*Research of Pattern Formation,* edited by R. Takaki, pp. 91–102.


# A Note on Intrinsic Geometry of Origami*

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**Abstract.** In this paper, the author presents a proposition of intrinsic geometry of origami. It treats the geometric properties which are dependent only on the vicinity of an arbitrary point on the surface of origami works. Based on the theory of curved surfaces, the basic theorems of intrinsic geometry of origami are obtained. Using the theorems, the relations of convexity/concavity of folds and the vertices are studied. The result is also used to explain the particular characteristics of the so-called Miura-ori.

## 1. Introduction

In general, the geometry of origami treats rather macroscopic aspects of the form of origami works. The geometry which the author presents here is to treat the theorems which are dependent only in the vicinity of an arbitrary point on the surface of origami works, and it can be called intrinsic geometry of origami.

The Japanese word "origami" consists of two parts, that is, "ori = fold" and "kami (gami) = paper". In this sense of the word, origami is a mathematical process giving a flat piece of paper appropriate folds and vertexes joining several folds, which results in a polyhedral surface.

We study a curved surface by means of the fundamental magnitudes of the first order and the consequent Christoffel symbols and Gaussian curvature, and we treat it "intrinsically". In the same way, we are able to study a polyhedral surface of origami intrinsically, and can obtain such theorems which govern the relations of

* This paper was presented at the First International Meeting of Origami Science and Technology, Ferrara, Italy, December 6–7, 1989.

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convexity/concavity of folds and the vertices of origami. Through this approach, we can consequently establish a science of intrinsic geometry of origami.

In this paper, the author presents the basic theorems on the surface of origami. The theorems are then used to exploit the properties of folds around a vertex. This is the first approach toward the science of intrinsic geometry of origami. Many drawings are used to provide the readers a visual glimpse on intrinsic geometry of origami. The theorems also gives mathematical foundation of the map fold a la Miura style, or Miura-ori, presented also in this book (Miura, 1993).

## 2. Gauss's Spherical Representation

The Gauss's spherical representation is very useful for the interpretation for the Gaussian curvature and the excellent introduction was made by Hilbert and Cohn-Vossen (Hilbert and Cohn-Vossen, 1932) and Coxeter (Coxeter, 1961). The author would like to borrow the following description from Coxeter.

To obtain his spherical representation of a surface, Gauss considered the locus of the end Q of a vector OQ = n where O is a fixed point and n is the unit normal at a point P which varies on the given surface (Fig. 1).

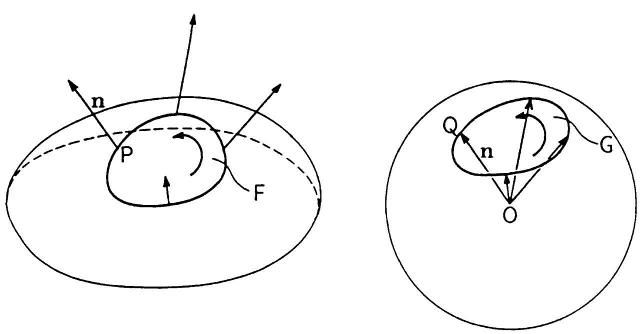


Fig. 1

When P travels over a sufficient small region $F$, bounded by a simple closed curve on the surface, Q travels over a corresponding region $G$ of the unit sphere with center O. Gauss defined the total curvature of the surface at P to be the limit of the area of $G$ and $F$ where these regions are shrunk to single points.

$$\lim_{F\to 0} \frac{G}{F} = K \tag{1}$$

The quantity $K$ thus defined is called the Gaussian curvature of the surface at the point P. It has also the well-known important property as follows: "The Gaussian curvature is invariant by any inextensional deformation."

## 3. Spherical Representation of Origami

The spherical representation can also be used for a polyhedral surface as shown in Fig. 2 (Hilbert and Cohn-Vossen, 1932). In this figure, p, q, r, and s are the unit normals of the sides P, Q, R, and S. In the right figure, p′, q′, r′, and s′ are the ends of corresponding normals drawn from the fixed point O. Connecting these points by large circles, we have a spherical polygon p′q′r′s′ which is the spherical representation of a closed curve on a polyhedral surface. We assumed here that the closed curve on a polyhedral surface is constructed so that each curve connecting the base of unit normals is orthogonal to the ridge line.

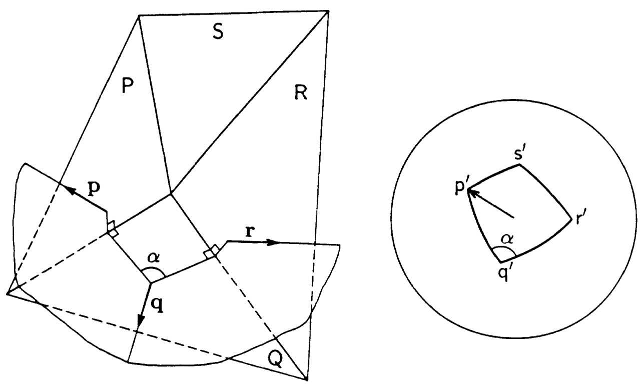


Fig. 2

Now we are going to investigate the surface of origami based on the above preparation. As is mentioned before, an origami work can be mathematically described by a polyhedral surface. The mathematical expression of an origami process is a transformation of a flat piece of paper into a polyhedral surface which expresses something. We assume, of course, the surfaces of a thin paper are sufficiently accurate model of the abstract definition of a surface, and thus hereafter both surface and paper may be used in the same meaning.

The Gaussian curvature $K$ is zero for a flat piece of paper. This is the initial condition before we start to fold. The folding process, that is, the transformation of a piece of paper does not include any extensional deformation. (The stretching, cutting, and gluing are prohibited for the orthodox origami.) Therefore, the zero

Gaussian curvature is invariant throughout the process and of course throughout the surface.

If we assume the area of the spherical representation $G$ as

$$G = 0$$

then, the zero Gaussian curvature condition is satisfied.

$$K = 0$$

On the contrary, if we assume

$$G \neq 0$$

then,

$$K = \lim_{F \to 0} \frac{G}{F} = \frac{\lim_{F \to 0} G}{\lim_{F \to 0} F} = \frac{G}{\lim_{F \to 0} F} = \pm\infty$$

because $G$ is determined solely by the normals, and that the normals in turn are independent on the change in the closed curve ($F \to 0$). Therefore, the latter assumption is irrational and the quantity $G$ must be vanished at any point of the surface. Thus, it is proved that

$$G = 0 \tag{2}$$

In this way, we have arrived at the following theorem.

*Theorem I*

*"For a sufficiently small closed curve on the surface of an origami, the area of the corresponding spherical representation G is zero."*

The proof of the above theorem can be done by another way. Using the wellknown expression for $G$, Eq. (2) can be written by the following formula,

$$G = \int\int |K| \sqrt{g} du^1 du^2 = 0 \tag{3}$$

where $g$ is the determinant of the first fundamental quantities of the surface, and $u^1$, $u^2$ are the coordinates. Because $K = 0$ on $(u^1, u^2)$ and $g$ is finite, $G$ must be zero. As a matter of fact, the above theorem is valid for any surfaces of zero Gaussian curvature, such as origami including curved folds.

The invariance of $G$ can also be proved and the following theorem is obtained.

*Theorem II*

*"For a sufficiently small closed curve on the surface of an origami, the area of the corresponding spherical representation G is invariant with fold angles."*

It is shown by this analysis that the intrinsic properties of origami can now be studied by means of the quantity $G$ as well as $K$, both are zero everywhere on the surface of an origami. Especially, the use of the quantity $G$ is most convenient, as we do not need to calculate a limiting value of a certain quantity.

## 4. One, Two, and Three Fold-Lines Join a Vertex

By means of the above theorems, we now study the intrinsic properties in the vicinity of an arbitrary point on the surface of an origami. The main subject is the relation of the number of folds at a vertex and their convexity/concavity.

### *4.1 One fold-line*

We assume the case that there is only a single fold-line (like a half-line) on the surface of an origami as shown in Fig. 3.

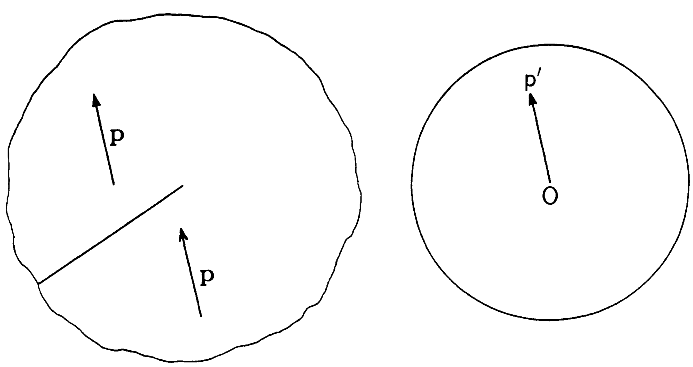


Fig. 3

We do not need to use the theorems in this case. The single fold-line does not divide a plane into two parts (like a half-line), and then apparent two planes at both sides of the fold-line are identical. Thus, there is no fold-line and the assumption is wrong. Therefore, this case does not exist.

*4.2 Two fold-lines join a vertex*

We assume two fold-lines, a and b, joining a vertex as shown in Fig. 4.

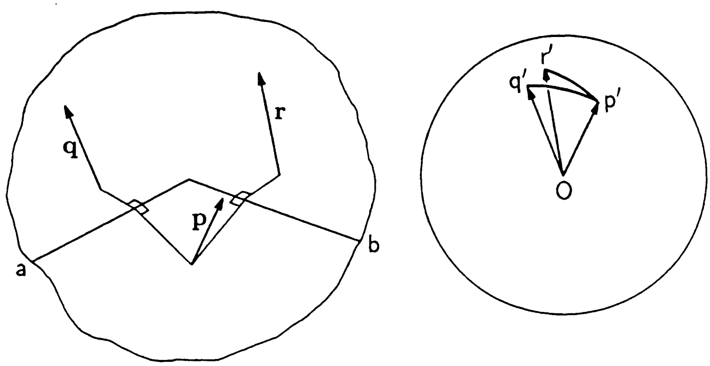


Fig. 4

In reference to the fold-line a, there exist two planes P and Q represented by unit normals p and q in the normal plane of a. The spherical representation of it is a segment p′q′ as shown in the right figure. In the same way, in reference to the fold-line b, the spherical representation p′r′ is obtained. Because the normals q and r are on the identical plane Q, p′q′ and p′r′ are identical, too. Because p′q′ is normal to a and b, the fold-lines a and b must be collinear. The area bounded by the spherical representation p′q′ is zero and thus $G = 0$ is satisfied. It turns out that this is the simplest of any origami works.

*4.3 Three fold-lines join a vertex*

We assume that the three fold-lines join a vertex as shown in Fig. 5.

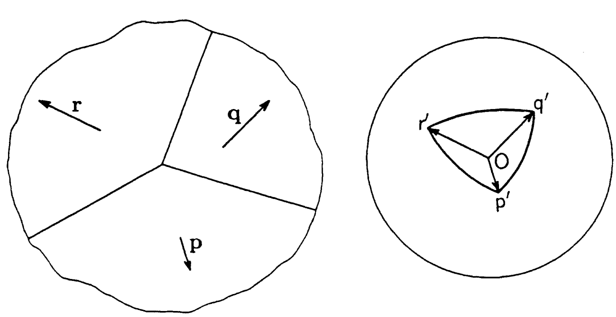


Fig. 5

Then there exist three different unit normals p, q, r, representing three planes P, Q, R. The resultant spherical representation is a spherical triangle p′q′r′. As far as p, q, and r are different from each other, the area of the triangle can not be vanished. In conclusion, $G = 0$ is not satisfied, the assumption is irrational, and this case does not exist.

## 5. Four Fold-Lines Joining a Vertex

### *5.1 Four fold-lines have a common sign (convex or concave)*

We assume that all of the four fold-lines are convex, and four planes P, Q, R, S are different from each other, then the spherical representation is a spherical quadrangle p′q′r′s′, as shown in Fig. 6. The area of spherical polygon is given by the following formula.

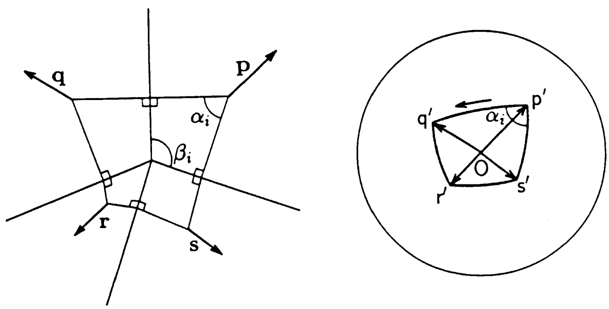


Fig. 6

$$A = \left[ \sum_{1}^{n} \alpha_i - (n-2)\pi \right] \rho^2 \tag{4}$$

where $\alpha_i$ are the internal angles of the spherical polygon, and $\rho$ is the radius of the sphere. The first term in the bracket is called the spherical excess and it must be positive if a polygon is a spherical polygon. The radius $\rho$ is equal to 1 for the spherical representation. The internal angles $\alpha_i$ (i = 1 ~ 4) and the vertex angles $\beta_i$ are supplementary. Then,

$$\sum^{4} \alpha_i + \sum^{4} \beta_i = 4\pi \tag{5}$$

and $K = 0$,

$$\sum^{4} \beta_i = 2\pi \qquad \sum^{4} \alpha_i = 2\pi$$

It results in a quadrangle whose sum of internal angles is $2\pi$ and the spherical excess is zero. There is no such a spherical quadrangle, except that, it is a single point on the surface. Therefore, the assumption is irrational. In conclusion, there is no four fold-lines joining a vertex if these fold-lines have a common sign.

The above theorem can be extended to the case of n fold-lines with a common sign as shown in Fig. 7. The equivalent equation to Eq. 5 for this case is

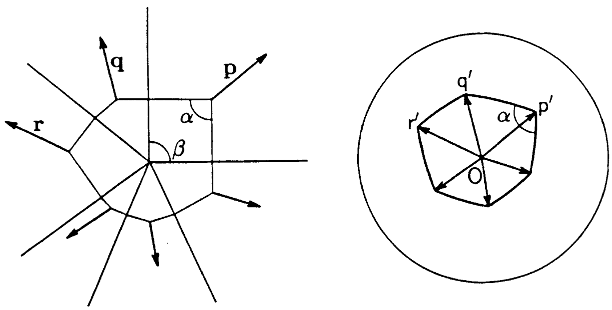


Fig. 7

$$\sum^{n} \alpha_i + \sum^{n} \beta_i = n\pi \tag{6}$$

Because of $K = 0$

$$\sum^{n} \beta_i = 2\pi \qquad \sum^{n} \alpha_i = (n-2)\pi$$

The sum of internal angles of this spherical $n$-polygon is the same as that of a planar $n$-polygon. Thus, the spherical excess is zero, and there is no such spherical polygon. Therefore, except $n = 2$, $n$ fold-lines with a common sign does not exist.

*5.2 Three fold-lines have a common sign, one fold-line has the other sign*

We assume that three fold-lines are convex and the other fold-line is concave, and resulting four planes P, Q, R, S are different from each other. The spherical representation is a skewed quadrangle p′q′r′s′, as shown in Fig. 8. In this figure the direction of the closed curve, as indicated by the arrows, is to be noted. Due to usual mathematical premise, the area enclosed by a closed curve in counter-clock-wise is taken positive and the one in clockwise is taken negative.

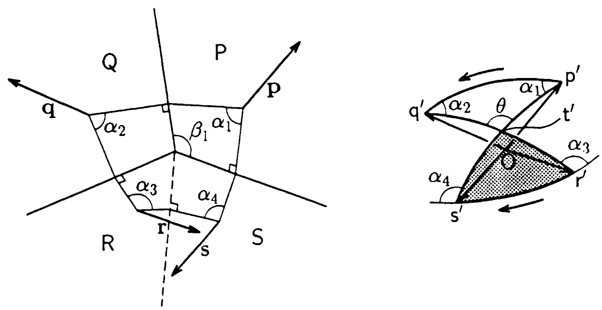


Fig. 8

In Fig. 8, the spherical triangle p′q′t′ has the positive area, while the shaded spherical triangle r′s′t′ has the negative area. The area of a spherical triangle depends only on the sum of internal angles. Then, the sum of internal angles of the triangle p′q′t′ is

$$\alpha_1 + \alpha_2 + \theta \tag{7}$$

while that of the triangle r′s′t′ is

$$(\pi - \alpha_3) + (\pi - \alpha_4) + \theta = 2\pi - (\alpha_3 + \alpha_4) + \theta = \alpha_1 + \alpha_2 + \theta \tag{8}$$

Therefore, the total area of the spherical representation $G$ vanishes.

This relation is valid for any combination of internal angles and thus any combination of vertex angles. That means it is valid for any three-to-one fold-lines.

*5.3 Orthogonal fold-lines*

This is the most popular fold pattern seen everywhere in our daily life. The structure of this fold consists of a pair of collinear positive fold-lines and a pair of collinear positive and negative fold-lines, and that both intersect at a right angle

(Fig. 9).

The spherical representation is the skewed spherical rectangle because of the condition

$$\alpha_i = \pi / 2. \tag{9}$$

The quadrangle consists of a pair of isosceles spherical triangle with right base angles. It should be noted that p′, q′, r′, and s′ lines on the equator of the sphere. Also, p′ and s′, q′ and r′ are in the opposite positions, respectively. It means that, the direction of normals p and s, q and r are in opposite directions respectively. It is easy to understand this situation by folding a piece of paper. We find that the paper is already folded up along the collinear convex fold-lines which divide the paper convex to concave regions.

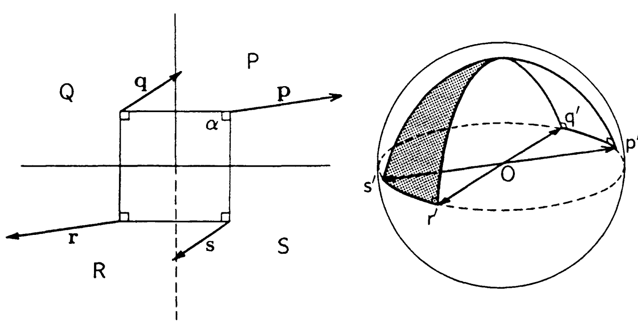


Fig. 9

This remarkable result indicates that the orthogonal folding is not a single step, but the sequential two steps of the simplest folding. In conclusion, there is no simultaneous orthogonal folding.

*5.4 Two convex and two concave fold-lines join a vertex*

If two fold-lines are convex and other two are concave, as shown in Fig. 10, and resulting four planes P, Q, R, and S are different from each other, the spherical representation is again a spherical quadrangle p′q′r′s′.

This figure is quite similar to the case Subsection 5.1 (Fig. 6), where four fold-lines with a common sign join a vertex. It should be noted, however, that the order of the normals p′q′r′s′ is clockwise in this quadrangle. It means the area of the quadrangle is negative and thus the surface consists of hyperbolic points.

The result is that the area of the quadrangle can be vanished only if the four normals are identical. Therefore, the initial assumption is irrational and there is no

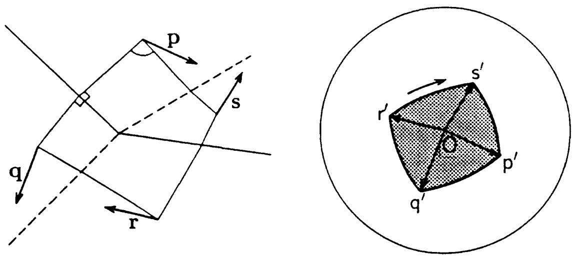


Fig. 10

two-to-two fold-lines joining a vertex.

## 6. Miura-Ori

The so-called Miura-ori or Miura-map consists of repetition of a fundamental region, which in turn is composed of four congruent parallelograms (cf. Figs. 3, 4, 5 of the reference 1 (Miura, 1970) in this book (Miura, 1993)).

In the light of intrinsic geometry of origami derived in this paper, it turned out that the Miura-ori consists solely of a single species of fold, the three-to-one fold-lines joining a vertex (cf. Fig. 8 of reference 1 (Miura, 1970; see Miura, 1993)). It is true that the Miura-ori is the second simplest origami only after the simplest, collinear two fold-lines at a vertex.

The behavior of simultaneous deployment and folding of a Miura-ori can also be explained by the properties inherited with the three-to-one fold-lines at a vertex. In marked contrast to the Miura-ori, the orthogonal fold is actually the succession of two simple steps. Furthermore, you can not reverse their order of deploying. This may be the reason why you never observe any orthogonal fold patterns in a randomly crushed piece of paper. It seems Nature likes angular folds rather than the orthogonal one.

## 7. Conclusion

In this paper the author presents a proposition of intrinsic geometry of origami. It treats the geometric properties which are dependent only on the vicinity of an arbitrary point on the surface of origami works. Based on the theory of curved surfaces, the basic theorems of intrinsic geometry of origami are obtained. Using the theorems, the relations of convexity/concavity of folds and the vertices are studied. The result is also used to explain the particular characteristics of the so-called Miura-ori.

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