delta_{\theta}(1/\beta) + \Delta_{2\theta}(1/\beta) + \Delta_{3\theta}(1/\beta).$$

It is convenient to introduce the following two matrices A and B,

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad (3-10)$$

$$\text{cf.}(2-8)$$

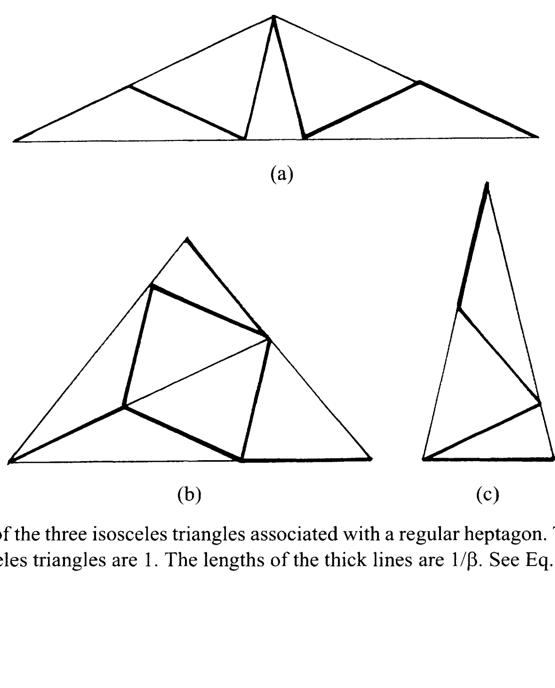

Fig. 3. Division of the three isosceles triangles associated with a regular heptagon. The side lengths of three isosceles triangles are 1. The lengths of the thick lines are $1/\beta$. See Eq. (3-9).

whose eigenvalue equations are respectively

$$\lambda^3 - \lambda^2 - 2\lambda + 1 = 0 \text{ for A,} \qquad (3-11)$$

cf.(2 − 1)

$$\text{and } \lambda^3 - 2\lambda^2 - \lambda + 1 = 0 \text{ for B.}$$

It is noted that the structure of matrices A and B are very similar to that of matrix T in Eq. (2-8) in the pentagonal case.

It is remarkable that all of the three vectors v_1, v_2 and v_3 defined by

$$v_1 = \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix}, \quad v_2 = \begin{bmatrix} \beta \\ 1 \\ -\alpha \end{bmatrix}, \quad v_3 = \begin{bmatrix} \alpha \\ -\beta \\ 1 \end{bmatrix}, \qquad (3-12)$$

cf.(2 − 9)

are common eigenvectors of A and B, respectively associated with the eigenvalues (α, $\beta-\alpha = 1/\beta$ and $1-\beta = -\beta/\alpha$) and (β, $1-\alpha = -\alpha/\beta$ and $1+\alpha-\beta = 1/\alpha$)

$$\mathrm{A}v_1 = \alpha v_1, \ \mathrm{A}v_2 = (\beta - \alpha)v_2, \ \mathrm{A}v_3 = -(\beta - 1)v_3, \qquad (3-13)$$

cf.(2 − 10)

$$\mathrm{B}v_1 = \beta v_1, \ \mathrm{B}v_2 = -(\alpha - 1)v_2, \ \mathrm{B}v_3 = (1 + \alpha - \beta)v_3.$$

It can be easily seen by making use of the relations in Eq. (3-3). It follows that all the relations with α and β are valid when they are replaced respectively by matrices A and B.

A similar semi-infinite sequences to Fibonacci sequence are defined for matrices A and B,

$$\mathrm{A}^n = \begin{bmatrix} a_{n-1} & a_n & a_{n+1} - a_{n-1} \\ a_n & a_{n+1} & a_{n+2} - a_n \\ a_{n+1} - a_{n-1} & a_{n+2} - a_n & a_{n+1} + a_n \end{bmatrix},$$

$$(3-14)$$

$$\mathrm{B}^n = \begin{bmatrix} b_{n-1} & b_n - b_{n-2} & b_n \\ b_n - b_{n-2} & b_n + b_{n-1} & b_{n+1} - b_{n-1} \\ b_n & b_{n+1} - b_{n-1} & b_{n+1} \end{bmatrix}. \qquad \text{cf.}(2-14)$$

It is remarked that these sequences are thus defined and the role of the forgoing matrix in this case is different from in Eq. (2-8). The recursion formulae for $\{a_n\}$ and $\{b_n\}$ are given by

$$a_{n+1} = a_n + 2a_{n-1} - a_{n-2}, \qquad (3-15)$$

$$\text{cf.}(2-6)$$

$$b_{n+1} = 2b_n + b_{n-1} - b_{n-2}$$

It is reasonable that there are some common features between Eqs. (3-15) and (3-11). Some starting terms of $\{a_n\}$ and $\{b_n\}$ are respectively

$$\{a_n;\ n \ge 1\} = \{1,0,2,1,5,5,14,19,42,66,131,221,417,728,\ldots\} \qquad (3-16)$$

$$\text{cf.}(2-7)$$

$$\{b_n;\ n \ge 1\} = \{1,1,3,6,14,31,70,157,353,793,\ldots\}.$$

The forward matrix for these sequences are respectively

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} \text{ for } \begin{bmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} \text{ for } \begin{bmatrix} b_{n-2} \\ b_{n-1} \\ b_n \end{bmatrix}. \qquad (3-17)$$

$$\text{cf.}(2-8)$$

In the limit of $n \to \infty$, they tend to

$$a_{n+1} / a_n \to \alpha, \ \ b_{n+1} / b_n \to \beta, \qquad (3-18)$$

$$\text{cf.}(2-13)$$

and

$$\frac{\mathrm{A}^n \alpha}{a_n} \to \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & \alpha^2 & \alpha\beta \\ \beta & \alpha\beta & \beta^2 \end{bmatrix}, \ \frac{\mathrm{B}^n \beta}{b_n} \to \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & \alpha^2 & \alpha\beta \\ \beta & \alpha\beta & \beta^2 \end{bmatrix}. \qquad (3-19)$$

$$\text{cf.}(2-15)$$

It follows from the relation that the sequences of the ratio of three integers $[a_{n-1}: a_n: (a_{n+1} - a_{n-1})]$ and $[b_{n-1}: (b_n - b_{n-2}): b_n]$ gives systematic rational approximation of an irrational ratio of three numbers $1:\alpha:\beta$. It is noted that matrices A and B are independently defined and sequences $\{a_n\}$ and $\{b_n\}$ are defined respectively only through matrices A and B. Although the sequences $\{a_n\}$ and $\{b_n\}$ were independently defined, they are closely connected with each other not only in their limit of $n \to \infty$ but also in their negative n's as shown below. By making use of the inverse matrices

$$\mathrm{A}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathrm{B}^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad (3-20)$$

cf. (2 – 12)

the sequences are naturally extended to the negative indices

$$\mathrm{A}^{-n} = \begin{bmatrix} a_{-n-1} & a_{-n} & a_{-n+1} - a_{-n-1} \\ a_{-n} & a_{-n+1} & a_{-n+2} - a_{-n} \\ a_{-n+1} - a_{-n-1} & a_{-n+2} - a_{-n} & a_{-n+1} + a_{-n} \end{bmatrix}. \qquad (3-21)$$

cf. (2 – 14)

It is obtained by replacing n routinely by $-n$ in Eq. (3-14). By the calculation of n-th power of A^{-1}, it is turned out that their values are expressed in terms of b's for positive n.

$$= \begin{bmatrix} b_{n+1} & b_n & b_{n-1} - b_{n+1} \\ b_n & b_{n-1} & b_{n-2} - b_n \\ b_{n-1} - b_{n+1} & b_{n-2} - b_n & b_{n-1} - b_n \end{bmatrix}, \qquad (3-22)$$

and similarly

$$\mathrm{B}^{-n} = \begin{bmatrix} b_{-n-1} & b_{-n} - b_{-n-2} & b_{-n} \\ b_{-n} - b_{-n-2} & b_{-n-1} + b_{-n} & b_{-n+1} - b_{-n-1} \\ b_{-n} & b_{-n+1} - b_{-n-1} & b_{-n+1} \end{bmatrix} \qquad (3-23)$$

$$= \begin{bmatrix} a_{n+1} & a_n - a_{n+2} & a_n \\ a_n - a_{n+2} & a_{n+1} + a_n & a_{n-1} - a_{n+1} \\ a_n & a_{n-1} - a_{n+1} & a_{n-1} \end{bmatrix}.$$

Therefore sequences $\{a_n\}$ and $\{b_n\}$ are unified into an infinite sequence (infinite in both directions) and they are mutually reversal order

$$a_n = b_{-n}, \ a_0 = b_0 = 0. \qquad (3-24)$$

cf.(2 – 11)

Finally, α^n and β^n are generally expressed as linear combination of 1, α and β

$$\alpha^n = a_{n-1} + a_n\alpha + (a_{n+1} - a_{n-1})\beta, \qquad (3-25)$$

cf.(2 – 16)

$$\beta^n = b_{n-1} + (b_{n+1} - b_{n-1})\alpha + b_n\beta$$

and the general expressions of a_n and b_n are

$$a_n = \frac{\alpha\beta - 1}{7}\left[\frac{1}{\beta}\alpha^n + \frac{1}{\alpha}\left(\frac{1}{\beta}\right)^n - \left(-\frac{\beta}{\alpha}\right)^n\right], \qquad (3-26)$$

cf.(2 – 17)

$$b_n = \frac{\alpha\beta - 1}{7}\left[\frac{1}{\alpha}\beta^n + \frac{1}{\beta}\left(\frac{1}{\alpha}\right)^n - \left(-\frac{\alpha}{\beta}\right)^n\right]$$

4. Arbitrary Regular Polygon

In a regular G-gon P $(P_1P_2P_3..., P_G)$ $(G \geq 3)$ of side length 1, there are $N_d = G(G-3)/2$ diagonals altogether. They are classified into D $(= g - 1)$ kinds, where $g \equiv [G/2]$ and the symbol $[x]$ stands for Gaussian operation of taking the maximum integer not exceeding x. For a while, for simplicity, G is confined to a prime number and then $D = (G - 3)/2$.

The unit of angle is chosen as $\theta \equiv \pi/G$. The circumradius R and the length of nth diagonals d_n are respectively given by

$$R = \frac{1}{2\sin\theta} \text{ and } d_n = \frac{\sin(n+1)\theta}{\sin\theta} \ (n = 1, 2, \ldots, D). \qquad (4-1)$$

An associated star polygon, (G/n)-gon $P^{(n)}$ $(P_1 P_{1+n} P_{1+2n} \ldots, P_{G-n} P_G)$, is defined as the regular figure obtained by connecting vertices with all the diagonal of $(n-1)$th kind when counted from the shortest.

Draw all the diagonals of a regular G-gon where G is an integer. Any two of N_d diagonals cross each other. The number of such cross points inside P is ${}_GC_4$. It is obvious that the angles and the lengths of segments appearing in it is very restricted as in the cases of pentagon in Section 2 and heptagon in Section 3. The similar relations are expected for any G.

Now, G is not necessarily a prime number. It is convenient to regard a side length 1 as the 0-th diagonal and take the first diagonal d_1 as the variable d.

$$d_0 = 1 \text{ and } d = d_1 = \frac{\sin 2\theta}{\sin\theta} = 2\cos\theta. \qquad (4-2)$$

By repeating application of addition theorem of trigonometry, an arbitrary d_n is written as a polynomial of d as

$$d_n = \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m)!}{m!(n-2m)!} d^{n-2m}. \qquad (4-3)$$

It should be pointed out that sequences $\{f_n k)\}$ in Eq. (1-6) and the length d_n as a function of d given in Eq. (4-3) are very similar to each other. Comparing $d_n(d)$ with $f_{n+1}(k)$, the only difference is that the coefficients in Eq. (1-6) are positive definite and those in Eq. (4-3) are alternating in signs. The explicit form of d_n are

$$\begin{aligned} d_2 &= d^2 - 1, \\ d_3 &= d^3 - 2d, \\ d_4 &= d^4 - 3d^2 + 1, \\ d_5 &= d^5 - 4d^3 + 3d, \\ d_6 &= d^6 - 5d^4 + 6d^2 - 1, \end{aligned} \qquad (4-4)$$

$$d_7 = d^7 - 6d^5 + 10d^3 - 4d.$$

These polynomials are unified in the form of a generating function

$$\frac{1}{1 - dz + z^2} = \sum_{n=0}^{\infty} d_n z^n. \qquad (4-5)$$

It is a special case of Gegenbauer polynomials whose generating function is given by

$$\left[1 - 2tz + z^2\right]^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(t) z^n, \qquad (4-6)$$

and the general form is

$$C_n^{\nu}(t) = \frac{(-1)^n}{2^n} \frac{\Gamma\left(\nu + \frac{1}{2}\right)\Gamma(n+\nu)}{\Gamma(2\nu)\Gamma\left(n + \nu + \frac{1}{2}\right)} \frac{\left(1 - t^2\right)^{(1/2)-\nu}}{n!} \frac{d^n}{dt^n}\left[\left(1 - t^2\right)^{n+\nu-(1/2)}\right]. \qquad (4-7)$$

The function is closely connected with the Chebychev's polynomials of the second kind $U_n(t)$ which is defined by

$$\frac{\sqrt{1 - t^2}}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} U_n(t) z^n. \qquad (4-8)$$

Our d_n can be written in terms of Gegenbauer function of $\nu = 1$ and $t = d/2$ as well as in terms of the Chebycheff's function of second kind

$$d_n = C_n^1(d/2) = \frac{U_n(d/2)}{\sqrt{1 - (d/2)^2}}. \qquad (4-9)$$

An identity

$$d_{n+k} + d_{n-k} = 2d_n \cos k\theta \qquad (4-10)$$

is easily proved for any integer k as

$$l.h.s. = \frac{\sin(n+k+1)\theta}{\sin\theta} + \frac{\sin(n-k+1)\theta}{\sin\theta} = \frac{2\cos k\theta\sin(n+1)\theta}{\sin\theta} = r.h.s. \quad (4-11)$$

The recursion formula Eq. (4-10) for $k = 1$ is

$$d_{n+1} + d_{n-1} = dd_n. \quad (4-12)$$

It should be compared with Eq. (1-7). It is another expression of the fact which was pointed out below Eq. (4-3). However, more essential fact behind them is still open question.

So far, n is regarded as no upper bound. The relations are valid in general for any regular G-gon. Now, let us specify the value of an integer G, then

$$d_{G-1} = 0 \quad (4-13)$$

which are equivalent to

$$\begin{aligned} d_{D-n} &= d_{D+1+n} \quad \text{if } G \text{ is odd,} \\ d_{D-n} &= d_{D+n} \quad \text{if } G \text{ is even,} \end{aligned} \quad (4-14)$$

for any n. Explicitly, as

$$\begin{aligned}
&G=5, \quad d_1 = d_2, \quad d = d^2 - 1, \quad d = \tau = 1.6180, \\
&G=6, \quad d_1 = d_3, \quad d = d^3 - 2d, \quad d = \sqrt{3} = 1.7321, \quad d_2 = 2, \\
&G=7, \quad d_2 = d_3, \quad d^2 - 1 = d^3 - 2d, \quad d = \alpha = 1.8019, \quad d_2 = \beta = 2.2470, \\
&G=8, \quad d_2 = d_4, \quad d^2 - 1 = d^4 - 3d^2 + 1, \quad d = \sqrt{2+\sqrt{2}} = 1.8478, \\
&\qquad d_2 = 1 + \sqrt{2}, \quad d_3 = \sqrt{4 + 2\sqrt{2}}.
\end{aligned} \quad (4-15)$$

The explicit forms of the generating functions for some values of G are summarized in Table 1.

Let us introduce the matrix representation as the previous cases in Sections 2 and 3. For a given G, $(D+1)$-dimensional space is convenient, where $D = [G/2] - 1$ was introduced at the beginning of this section. *A* $[(D+1)\times 1]$-matrix d is defined

Table 1. Some examples of the generating function of diagonal lengths for G-gon.

$G=3$ ($d=1$)	$\frac{1}{1-z+z^2}$	$=1+z-z^3-z^4+$	$=\frac{1+z}{1+z^3}$
$G=4$ ($d=\sqrt{2}$)	$\frac{1}{1-\sqrt{2}z+z^2}$	$=1+\sqrt{2}z+z^2-z^4-\sqrt{2}z^5-z^6+$	$=\frac{1+\sqrt{2}z+z^2}{1+z^4}$
$G=5$ ($d=\tau$)	$\frac{1}{1-\tau z+z^2}$	$=1+\tau z+\tau z^2+z^3-z^5-\tau z^6-\tau z^7-z^8+\ldots$	$=\frac{1+\tau z+\tau z^2+z^3}{1+z^5}$
$G=5/2$ ($d_0=\tau$) ($d=1$)	$\frac{1}{1-\frac{1}{\tau}z+z^2}$	$=\tau+z-z^2-\tau z^3+\tau z^5+z^6-z^7-\tau z^8+\ldots$	$=\frac{\tau+z-z^2-\tau z^3}{1-z^5}$
$G=6$ ($d=\sqrt{3}$)	$\frac{1}{1-\sqrt{3}z+z^2}$	$=1+\sqrt{3}z+2z^2+\sqrt{3}z^3+z^4-z^6-$	$=\frac{1+\sqrt{3}z+2z^2+\sqrt{3}z^3+z^4}{1+z^6}$
$G=7$ ($d=\alpha$)	$\frac{1}{1-\alpha z+z^2}$	$=1+\alpha z+\beta z^2+\beta z^3+\alpha z^4+z^5-z^7-$	$=\frac{1+\alpha z+\beta z^2+\beta z^3+\alpha z^4+z^5}{1+z^7}$
$G=7/2$ ($d_0=\alpha$) ($d=\beta$)	$\frac{\alpha}{1-\frac{\beta}{\alpha}z+z^2}$	$=\alpha+\beta z+z^2-z^3-\beta z^4-\alpha z^5+\alpha z^7+$	$=\frac{\alpha+\beta z+z^2-z^3-\beta z^4-\alpha z^5}{1-z^7}$
$G=7/3$ ($d_0=\beta$) ($d=1$)	$\frac{\beta}{1-\frac{1}{\beta}z+z^2}$	$=\beta+z-\alpha z^2-\alpha z^3+z^4+\beta z^5-\beta z^7-$	$=\frac{\beta+z-\alpha z^2-\alpha z^3+z^4+\beta z^5}{1+z^7}$

as

$$\mathrm{d}=\begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_D \end{pmatrix} \qquad (4-16)$$

D forward matrices $\mathrm{A_n}$ (n = 1, 2, ..., D) of [(D+1) × (D+1)] are defined as

$$\mathrm{A}_1=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & 1 & 1 \end{bmatrix} \qquad \mathrm{A}_2=\begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 1 \\ 0 & 0 & 0 & \cdot & \cdot & 1 & 1 \\ 0 & 0 & 0 & \cdot & 1 & 1 & 1 \end{bmatrix} \qquad \cdots$$

$$\mathrm{A}_D = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 1 \\ 0 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 1 & 1 & \cdot & 1 & 1 & 1 \end{bmatrix}. \qquad (4-17)$$

cf.(2 – 8)
cf.(3 – 10)

It is easily seen that they are natural extension of T in Eq. (2-8) and A and B in Eq. (3-10). It is noted that these matrices can be regarded as the difference equations which describe the wave propagation on a one-dimensional lattice consists of just G-1 sites. If $(G–1) \times (G–1)$ matrices were used, then the situation is more transparent to see. Taking the symmetry into account, they are reduced into Eq. (4-17). It corresponds to take only symmetrical waves with respect to the mid point. Now, the form of Eq. (4-1) will be more easy to see: d_n can be regarded as the value of the wavefunction at the site n.

All the matrices A_n's have common eigenvectors. For example, the vector d given in Eq. (4-16) is an eigenvector of A_n associated with the eigenvalue d_n

$$\mathrm{A}_n \mathrm{d} = d_n \mathrm{d}. \qquad (4-18)$$

It is equivalent to the fact that any product and any quotient of d_ns expressed as a linear combination of d_n's and 1 with only integral coefficients as in Eq. (3-3).

Some infinite sequences are generated in powers of A_n though no description is given any more.

5. Concluding remarks

The self-similar nature among the set of the diagonals of a regular polygon has been mainly investigated from a general point of view. Two remarks were given below, one was connected with beauty and the other was connected with a quasicrystal which gave the motivation of this study to the author as mentioned at first.

5.1 Golden ratio and aesthetics

The relation between the golden ratio and aesthetics has been pointed out since ancient time. For example, the golden rectangles are believed as the most beautiful rectangle. How do constants α and β relate with beauty from this point of view? An attempt is given in Fig. 4. It is a cube seen from an infinite distance in the direction $1:\alpha:\beta$ relative to three edges. It is noted that the ratio $1:\alpha:\beta$ appears in the ratio of the areas of three rhombuses which are seen as the faces of a cube in Fig. 4.

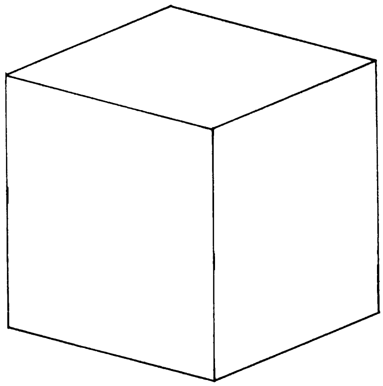

Fig. 4. A view of a cube from an infinite distance in the direction of $(1, \alpha, \beta)$. Is it the most beautiful direction? The ratio of the areas of three rhombuses that can be seen three faces of a cube is $1:\alpha:\beta$. The figure is useful in constructing the arrangement (5-4) by so-called projection method.

The personal view by the author is the following. Watching a golden rectangle and another one which is slightly different, one can not tell which is the golden rectangle. In a more complicated figure, however, if the golden ratio is used somewhere in it then the ratio can be expected to appear everywhere in the same figure because of the self similarity associated with golden ratio. In such cases, difference can be told easily. The beauty connected with the golden ratio lies in the mathematical facts. Therefore, it is natural that Fig. 4 does not look so beautiful even if the ratio $1:\alpha:\beta$ really associates some beauty.

5.2 A tiling with three elements

Before closing this paper, an attempt concerning with a one-dimensional aperiodic structure is mentioned. The technique of obtaining the arrangement is a standard one in the study of quasicrystals, a projection method.

Chose a point (a, b, c). An infinitely long hexagonal cylinder is defined by six planes as

$$(x-a)-\frac{y-b}{\alpha}<1,\quad \frac{y-b}{\alpha}-\frac{z-c}{\beta}<\frac{1}{\alpha},\quad \frac{z-c}{\beta}-(x-a)<\frac{1}{\beta},$$

$$\frac{y-b}{\alpha}-(x-a)<\frac{1}{\alpha},\quad \frac{z-c}{\beta}-\frac{y-b}{\alpha}<\frac{1}{\beta},\quad (x-a)-\frac{z-c}{\beta}<1. \tag{5-1}$$

Taking all the lattice point inside of the cylinder, throw all the others away. Then only one site

$$x+y+z=n \tag{5-2}$$

remains for every integer n. Project the remaining sites to a straight line

$$x-a=\frac{y-b}{\alpha}=\frac{z-c}{\beta}. \tag{5-3}$$

Now an aperiodic arrangement was obtained. There are three kinds of separations $S=(3+\alpha+2\beta)^{-1/2}$, $M=\alpha S$ and $L\equiv\beta S$. Starting from the site for a proper value of n, the remaining sites can be described by the direction of the step corresponding to $\Delta n = 1$. Separation S corresponds to the step in x direction. Separation M corresponds to the step in y direction. Separation L corresponds to the step in z direction. A part of an infinite arrangement with 100 separations is shown below

... *LMLSM LMLSM LLMSL MLMSL MLLMS LMLSM LMLSL*
MLMSL MLMLS LMLMS LMLSM LLMSL MLMLS MLLMS
LMLMS LMLSL MLMLS MLMLS MLMLS LMLMS ...

(5 – 4)

where the values of three constants a, b and c are properly chosen. The composition of three composition, S, M and L tends to $1:\alpha:\beta$ in the limit of infinitely large system. The fact is reflected in that the ratio of the area of three rhombuses in figure, is $1:\alpha:\beta$.

Only a single topic was described in this paper among three topics reported at the symposium. The other two have been already published (Ogawa, 1987, 1989).

Acknowledgment

The author is indebted for illuminative discussions to Dr. K. N. Ishihara, KyotoUniversity, Prof. K. Niizeki, Tohoku University, and Prof. Masaharu Inoue, University of Tsukuba.

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