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Projections of Four-Dimensional Regular Space Fillings

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Abstract. There are a total of 3 kinds of 4-dimensional space fillings, each of which is composed of congruent 4-dimensional regular polytopes; 4-dimensional cubes, 16-cells, and 24-cells. We call them "4-dimensional regular space fillings".

When they are orthogonally projected into 3- or 2-dimensional cartesian space (3- or 2-space, hereafter), they show various 3-dimensional space fillings or 2-dimensional planar tessellations whose appearances change continuously, according to a small perturbation of the direction of the projection.

Similar variable patterns can be obtained from the orthogonal projections of $n(\geq 5)$-dimensional regular space fillings, each of which is composed of congruent n-dimensional cubes.

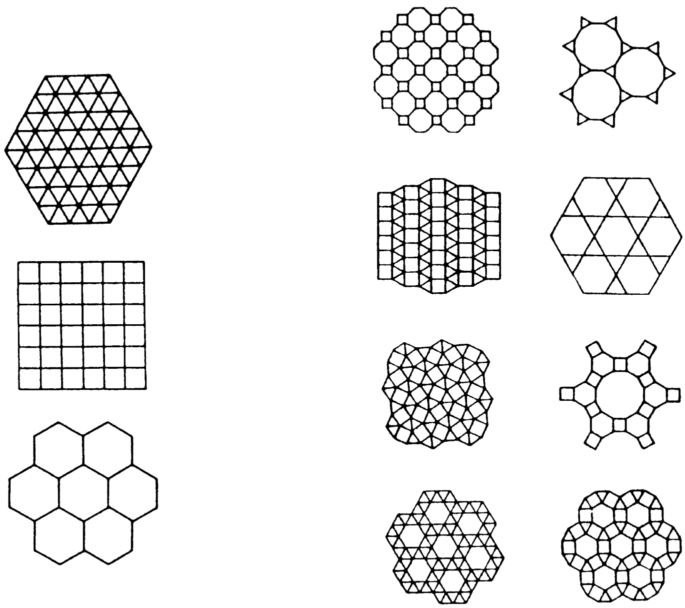
It is expected that, whether they have periodic features or not, many geometrical patterns in nature and art can be derived from such projections into 3- or 2-space of n-dimensional regular space fillings.

1. Introduction

In nature and art, we can observe various planar tessellations by polygons and space fillings by polyhedra, both of which have periodic and non-periodic features.

As far as a periodic feature is concerned, the best known are planar tessellations by regular polygons and space fillings by regular polyhedra.

In 2-space, there is a total of 3 kinds of regular tessellations, each of which has only one kind of regular polygon fitted together similarly around each vertex (Fig. 1(a)), and there is a total of 8 kinds of semi-regular tessellations, each of



(a) (b)

Fig. 1. (a) Regular tessellations. (b) Semi-regular tessellations.

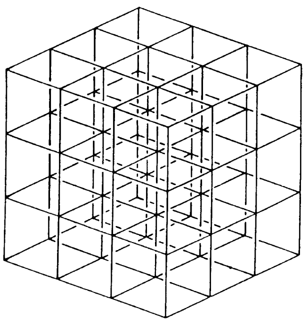
which has 2 or more kinds of regular polygons fitted together similarly around each vertex (Fig. 1(b)).

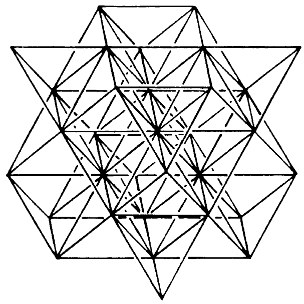
In 3-space, there is only one regular space filling, the polycube, which has congruent cubes fitted together similarly around each edge (Fig. 2(a)), and there is only one semi-regular space filling, the octet-truss, which has regular tetrahedra and regular octahedra fitted together similarly around each edge (Fig. 2(b)).

In this paper, firstly, a total of 3 kinds of 4-dimensional regular space fillings were considered as an analogue of the 3-dimensional regular space fillings. Next, their various orthogonal projections into 3- and 2-space were constructed by using successively changed directions of the projection. Lastly, the operation will be extended to $n(\geq 5)$-dimensional regular space fillings by n-dimensional cubes. There may be no 4- or higher-dimensional semi-regular space filling which is an analogue of the octet-truss in 3-space.

2. The Orthogonal Projection in 4-Space

In the 4-space R^4 (XYZU) having 4 coordinate axes X, Y, Z, and U, the orthogonal projection a of the point A into R^3 (XYZ) means the intersection of R^3





(a) (b)

Fig. 2. (a) The polycube. (b) The octet-truss.

(XYZ) and the perpendicular *Aa* (Fig. 3). Also, the orthogonal projection *a'* of the point *A* into R^2 (XY) means the intersection of R^2 (XY) and the perpendicular *aa'* (Miyazaki 1990).

Their cartesian coordinates are determined as follows: When the coordinates of *A* is (x, y, z, u), then the coordinates of the orthogonal projection into R^3 (XYZ) become to (x, y, z, 0), and those into R^3 (XYU), (XZU), and (YZU) become to (x, y, 0, u), (x, 0, z, u), and (0, y, z, u) respectively. Also, the coordinates of the orthogonal projection into R^2 (XY) become to (x, y, 0, 0), and those into R^2 (XZ), (XU), (YZ), (YU), and (ZU) become to (x, 0, z, 0), (x, 0, 0, u), (0, y, z, 0), (0, y, 0, u), and (0, 0, z, u) respectively.

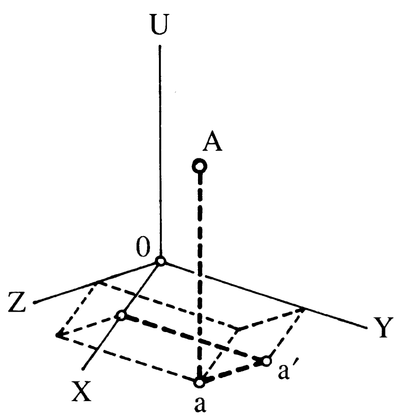


Fig. 3. The orthogonal projection of the point *A* in the 4-space R^4 (XYZU). *a* is in R^3 (XYZ) and *a'* in R^2 (XY).

3. Four-Dimensional Regular Polytopes

A 4-dimensional polytope is constructed of polyhedral cells (polyhedral parts of 3-spaces), certain pairs of which have a face in common.

A 4-dimensional regular polytope has only one kind of regular polyhedral cell, a fixed number of which fits together around every edge and any two of which have a face in common. They correspond to infinite kinds of regular polygons in 2-space or 5 kinds of regular polyhedra in 3-space.

A total of 6 kinds of 4-dimensional regular polytopes are as follows; the 5-cell (a 4-dimensional regular tetrahedron) having 5 regular tetrahedral cells, 3 of which fit together around every edge, the 8-cell (a 4-dimensional cube) having 8 cubic cells, 3 of which fit together around every edge, the 16-cell (a 4-dimensional regular octahedron) having 16 regular tetrahedral cells, 4 of which fit together around every edge, the 24-cell having 24 regular octahedral cells, 3 of which fit together around every edge (there is no corresponding regular polyhedron), the 120-cell (a 4-dimensional regular dodecahedron) having 120 regular dodecahedral cells, 3 of which fit together around every edge, and the 600-cell (a 4-dimensional regular icosahedron) having 600 regular tetrahedral cells, 5 of which fit together around every edge.

In this paper, we treat only three of them; a 4-dimensional cube, the 16-cell, and the 24-cell.

Their vertices have the following cartesian coordinates in 4-space: The 16 vertices of the 4-cube whose edge length is 2 are the permutation of $(\pm 1, \pm 1, \pm 1, \pm 1)$. The 8 vertices of the 16-cell whose edge length is $\sqrt{2}$ are the permutation of $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$, and $(0, 0, 0, \pm 1)$. The 24 vertices of the 24-cell whose edge length is 2 are the permutation of $(\pm 1, \pm 1, \pm 1, \pm 1)$, $(\pm 2, 0, 0, 0)$, $(0, \pm 2, 0, 0)$, $(0, 0, \pm 2, 0)$, and $(0, 0, 0, \pm 2)$. On the other types of these coordinates, refer to Coxeter (1973) and Banchoff (1990), etc.

Their typical orthogonal projections into 3-space are shown in Table 1 by using axometric drawings and plan-elavation pairs. Each row shows one kind of the regular polytope; (a) the 4-cube, (b) the 16-cell, and (c) the 24-cell. Each column shows one kind of the orthogonal projection; (1) the vertex-centered projection, (2) the edge-centered projection, (3) the face-centered projection, and (4) the cell-centered projection, where the vertex centered projection means a projection having a vertex in the center, and so on. On their practical models, refer to Miyazaki (1982, 1986).

4. Four-Dimensional Regular Space Fillings

In this paper, a 4-dimensional regular space filling means a 4-dimensional space filling by congruent 4-dimensional regular polytopes fitted together similarly around each face.

Table 1. Typical orthogonal projections into 3-space of 3 kinds of 4-dimensional regular polytopes. Shown by axometric drawings and plan-elevation pairs. (a) the 4-dimensional cube, (b) the 16-cell, (c) the 24-cell. (1) the vertex-centered projection, (2) the edge-centered projection, (3) the face-centered projection, (4) the cell-centered projection.

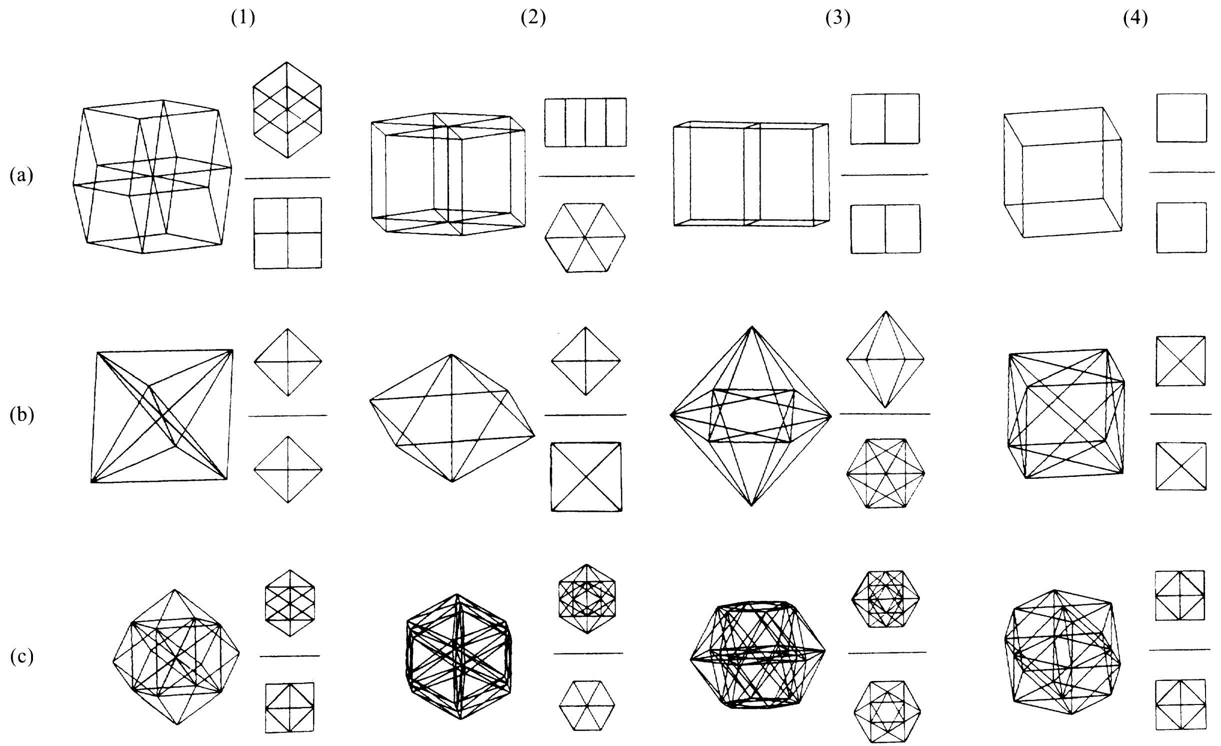
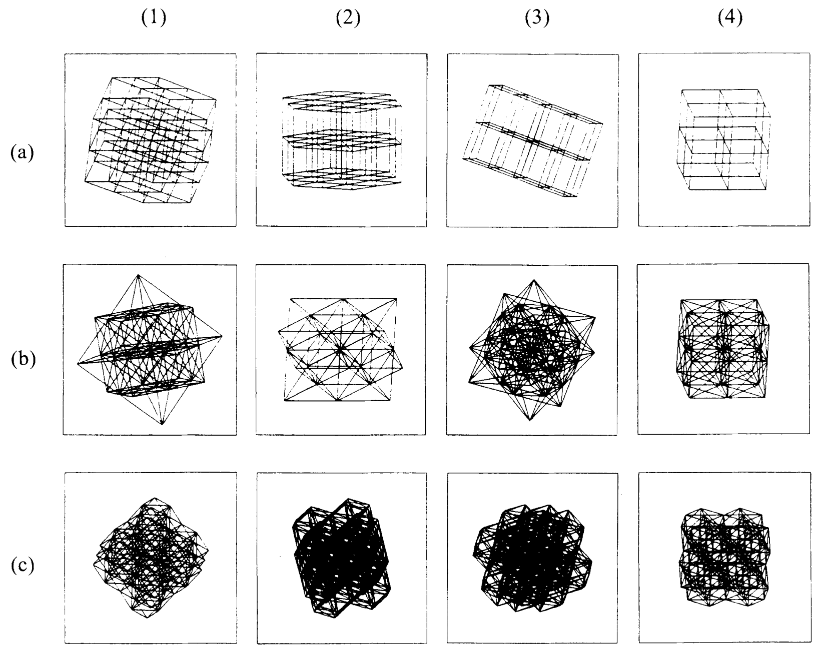


Table 2. Typical orthogonal projections into 3-space of 4-dimensional regular space fillings shown by axometric drawings. The arrangement of each projection coincides with that in Table 1.



According to Coxeter (1973), there is a total of 3 kinds of 4-dimensional regular space fillings (4-dimensionl regular honeycombs). They are by 4-dimensional cubes each two of which have a cubic cell in common, by the 16-cells each two of which have a regular tetrahedral cell in common, and by the 24-cells each two of which have a regular octahedral cell in common. Each has a periodic feature in 4-space.

Table 2 shows their typical orthogonal projections into 3-space whose units are already shown in Table 1 by axometric drawings. Various well-known 3-dimensional space fillings such as the polycube and the octet-truss are hidden in these figures.

The other complicated 3-dimensional patterns can be continuously and systematically derived according to the changes in the direction of the projection.

5. Linear Patterns Derived From 4-Dimensional Regular Space Fillings

Each figure in Table 2 shows a complicated linear pattern in 2-space.

Various simpler linear patterns can be derived from the orthogonal projec-

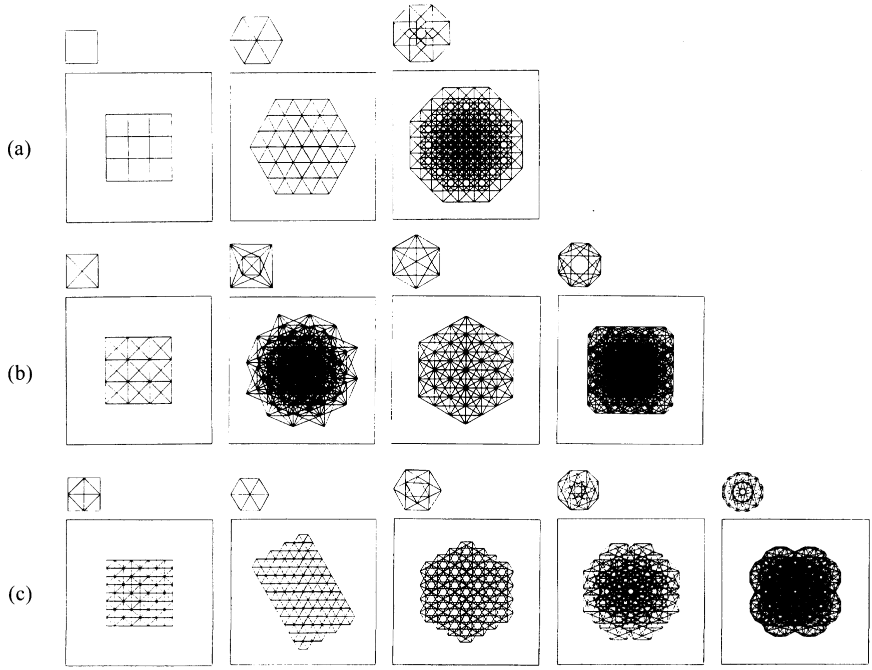


Fig. 4. Linear patterns derived from 4-dimensional regular space fillings each of whose units has a regular polygonal outline as shown in each upper position. (a) was derived from 4-dimensional cubes, (b) from the 16-cells, and (c) from the 24-cells.

tion of 4-dimensional regular space fillings. Figure 4 shows their examples each of whose units has a regular polygonal outline shown in each upper position. The pattern in (a) is derived from 4-cubes, (b) from the 16-cells, and (c) from the 24-cells.

Similarly, we can derive the other simple patterns from each polygonal unit shown in Table 1 as a plan or elevation. From these linear patterns, we can extract various 2-dimensional planar tessellations such as regular and semi-regular tessellations. Especially, a non-periodic tessellation by 2 kinds of rhombi (Fig. 6(a), bottom) is derived from the case using a regular octagonal unit as a projection of a 4-cube (Fig. 4(a), right end). On the detail of this pattern, refer to Grünbaum (1989).

6. Dotted Patterns Derived From 4-Dimensional Regular Space Fillings

Each pattern in Fig. 4 shows an arrangement of edges of each 4-dimensional regular space filling.

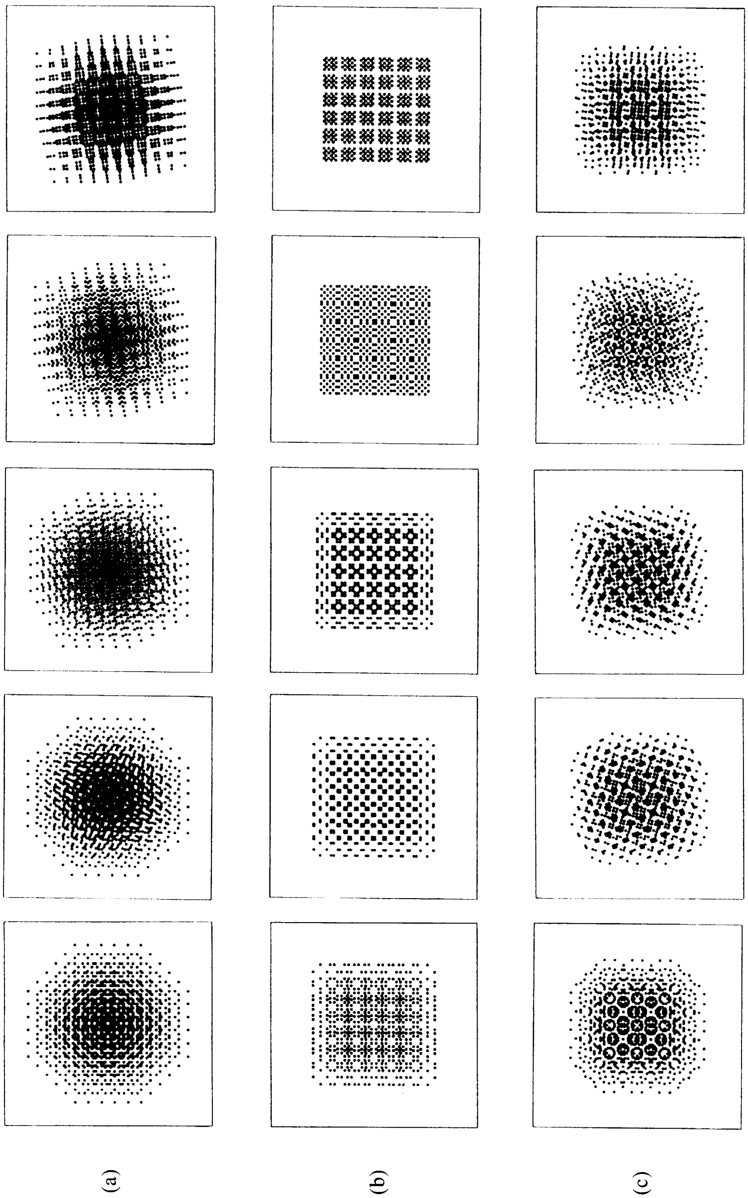


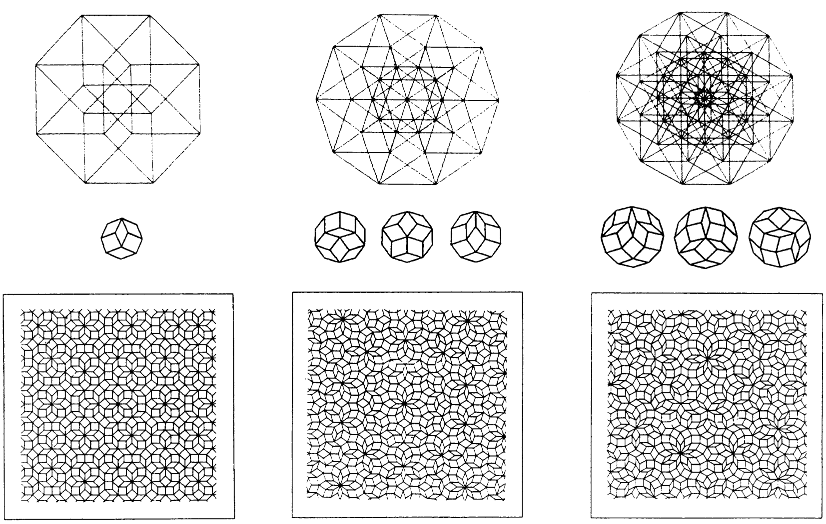
Fig. 5. Transformations of arrangements of vertices of the 4-dimensional regular space fillings. (a) is derived from 4-dimensional cubes, (b) from the 16-cell, and (c) from the 24-cell.

Similarly, arrangements of their vertices can be shown by dotted patterns (Fig. 5). The pattern shown in each left end represents the arrangement of vertices of the space fillings, each of which is shown in right end of each row in Fig. 4; (a) is derived from the regular octagonal projection of a 4-cube, (b) from the regular octagonal projection of the 16-cell, and (c) from the regular 12-gonal projection of the 24-cell. They are gradually transformed from left to right according to a small perturbation of the direction of the projection.

7. $n(\geq 5)$-Dimensional Regular Space Fillings by n-Dimensional Cubes

There is a total of 3 kinds of regular n-polytopes in each $n(\geq 5)$ space; an $(n+1)$-tope having regular triangular faces (an n-dimensional regular tetrahedron), a $2n$-tope having square faces (an n-dimensional cube), a 2^n-tope having regular triangular faces (an n-dimensional regular octahedron). Of them, only each n-dimensional cube can periodically fill n-space having an $(n-1)$ dimensional cubic cell in common.

An n-dimensional cube can be orthogonally projected into 2-space such that the outline forms a regular $2n$-gon. For example, a 4-dimensional cube can be projected having a regular octagonal outline as already shown in Fig. 4(a), right



(a) (b) (c)

Fig. 6. Non-periodic tessellations by rhombi (bottom row) which are derived from the regular polygonal projections of the n-dimensional cubes (top row). Second row shows the examples of elimination of the hidden-lines of each cube. (a) is derived from 4-dimensional cubes, (b) from 5-dimensional cubes, and (c) from 6-dimensional cubes.

end. Such regular polygonal projections of a 4-, 5-, and 6-dimensional cube, are shown in Figs. 6(a), (b), and (c), respectively.

Each of these patterns is constructed of some kinds of mutually overlapping rhombi, all of whose edges are parallel to any edge of the $2n$-gonal outline. When some of these overlapping rhombi are eliminated, we can obtain various $2n$-gonal patterns filled by single layered rhombi. Some of them are shown in Fig. 6, second row.

The infinite numbers of such $2n$-gonal patterns can cover a 2-space such that each two of them has one or more rhombi in common. From this procedure, we can sometimes obtain even a non-periodic tessellation as is shown in Fig. 6, bottom row. They are derived from a unit shown in each upper position. The case in 5-space (bottom center) coincides with the famous Penrose pattern. As well-known, some of such non-periodic patterns were already found in nature and applied in art as will be shown in the other sections in this book.

The 3-dimensional periodic version is a 3-dimensional space fillings by a total of 5 kinds of parallelohedra already found by Fedorov. Three of them (a cube or cuboid, a hexagonal prism, and a rhombic dodecahedron) can be derived from the typical orthogonal projections of a 4-dimensional cube as already shown in Table 1(a). Each of the other two (an elongated rhombic dodecahedron and a truncated octahedron) can be derived as a projection of a 5- and 6-dimensional cube, respectively.

8. Conclusion

We can derive various 3-dimensional space fillings and 2-dimensional planar tessellations by the orthogonal projections into 3- and 2-space of 4-dimensional regular space fillings. They are systematically and successively varied according to a small perturbation of the direction of the projection. In the way, we can find various periodic and even non-periodic tessellations, including the well-known regular and semi-regular tessellations and space fillings. Their graphic features can be appreciated by the linear patterns and the dotted patterns.

Such procedure can be extended to any $n(\geq 5)$-dimensional space filling by n-dimensional cubes.

The obtained linear and dotted patterns or their transformations may coincide with a design in nature and art.

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