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Derivation of Some Equilateral Zonohedra and Star Zonohedra

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Abstract. Some equilateral-zonogon (EQZG) and star-zonogon (STZG) in the 2D space, and equilateral-zonohedra (EQZH) or star-zonohedra (STZH) in the 3D space is obtained by using the base vectors of the regular polygon (RPG), regular polyhedra (RPH; Plato polyhedra) or quasi-regular polyhedra (QRPH; Archimedes quasi-polyhedra) and RPH-QRPH complex respectively. A sense of base vector was defined using arrow from the centre to vertex of a RPG (2D) or RPH, QRPH and RPH-QRPH complex (3D). The building of EQZG, STZG, EQZH and STZH was carried out by the successive generation of the base vectors at the vertices of RPG (2D) or RPH, QRPH and RPH-QRPH complex (3D). The symmetry of these EQZG, STZH and EQZH composed of the base vectors was discussed.

1. Introduction

Some EQZG and STZG in the 2D space is well known as a projection of a hyper cube in the higher dimensional space to 2D space. In the same way as in 2D space, the EQZH or STZH was also obtained by the projection of hyper cube to 3D space. The geometrical and algebraic properties of these solids were summerized by Coxeter (1973). Some zonohedron like a triacontahedron which was associated with the icosahedral symmetry group was obtained by the projection method in relation to the study of quasicrystal space groups (Haase *et al.*, 1987). In this paper a EQZG or EQZH building procedure is proposed, where the composition of these EQZG or EQZH is based only on the base vectors of RPH, QRPH or RPH-QRPH complex.

2. Derivation of 2D EQZG

A building procedure of EQZG is illustrated using typical six examples of RPG, regular-triangle, square, pentagon, hexagon, octagon and decagon as shown in Fig. 1. Each base vectors of RPG is represented by an arrow from the centre to vertex. The following vector is generated at the vertex so as to form a rhombus which is constituent unit of EQZG. STZG appears by erasing the edge of RPG which corresponds to a diagonal line of a rhombus, then RPG was replaced by STZG. In a triangle and a square case ($n = 3, 4$) STZG coincide with EQZG as shown in Fig. 1. At the outside vertex of the STZG, an operation of generating the base vector was repeated successively to build EQZG. EQZG thus obtained was classified into two types according to odd or even number of the edge of a RPG. While, in the even case, EQZG obtained forms a similar polygon with twice as long as edge length of RPG, in the odd case, an EQZG has twice as many as edge number of RPG. If base vector set in the hemi-polygon of the even case is used, EQZG contains self-similar RPG in itself as shown in Fig. 1 by shaded regular polygon which is subdivided assymmetricaly by rhombus for $n = 8, 10$ and symmetricaly for $n = 6$. The kind of rhombus constituent EQZG which was derived by choosing two vectors pair from n-gonal basis. The combination of the two base vectors is restricted to one kind for $n = 3, 4, 6$ and two kinds for $n = 5, 8, 10$ respectively.

3. Derivation of 3D EQZH of Lower Symmetry

The procedure to obtain 3D EQZH is the same as the 2D case. In the 3D EQZH building RPH, QRPH and a RPH-QRPH complex are set up as basic polyhedra corresponding to RPG in 2D case. Base vector set of each basic polyhedra is represented by the arrow from the centre to the vertex of a basic-polyhedron. Consider N dimensional base vector, ${}_NC_3$ pieces of rhombohedron cell pack inside EQZH, but their arrangement is not uniquely determined.

In preparation of building STZH and EQZH, consider a synthesis of the vectors of a n-gonal pyramid which stretch from an apex to the vertexes of base polygon (Kramer, 1982). Basic polyhedron was decomposed into n-gonal pyramids of which base polygon is coincident with a self-facet. There are two processes to construct EQZH, one is a building by the base vectors in a basic whole PRH or whole PRH-QPRH-complex and the other, that in a basic hemi PRH or a hemi PRH-QPRH complex. The former and the latter vector set situation is equivalent to the case of odd number edge RPG and that of even edge RPG in 2D case respectively. Examples of each building step are illustrated in Fig. 2 for octahedron, tetrahedron and hexahedron (cube). Hemi vector set stretching from origin to three vertexes (V6/2) for the octahedron and four vertexes (V8/2) for the cube is set up respectively as shown in Fig. 2 with the cutting plane of the base vectors. Two new basis added to an apex of the base vectors form rhombic facet of EQZH and the same operation is continued at the new apex successively until the space surrounded by the facet was

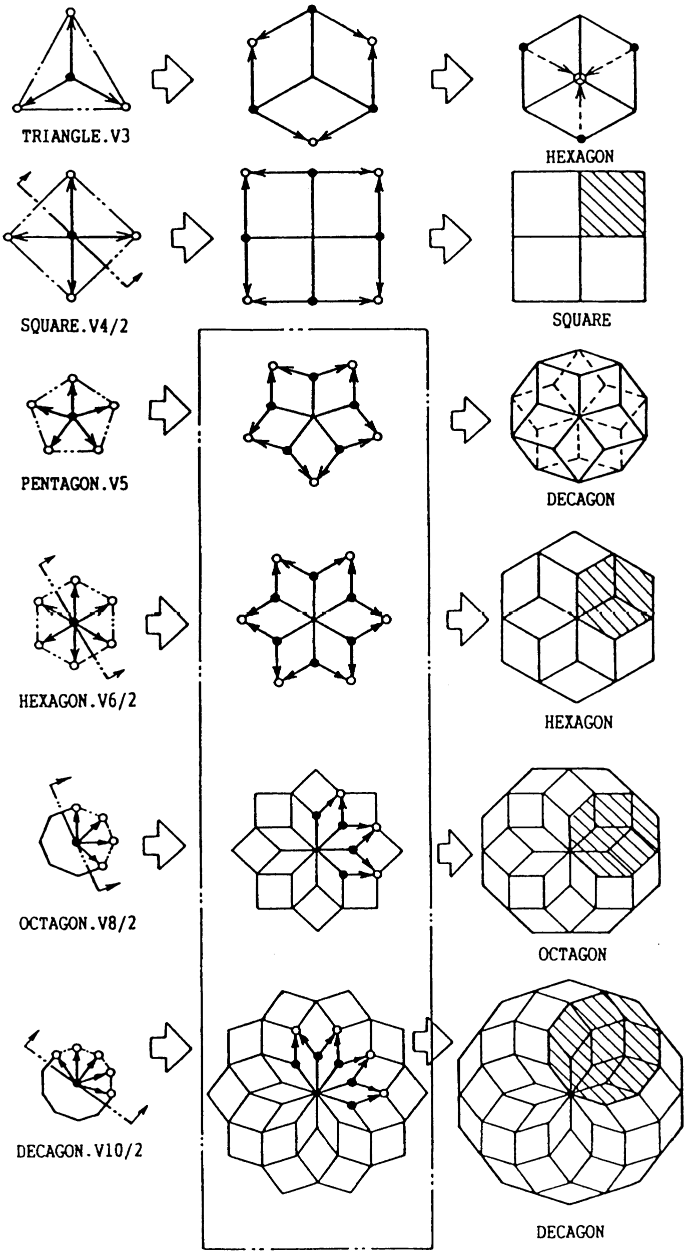

Fig. 1. Generating process of 2D EQZG. Base vector is shown with arrow. STZG is shown surrounding by two-dot chain line. Shaded area of EQZG is sub-EQZG obtained by hemi RPG.

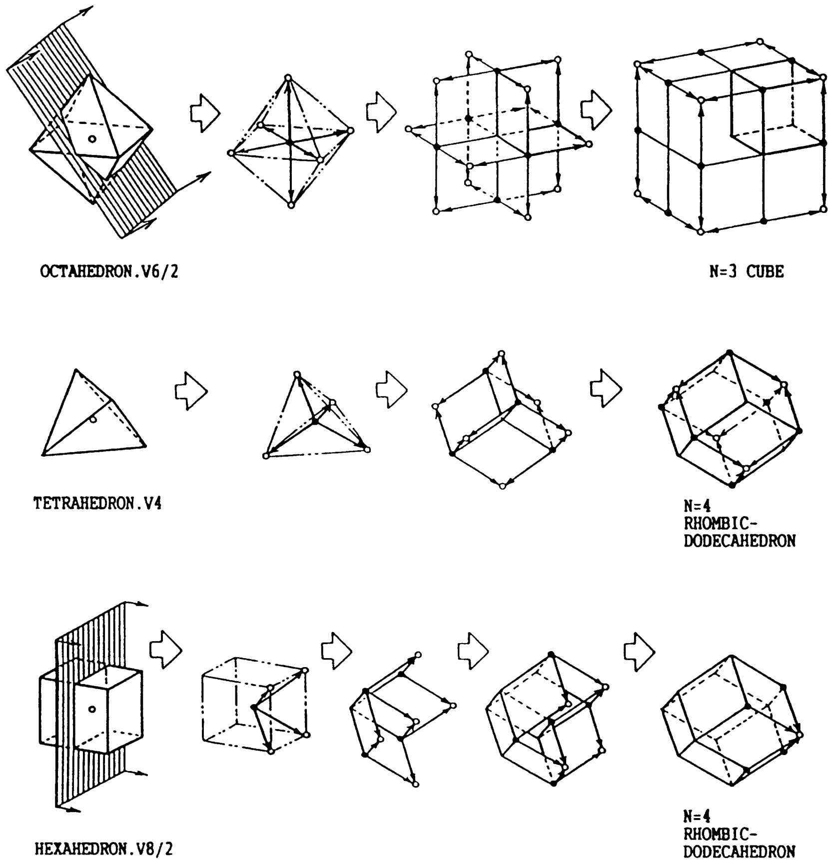

Fig. 2. Generating process of 3D EQZH. Demonstration of tetrahedron, octahedron and hexahedron of RPH. Hemi vector set was used for the latter two RPH. Vn, Vn/2 is number of vertex of whole-RPH and hemi-RPH respectively. N is dimension number.

closed to complete EQZH. Thus, unit cube and rhombicdodecahedron with unit edge length was obtained from octahedron and cube respectively. If the whole vector set of octahedron or cube was selected instead of hemi vector set, an EQZH with twice as long as edge length of EQZH derived from hemi-set could be obtained. The typical example of the lengthening effect by the base vector selection on the edge length of EQZH is presented for the basic octahedron (V6). All the base vector set have a centre of symmetry except a tetrahedron (V4) or a truncated tetrahedron (V12) vector set which is coincident with hemi set of small rhombicubocta hedron (V24/2), then it is enough to use hemi vector set to build an EQZH. It was noticed that a rhombicdodecahedron, one of EQZH, was derived from either tetrahedron or hemi hexahedron. If the building begin with a tetrahedron or hemihexahedron, the corresponding starting point is 3-fold or 4-fold axis of rhombic dodecahedron respectively.

4. 3D STZH of Lower Symmetry and Icosahedral Symmetry

3D Star equilateral zonohedron (STZH) necessarily appear in the building process from RPH, QRPH or RPH-QRPH complex to EQZH. The constituent unit of STZH is rhombohedron or rhombic parallelopiped. If RPH takes octahedron or tetrahedron or cube, the rhombohedral cell of STZH degenerates to plane as shown in Fig. 2. This situation corresponds to the triangle or square of RPG in 2D case. In degenerated case STZH should not have the volume. There exists a rotation axis like a 3-, 4-, 5-fold axis in the centre of star pattern, which originates from a symmetry of base vector set. Since EQZH is generated to match a concave star pattern, a symmetry axis of EQZH should be coincident with that of STZH.

A representative example of STZH is presented for icosahedron of RPH. Figure 3 shows a building process from RPH (icosahedron) to EQZH (triacontahedron) through STZH (equilateral star dodecahedron).

The constituent unit cell of STZH and EQZH is derived by choosing a three vectors from the basis of RPH, QRPH or RPH-QRPH complex. In the case of an icosahedral RPH, the two kinds of unit cells were obtained, namely acute and obtuse unit cell (Mackay, 1984; Kramer *et al.*, 1984). The basis of an acute cell of STZH is equivalent to the three base vectors from an origin to the vertexes of the regular triangle of an icosahedron. On each regular triangle facet of an icosa hedron, an acute cell grows into dodecagonal STZH. This STZH has the same symmetry as an original icosahedron of RPH.

In order to obtain EQZH, a rhombic-icosahedron which is a projection of 5D cube to 3D space is fitted into concave star facet. At the next step, an assembly of the acute cells contacting to a rhomgic-icosahedron is apart from STZH, then a

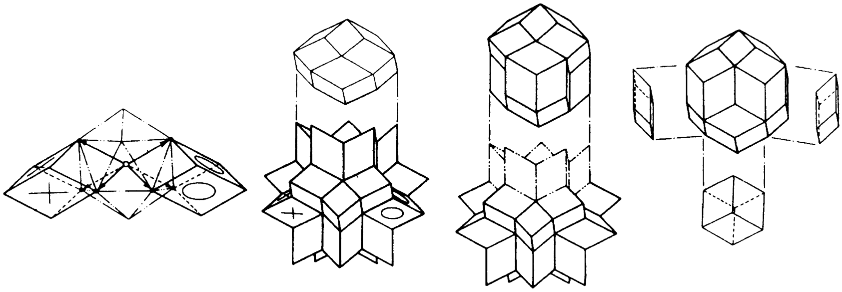

Fig. 3. Example from icosahedron (two-dot chain line) to triacontahedron through star rhombic dodecahedron.

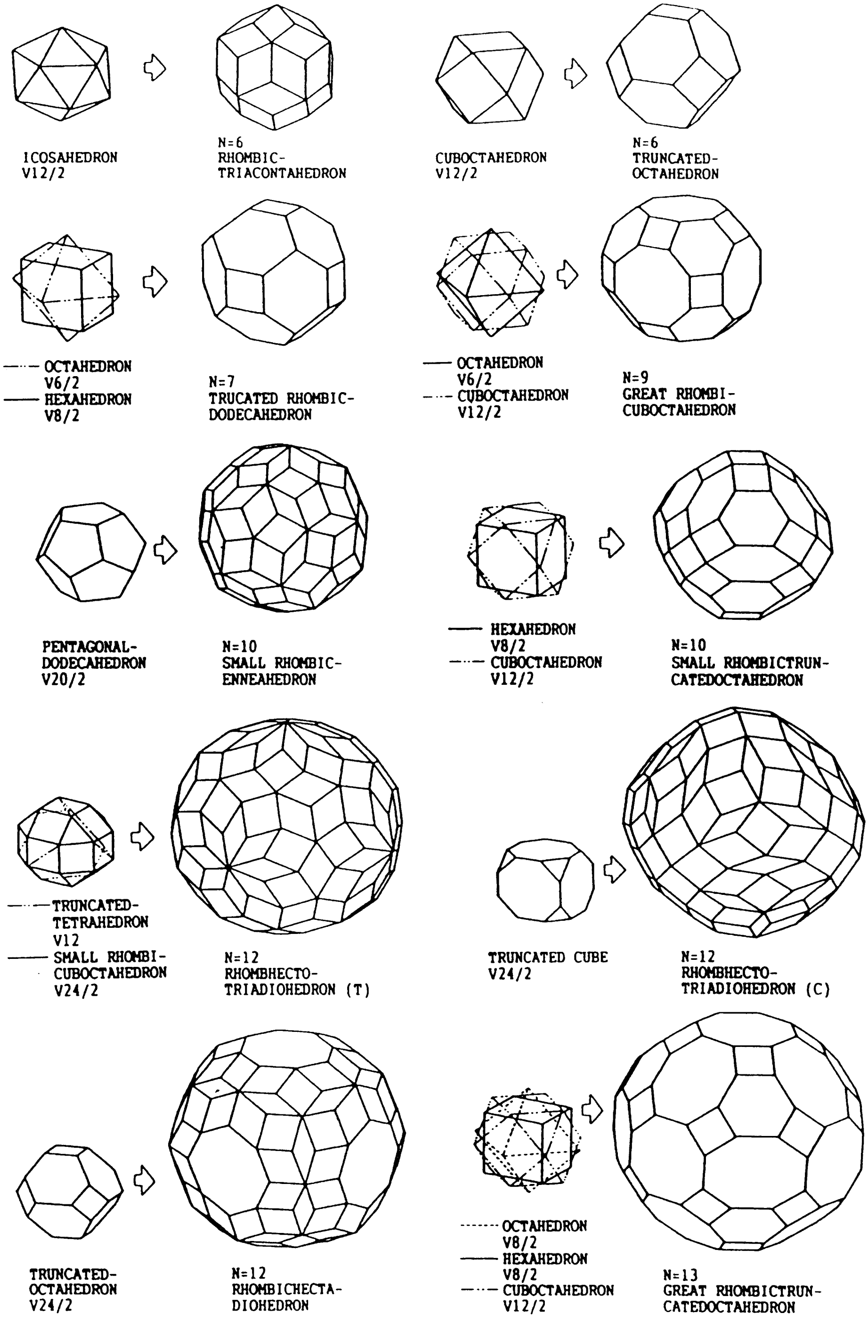
ICOSAHEDRON
V12/2
N=6
RHOMBIC-
TRIACONTAHEDRON
CUBOCTAHEDRON
V12/2
N=6
TRUNCATED-
OCTAHEDRON
OCTAHEDRON
V6/2
HEXAHEDRON
V8/2
N=7
TRUCATED RHOMBIC-
DODECAHEDRON
OCTAHEDRON
V6/2
CUBOCTAHEDRON
V12/2
N=9
GREAT RHOMBI-
CUBOCTAHEDRON
PENTAGONAL-
DODECAHEDRON
V20/2
N=10
SMALL RHOMBIC-
ENNEAHEDRON
HEXAHEDRON
V8/2
CUBOCTAHEDRON
V12/2
N=10
SMALL RHOMBICTRUN-
CATEDOCTAHEDRON
TRUNCATED-
TETRAHEDRON
V12
SMALL RHOMBI-
CUBOCTAHEDRON
V24/2
N=12
RHOMBHECTO-
TRIADIOHEDRON (T)
TRUNCATED CUBE
V24/2
N=12
RHOMBHECTO-
TRIADIOHEDRON (C)
TRUNCATED-
OCTAHEDRON
V24/2
N=12
RHOMBICHECTA-
DIOHEDRON
OCTAHEDRON
V8/2
HEXAHEDRON
V8/2
CUBOCTAHEDRON
V12/2
N=13
GREAT RHOMBICTRUN-
CATEDOCTAHEDRON

Fig. 4

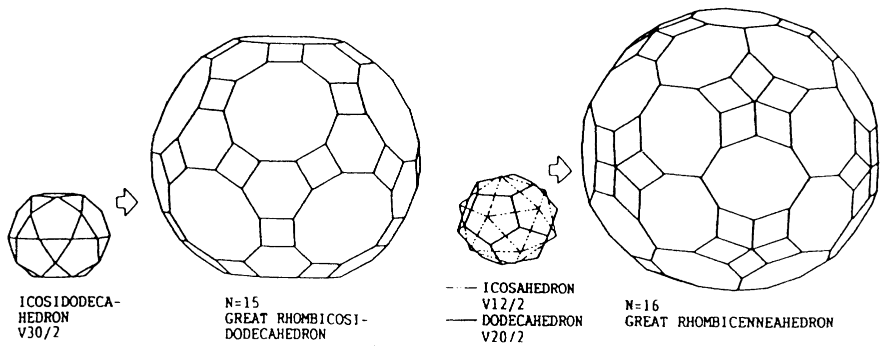

Fig. 4. Some generalized EQZH and original RPH, QRPH and RPH-QRPH complex which was represented a solid line, a dashed line and a two-dot chain line.

triacontahedron lacking in three obtuse cells was obtained. The last, each three obtuse cell fits into a cavity of a triacontahedron such that 3-fold axis of both solids coincide.

5. Some Generalized EQZH

As a general rule, EQZH have a zone which is composed of N pieces of rhombus and each zone crosses N-1 zone, then N(N-1) facet exists on the surface of EQZH. N is called dimension number. Figure 4 shows ten kinds of EQZH derived from RPH, QRPH or PRH-QRPH complex abbreviating the STZH building process, VN/2 indicate the number of vertex of the hemi vector set used, the name of the composite polyhedron (RPH, QRPH and RPH-QRPH complex) and that of EQZH is given.

In the complex case, double or triple set of basic polyhedron is drawn with solid and dashed chain line or solid, dashed chain and two dot-chain line respectively. In single basic polyhedron case, five RPH and QRPH (cuboctahedron, truncated-cube, truncated-octahedron, icosidodecahedron) is drawn clearly in shape. However, since the geometry of the base vector in complex case is complicated on account of the duplication of the vectors, some explanatory notes are necessary for the geometry of the complex of basic polyhedron. From a concentric hemi regular octahedron and hemi hexahedron which inscribe a sphere, a truncated rhombic dodecahedron (N = 7) could be obtained in which three rhombi exist on the same hexagon plane. This hexagon plane is generated by the edge centre truncation of rhombic dodecahedron by the plane normal to the 4-fold axis.

If a concentric hemi hexahedron is replaced by a hemi cuboctahedron, a great rhombicuboctahedron (N = 9) could be obtained, and a concentric hemi octahedron is replaced by a hemi cuboctahedron, a small rhombictruncatedoctahedron (N = 10) could be obtained, on which two kinds of hexagon and square appear.

Either a whole truncated tetrahedron or hemi small rhombicuboctahedron could give a rhombihectotriadiohedron independently (N = 12). There exists a relation between these two polyhedron that four truncated triangle sections of a tetrahedron coincide with those of small rhombicuboctahedron. The edge of truncated tetrahedron have unit length for triangle facet and $\sqrt{2}$ for hexagon facet. Obtained rhombihectotriadiohedron have two kinds of rhombic facet, thick and thin rhombus, the former takes together around the 3-fold axis and the latter assembles around the 4-fold axis to form a local 8-fold pattern of thin rhombi. A hemi octahedron, hemi hexahedron and hemi cuboctahedron complex grows into great rhombictruncatedoctahedron (N = 13), the surface of which square, hexagon and octagon appear.

The last basic polyhedron complex, a hemi icosahedron-dodecahedron group grows into a great rhombicenneahedron (N = 16), on the surface of which thick rhombus of small rhombic enneahedron (N = 10) and octagon assemble around the 5-fold and 3-fold axis respectively. The thick rhombus is the same as that in triaconta hedron. A geometry of pattern and symmetry of EQZH in the single case is the same as in the complex case.

The method of EQZG or EQZH building is derived using base vector set of RPG (2D) or RPH, QRPH and RPH-QRPH complex (3D). EQZH obtained have the unit length from the origin to the vertex and its facet pattern is restricted to rhombus (including cube). Thus, obtained EQZH plays an important role in the quasilattice structure, in which EQZH and STZH is associated to each other to form space filling packing with other pieces of rhombohedral unit. A typical example is found in the quasicrystal like Al-Li-Cu alloy, the structure unit exists as a cluster of tricontahedron derived here. A lot of stable quasicrystal structure of binary or ternary alloy was discovered by recent work and the structure model will become increasingly necessary. Most of EQZG or EQZH presented here is already obtained but, so far as we know, the process of obtaining EQZG or EQZH has not come out visualy using base vectors of RPH or QRPH or its complex. This paper illustrated the building process of EQZG and EQZH in the meaning of the geometric understanding of quasilattice or more complicated cluster model.

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