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Hidden Singularities, Phase Transitions and Conservation Law in Multifractal Patterns

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Abstract. A statistical-mechanical approach is developed to extract more detailed information from multifractal patterns. The analogy with spin systems is emphasized. The $f(\alpha)$ formalism proposed by Halsey *et al.* (Phys. Rev., **A33**, 1141, 1986) is generalized by introducing an "external field" to remove the degeneracy in $f(\alpha)$ and to obtain the complete spectrum of the singularities in the multifractal patterns. It is also seen that "phase transition" can occur under an appropriate condition. We take into account conserved quantities to analyze the pattern including lacunae. Some perspective are discussed as well.

1. Introduction

Multifractal sets (Mandelbrot, 1974; Hentschel and Procaccia, 1983) have been vigorously investigated as a common notion in many physical object such as strange attractors in chaotic systems (Cvitanovic *et al.*, 1985; Feigenbaum *et al.*, 1986; Jensen *et al.*, 1987; Katzen and Procaccia, 1987; Bohr and Jensen, 1987; Sato and Honda, 1990), fractal clusters in growth processes (Halsey *et al.*, 1986; Matsushita *et al.*, 1987; Hayakawa *et al.*, 1987; Ohta and Honjo, 1988), velocity distributions in turbulence (Benzi *et al.*, 1984; Meneveau and Sreenivasan, 1987), energy spectrum in quasicrystals (Kohmoto *et al.*, 1987) and so forth (de Arcangels *et al.*, 1986; Bene and Szepfalusy, 1988).

Let us consider a set which is covered by $N(l)$ boxes with a size of order l. Each box is assigned to a respective measure $P_i(l)$ ($i = 1, 2, ..., N(l)$). For growing fractal patterns, for example, harmonic measures on the growth surface can be regarded as the probability measure. The generalized dimension D_q for any real q is defined by (Hentschel and Procaccia, 1983)

$$D_q = \lim_{l \to 0} \frac{1}{q-1} \frac{\ln Z_l(q)}{\ln l}, \tag{1}$$

where the partition function $Z_l(q)$ is given by

$$Z_l(q) = \sum_i [P_i(l)]^q. \tag{2}$$

$D_{q=0}$ is equal to the Hausdorff or fractal dimension of the set itself (support of the measure), which is also obtained by averaging the measures over the set and putting $P_i(l) = 1/N(l)$. Halsey *et al.* (1986) clarified that the generalized dimension D_q are closely related to the spectrum $f(\alpha)$ of singularities, which characterizes the multifractal set: Suppose that the measure of the i-th box follows the power law as $P_i(l) \sim l^{\alpha_i}$, and the number of boxes with α_i taking on a value between α and α + $d\alpha$ is $\rho(\alpha) l^{-f(\alpha)} d\alpha$. The quantity $f(\alpha)$ is a fractal dimension of the interwoven subsets with the singularity α. The sum in Eq. (2) is then replaced by the integral

$$Z_l(q) = \int d\alpha \rho(\alpha) l^{q\alpha - f(\alpha)}. \tag{3}$$

Since l is very small, a steepest descent method is applied to obtain

$$Z_l(q) \sim l^{q\alpha(q) - f(\alpha(q))}. \tag{4}$$

$\alpha(q)$ minimizes the exponent $q\alpha - f(\alpha)$ with respect to α;

$$\left. \frac{df(\alpha)}{d\alpha} \right|_{\alpha = \alpha(q)} = q, \tag{5}$$

$$\frac{d^2 f(\alpha)}{d\alpha^2} < 0. \tag{6}$$

From Eqs. (1) and (4) we obtain $\tau(q)$ defined by $\tau(q) = (q-1)D_q$ as

$$\tau(q) = q\alpha(q) - f(\alpha(q)) \tag{7}$$

The Legendre transformation of $\tau(q)$ leads to the $f(\alpha)$ spectrum;

$$\alpha(q) = \frac{d\tau(q)}{dq} \tag{8}$$

$$f(q) = q\alpha(q) - \tau(q) \tag{9}$$

by making use of q as a parameter.

As reviewed briefly in Section 2, the multifractal patterns can be analyzed in the formalism of statistical mechanics by redefining partition processes. It is seen that the partition function $Z_l(q)$ in Eq. (2) has the same form as those for one-dimensional classical spin systems in equilibrium. Each divided part of the multifractal pattern is assigned to one of the spin configurations. Using well known technique for the spin systems, we can develop the method for analyzing the multifractal patterns and acquire some perspective as well.

In a usual scenario of obtaining $f(\alpha)$ experimentally (Ohta and Honjo, 1988; Glazier *et al.*, 1988; Atmanspacher *et al.*, 1988) or numerically (Hayakawa *et al.*, 1987) generated multifractal patterns, we first calculate $Z_l(q)$ in Eq. (2) and then evaluate $\tau(q)$ from the slope of log-log plots of $Z_l(q)$ versus box-size l. The numerical derivative of $\tau(q)$ gives $\alpha(q)$ from Eq. (8), and then $f(q)$ from Eq. (9). The transfer matrix method (Feigenbaum *et al.*, 1986; Jensen *et al.*, 1987; Katzen and Procaccia, 1987;Feigenbaum, 1987) using what is called daughter-to-mother ratio as a scaling function also follows the same procedure, though $\tau(q)$ or its inverse function $q(\tau)$ can be ingeniously obtained. Generally speaking, however, there exists in the conventional method described above such a serious shortcoming that most of the original singularities in the multifractal patterns cannot be extracted. We call them hidden singularities. In Sections 3 and 4 we consider simple models to illustrate the hidden singularities, which will be elucidated to arise from the "degeneracy in energy".

In Section 5 we introduce an order parameter to propose "phase transitions" associated with the generalized dimension D_q or the singularity spectrum $f(\alpha)$ in the multifractal patterns. To generate such patterns we use Husimi-Temperly model for ld Ising system. If, in the division process, there exist weak correlations among any steps, the phase transition can occur. Critical properties in patterns are very interesting to be studied.

Many multifractal patterns in nature include lacunae within themselves. To analyze these, we have to remove vacant parts from the sum in Eq. (2). In Section 6 we give a simple method with use of the conservation law, which will provide a basic concept as in other fields of physics.

The last section is devoted to discuss some generalization and to summarize the present paper, a part of which has been already presented by Honda (1989).

2. Statistical Mechanics Formalism

Let us consider a one-dimensional multifractal pattern at the n-th level of refinement covered by k^n (k; a positive integer) boxes of length $l = k^{-n}$. Extension to higher dimensions is straightforward as discussed in Section 7. Each box of the pattern is addressed by $\{s\}_n = (s_1, s_2, ..., s_n)$, where s_j takes on values $s_j = 1, 2, ..., k$. In the case for $k = n = 2$, for example, the address of a respective quarter is given in turn from left to right as (1, 1),(1, 2),(2, 1) and (2, 2). The measure of the i-th box $P_i(l)$ is reexpressed as $P(\{s\}_n)$. When a "Hamiltonian" is defined by the information,

$$H(\{s\}_n) = -\ln P(\{s\}_n), \tag{10}$$

we have the partition function in the same form as in the statistical physics;

$$Z_n(q) = \sum_{\{s\}_n} \exp\left[-qH(\{s\}_n)\right]. \tag{11}$$

The subscript n has been used in place of l. The quantity q corresponds to the inverse temperature β, although it takes on values $-\infty \le q \le \infty$. $\tau(q)$ is given by

$$\tau(q) = -\lim_{n\to\infty} n^{-1}\ln Z_n(q) / \ln k. \tag{12}$$

Suppose that the "Hamiltonian" has continuous energy spectrum E. Then the partition function is calculated as

$$Z_n(q) = \int dE\Omega(E)\mathrm{e}^{-qE}, \tag{13}$$

where $\Omega(E)$ is the density of states. Since the energy E and the entropy $S(E) = \ln \Omega(E)$ become larger in proportion to n, we can apply the steepest descent method for large n to Eq. (13) yielding

$$Z_n \cong \exp\left[S(E^*) - qE^*\right], \tag{14}$$

where E^* is a function of q satisfying

$$\left.\frac{dS(E)}{dE}\right|_{E=E^*} = q. \tag{15}$$

In addition E^* is obviously equal to the averaged energy

$$\langle E\rangle = Z_n(q)^{-1}\sum_{\{s\}_n} H(\{s\}_n)\exp\left[-qH(\{s\}_n)\right] \tag{16}$$

$$= -\frac{\partial \ln Z_n(q)}{\partial q}.$$

We can then identify a "free energy" with

$$F(q) = E^* - q^{-1}S(E^*). \tag{17}$$

From these analysis the well known thermodynamic formalism for the multifractal patterns can be derived straightforwardly. Comparing Eqs. (2) ~ (9) with Eqs. (11) ~ (17), we immediately see the following relations, aside from a trivial coefficient, ln k,

$$\alpha(q) = \langle E\rangle / n,$$

$$f(\alpha) = S(E) / n, \tag{18}$$

$$\tau(q) = qF(q) / n.$$

We will apply this formalism in the following sections. It should be noted in particular that Eq. (10) is the Hamiltonian of 1d Ising spin system with k states. Generally it should consist of terms proportional to n, which guarantees the scaling property of $Z_n(q)$ in Eq. (11).

3. Hidden Singularities

In order to illustrate the hidden singularities mentioned briefly in Section 1, we study the simplest but nontrivial case, a series of trisection of the segment [0, 1] as shown in Fig. 1. At each stage of division, the measures p_1, p_2, p_3 ($p_1 + p_2 + p_3 = 1$) are assigned to the segments in turn from left to right, respectively. To avoid confusing complexity (Halsey *et al.*, 1988; Kohmoto, 1988), the lengths of three

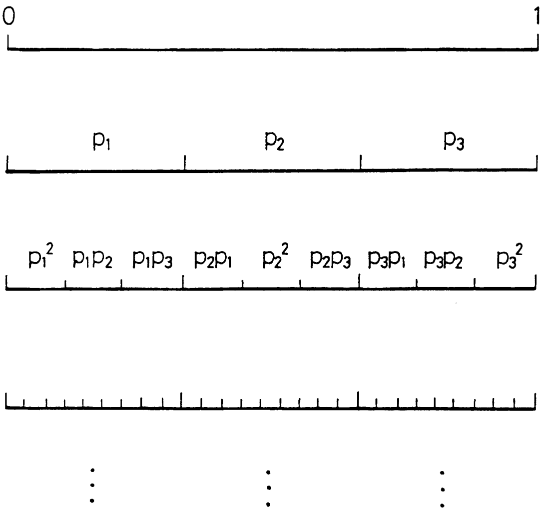

Fig. 1. A series of trisection of the segment [0, 1] with three measures rescaling $p_1 = 1/2, p_2 = 1/3$ and $p_3 = 1/6$. The division of the set continuous self-similarly.

segments are assumed to be equal to each other. More general cases will be discussed in Section 7. At the n-th step, the length of division is $l = 3^{-n}$ The usual scenario to obtain the singularity spectrum $f(\alpha)$ is as follows: Let us denote the measure of the box which takes p_i by m_i times ($i = 1, 2, 3$) at the n-th step ($m_1 + m_2 + m_3 = n$) as $P(m_1, m_2, m_3)$, which is

$$P(m_1, m_2, m_3) = p_1^{m_1} p_2^{m_2} p_3^{m_3}. \tag{19}$$

The number of those boxes is obviously given by

$$C_n(m_1, m_2, m_3) = n!/(m_1!\, m_2!\, m_3!). \tag{20}$$

Hence we have the partition function at the n-th step

$$Z_n(q) = \sum_{m_1}\sum_{m_2}\sum_{m_3} C_n(m_1, m_2, m_3)\left[P(m_1, m_2, m_3)\right]^q. \tag{21}$$

These three sums are carried out under the condition, $m_1 + m_2 + m_3 = n$. The substitution of Eqs. (19) and (20) into Eq. (21) immediately yields

$$Z_n(q) = \left(p_1^q + p_2^q + p_3^q\right)^n. \tag{22}$$

Using Eqs. (8), (9) and (12), we find

$$\alpha(q) = -\left(v_1 \ln p_1 + v_2 \ln p_2 + v_3 \ln p_3\right) / \ln 3, \tag{23}$$

$$f(q) = -\left(v_1 \ln v_1 + v_2 \ln v_2 + v_3 \ln v_3\right) / \ln 3, \tag{24}$$

where

$$v_i = p_i^q / \left(p_1^q + p_2^q + p_3^q\right), \quad (i = 1,2,3), \tag{25}$$

are functions of only one variable q. Then Eqs. (23) and (24) give the $f(\alpha)$ curve, as shown illustratively by a solid line in Fig. 2 for the case of $p_1 = 1/2, p_2 = 1/3, p_3 = 1/6$.

We apply the formalism described in Section 2 to the case. Its process of refinement is markovian

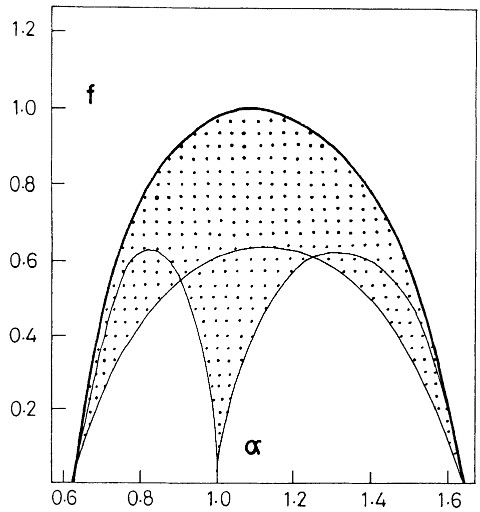

Fig. 2. The plot of f vs. α for the set in Fig. 1. Dots represent the singularities generated by Eqs. (31) and (33). Respective thin line corresponds to $r_i = 0$ ($i = 1, 2, 3$) in Eqs. (31) and (33), while an envelope bold line represents the f–α curve obtained from Eqs. (23) and (24).

$$P\left(\{s\}_n\right) = \prod_{j=1}^{n} p_{s_j}. \tag{26}$$

Here s_j takes on ternary values 1, 2 or 3. Hence the Hamiltonian $H(\{s\}_n)$ can be expressed as the sum of individual ones

$$h\left(s_j\right) = -\ln p_{s_j}, \tag{27}$$

which can be described generally in term of s_j as

$$h\left(s_j\right) = as_j^2 + bs_j + c, \tag{28}$$

by making use of appropriate coefficients a, b and c. Substituting Eqs. (26) and (27) into Eq. (10) and using it in Eq. (11), we get

$$Z_n(q) = \left\{e^{-qh(1)} + e^{-qh(2)} + e^{-qh(3)}\right\}^n, \tag{29}$$

which is equivalent to Eq. (22).

On the other hand, the direct approach to extract the singularity spectrum is also possible as follows: The measure given in Eq. (19) shows a scaling

$$P\left(m_1, m_2, m_3\right) = l^{\alpha(r_1, r_2, r_3)} \tag{30}$$

with the exponent

$$\alpha\left(r_1, r_2, r_3\right) = -\left(r_1 \ln p_1 + r_2 \ln p_2 + r_3 \ln p_3\right) / \ln 3. \tag{31}$$

Here we have used the respective ratio $r_i = m_i/n$ ($i = 1, 2, 3$; $r_1 + r_2 + r_3 = 1$). Using Stirling approximation for large n, the number of boxes $C_n(m_1, m_2, m_3)$ is also found to obey the scaling law

$$C_n\left(m_1, m_2, m_3\right) \sim l^{-f(r_1, r_2, r_3)}, \tag{32}$$

where

$$f\left(r_1, r_2, r_3\right) = -\left(r_1 \ln r_1 + r_2 \ln r_2 + r_3 \ln r_3\right) / \ln 3. \tag{33}$$

Equations (31) and (33) represent a region in the f–α plane, because the parameters r_1, r_2, r_3 move under the restrictions that $0 \leq r_1, r_2, r_3 \leq 1$ and $r_1 + r_2 + r_3 = 1$. The region for the above case is shown by dots in Fig. 2. This argument clearly indicates the fact that the usual scenario may miss the singularities corresponding to the inner part of the $f(\alpha)$ curve. They are the hidden singularities.

Returning to the statistical mechanics formalism, we notice that $<h(s_j)>= \alpha(q)$ consists inherently of independent quantities $<s_j>$ and $<s_j^2>$, so that a number of states giving different $\tau(q)$ or $f(\alpha(q))$ with the same $\alpha(q)$ can exist. In order to remove the degeneracy in the energy, we introduce an additional field η conjugate to an order parameter $\Psi(\{s\}_n)$, and add the term, $\eta\Psi(\{s\}_n)$, into the exponent in Eq. (11). The form of $\Psi(\{s\}_n)$ is not necessarily specified for our purpose, if it is independent of the energy $H(\{s\}_n)$ and extensive. The latter condition assures the scaling of the partition function. We adopt here the simplest case $\Psi(\{s\}_n) = \Sigma_j(s_j - 2)$. As is easily seen, η and Ψ correspond, respectively, to magnetic field and magnetization in the 3-state Ising system. Hence we have, in place of Eq. (11),

$$Z_n(q,\eta) = \sum_{\{s\}_n} \exp\left[-qH\left(\{s\}_n\right) - \eta\Psi\left(\{s\}_n\right)\right]. \tag{34}$$

Following the procedures familiar in statistical physics, we obtain

$$f\left(\alpha(q,\eta), \psi(q,\eta)\right) = q\alpha(q,\eta) + \eta\psi(q,\eta) - \tau(q,\eta), \tag{35}$$

where use was made of the relations

$$\tau(q,\eta) = -\lim_{n\to\infty} n^{-1}\ln Z_n(q,\eta), \tag{36}$$

$$\alpha(q,\eta) = \partial\tau(q,\eta)/\partial q, \tag{37}$$

$$\psi(q,\eta) \equiv <\Psi\left(\{s\}_n\right)>/n = \partial\tau(q,\eta)/\partial\eta. \tag{38}$$

Substituting Eq. (34) into Eqs. (35) ~ (38) and using Eq. (27), we finally obtain

$$\alpha(q,\eta) = h(1)u_1 + h(2)u_2 + h(3)u_3, \tag{39}$$

$$f(q,\eta) = -\left(u_1\ln u_1 + u_2\ln u_2 + u_3\ln u_3\right), \tag{40}$$

where u_is are defined respectively by

$$u_i = \frac{e^{-qh(i)-\eta(i-2)}}{e^{-qh(1)+\eta} + e^{-qh(2)} + e^{-qh(3)-\eta}} \tag{41}$$

They move in the same region of values as r_is appeared in Eqs. (31) and (33). The singularity spectrum $f(\alpha)$ is generalized into a function $f(\alpha, \psi)$ both of α and ψ.

We have thus succeeded to recover Eqs. (31) and (33), aside from a trivial coefficient ln3, by introducing a new pair of thermodynamic variables. The Eq. (22) or (29) corresponds to the case $\eta = 0$. Since $f(\alpha(q, 0), \psi(q, 0)) \geq f(\alpha(q, \eta), \psi(q, \eta)$, owing to the relations $\partial f/\partial\psi = \eta = 0$ and $\partial^2 f/\partial\psi^2 < 0$, the border in Fig. 2 is given by Eqs. (23) and (24).

4. Correlated Multifractal Patterns

From the arguments in Section 3 we can easily forecast that the hidden singularities can appear in more general cases. The hidden singularities arise from the degeneracy of states with the same energy. This situation can occur even when the refinement processes are binomial, where s_i takes on values ± 1. If the process is non-markovian and has a memory of the previous one step, for simplicity, the measure is expressed as

$$P(\{s\}_n) = \prod_i P_1(s_i) \prod_i P_2(s_i \;\; s_{i+1}) \tag{42}$$

Hence we define a generalized partition function

$$Z_n(q \;\; \eta) = \sum_{\{s\}_n} \left[P_1(s_i)\right]^\eta \left[P(\{s\}_n)\right]^q \tag{43}$$

Normalization condition requires $Z_n(1, 0) = 1$. From Eq. (10) the corresponding Hamiltonian is given by, which use of a coupling constant J between nearest neighboring spins and an external field h,

$$H(\{s\}_n) = -J \sum_i s_i s_{i+1} - h \sum_i s_i \tag{44}$$

except for a normalization constant. In Fig. 3 we show the patterns constructed by printing dots in proportion to $\exp[-H(\{s\}_n)]$ into the respective slit with of 2^{-n}, which corresponds to one of the spin configurations. The partition function via an easy exercise is

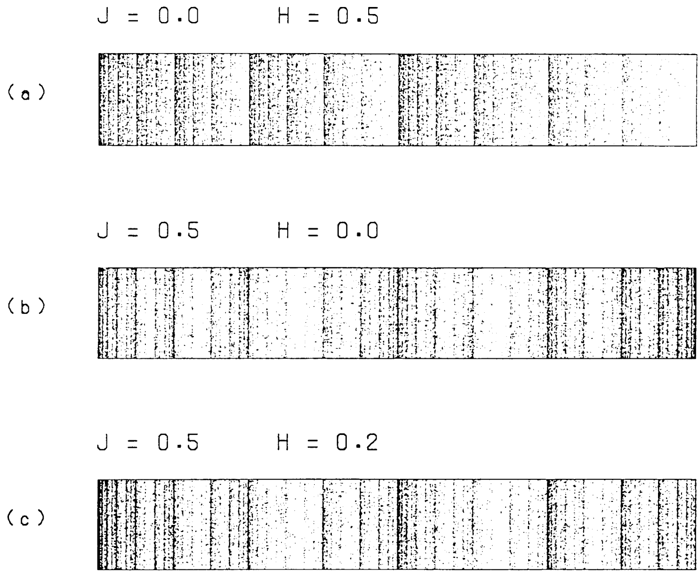

Fig. 3. One-dimensional multifractal patterns generated by the "Hamiltonian" Eq. (44). A number of points in a respective slit with a width of 1/1024 is proportional to $\exp[-H(\{s\}_n)]$.

$$Z_n(q,\eta)=\left[\frac{e^{qJ}\cosh(q+\eta)h+\left(e^{2qJ}\cosh^2(q+\eta)h-2\sinh 2qJ\right)^{1/2}}{e^{J}\cosh h+\left(e^{2J}\cosh^2 h-2\sinh 2J\right)^{1/2}}\right]^n. \tag{45}$$

The denominator is required due to the normalization. We here note that the system represented by the Hamiltonian Eq. (44) has two extensive quantities, that is, the energy $\alpha = \langle H(\{s\}_n)\rangle/n$ and the magnetization $\psi = \langle\Sigma_i s_i\rangle/n$. The two are independent of each other. As discussed in Section 3, therefore, there exist hidden singularities in the pattern in Fig. 3(c) as well. The singularity spectrum $f(\alpha, \psi)$ is obtained by using Eqs. (35) ~ (38) such as

$$\begin{aligned} f(\alpha,\psi)=\big[&-(1/2J)(\xi+h\psi)\ln(\xi+h\psi)+(1/4J)\xi_+\ln|\xi_+| \\ &+(1/4J)\xi_-\ln|\xi_-|+(1/2)(1+\psi)\ln(1+\psi) \\ &+(1/2)(1-\psi)\ln(1-\psi)+\ln 2J\big]/2, \end{aligned} \tag{46}$$

where

$$\xi = (\alpha - \alpha_0)\ln 2, \tag{47}$$

$$\xi_\pm = \xi + h\psi - 2J(1 \pm \psi), \tag{48}$$

$$\alpha_0 = \left[-J + \ln\left\{e^J \cosh h + \left(e^{2J}\cosh^2 h - 2\sinh 2J\right)^{1/2}\right\}\right] / \ln 2. \tag{49}$$

In Fig. 4 we show $f(\alpha, \psi)$, which is defined on the triangle surrounded by three lines, $\xi + h\psi - 2J(1 \pm \psi) = 0$, $\xi + h\psi = 0$. It resembles a sunshade cramped on the latter line and at a point $(2J + \alpha_0 \ln 2, 0)$.

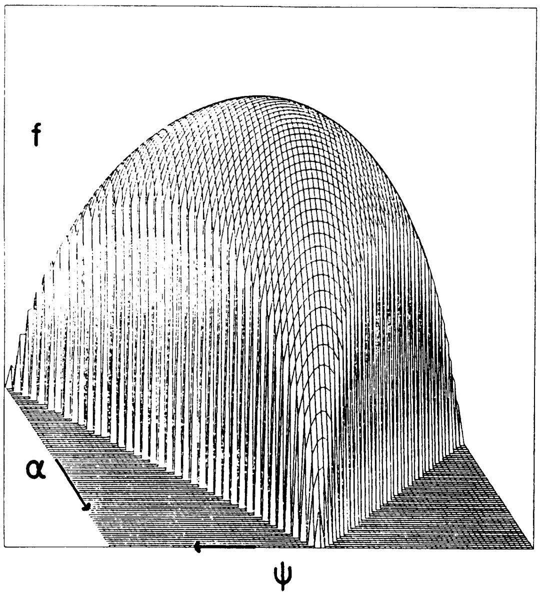

Fig. 4. Three dimensional pictures of $f(\alpha, \psi)$ in Eq. (46), which stand on the triangle surrounded by three lines and cramped at a point and on a line.

Before closing this section, let us discuss the method for analyzing the patterns obtained experimentally or generated numerically. We use the daughter-to-mother ratio defined by

$$\sigma(s_1, s_2, \ldots, s_n) = \frac{P(s_1, s_2, \ldots, s_n)}{P(s_1, s_2, \ldots, s_{n-1})}. \tag{50}$$

Since the measure is expressed as

$$P(\{s\}_n) = \prod_{j=1}^{n} \sigma(s_1, s_2, \ldots, s_j), \tag{51}$$

the Hamiltonian is given as a sum of ln σs. If the ratio for large n depend only on the last spin s_j, the partition can be divided in a markov type. When the ratio depends only on the last two spins, s_j and s_{j-1}, we have a Hamiltonian equivalent to Eq. (44). Whenever above conditions are fortunately satisfied, we can analyze the multifractal patterns rigorously.

5. Phase Transition in Multifractal Patterns

Critical phenomena have attracted many investigators to study their anomalous behaviors and their universal properties. In analyzing multifractal patterns, it is also of interest to introduce the concept of phase transitions. Since the refinement processes of the patterns are sequential, corresponding spin systems are in one-dimensional space, where, as is well known, phase transitions are absent. Only one exception is the case that the interaction between spins is very weak and long range, the Hamiltonian of which is given by

$$H(\{s\}_n) = -(J/2n)\sum_{i}^{n}\sum_{j\neq i}^{n} s_i s_j. \tag{52}$$

This is the Husimi-Temperly model, where a molecular field approximation gives rigorous results. The coupling constant is so weak that the expectation value $<H(\{s\}_n)>$ is extensive and proportional to n. Substituting Eq. (52) into Eq. (11) and following the usual scenario we easily obtain for $0 < J < 1$

$$\tau(q) = \left[(1+m^*)\ln(1+m^*) + (1-m^*)\ln(1-m^*) - qJ + 2(q-1)\ln 2\right]/(2\ln 2), \tag{53}$$

$$\alpha(q) = \left[-Jm^{*2} + 2\ln 2\right]/(2\ln 2), \tag{54}$$

$$f(q) = \left[-(1+m^*)\ln(1+m^*) - (1-m^*)\ln(1-m^*) + 2\ln 2\right]/(2\ln 2), \tag{55}$$

where m^* is the order parameter determined by the self-consistent equation

$$(1/2)\ln\left[\left(1+m^*\right)/\left(1-m^*\right)\right]=qJm^*. \tag{56}$$

It is well known that Eq. (56) has a trivial solution $m^* = 0$ for $q < q_c$ ($=J^{-1}$), while

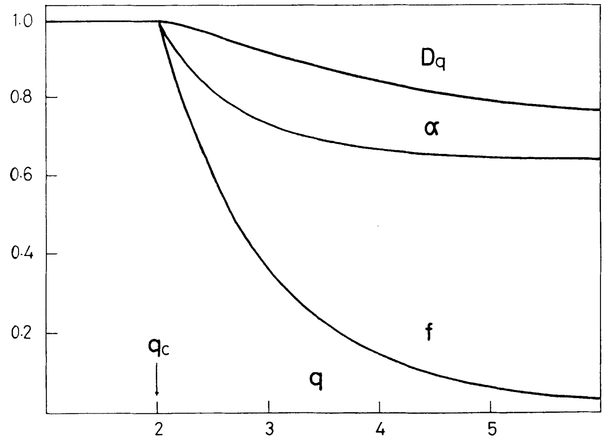

Fig. 5. Plots of D_q, $\alpha(q)$ and $f(q)$ as a function of q for the multifractal pattern generated by Eq. (52). It is noted that the second derivative of D_q is discontinuous at the transition point $q_c = 2$. Both D_q and $\alpha(q)$ approach a same value in the $q \to \infty$ limit.

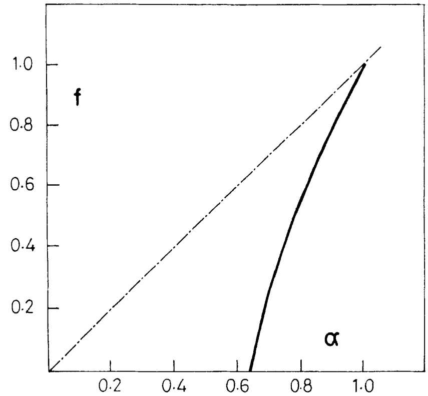

Fig. 6. Plot f vs. α represented by Eqs. (54) and (55) for the case of $J = 0.5$. Note that points for $q \le q_c = 2$ are accumulated at (1, 1).

for $q > q_c$ m^* increases with increasing q, starting from zero at q_c with a critical exponent 1/2. In Fig. 5 we give D_q, $\alpha(q)$ and $f(q)$ showing anomalous derivative at q_c. In particular a singular behavior of $f(\alpha)$ should be noted as shown in Fig. 6, where the points for $q \leq q_c$ (that is, $m^* = 0$) are accumulated at a point (1, 1).

The above argument leads us to the following; The phase transition in a multifractal pattern can occur if only if there exist infinitely long memories in the refinement process. It seems a cock-and-bull story to introduce the phase transition. Nevertheless, we believe that such a novel concept may play an effective role in characterizing the pattern in the future. In particular, the scaling relations and universal properties are very interesting to be studied.

6. Conservation Law in Multifractal Patterns

All cases discussed above have a fractal dimension D_0 of support identical with the space dimension d embedding the pattern. However, there exist many lacunae with size of hierarchical order in patterns of nature. To calculate the partition function for these cases, we have to exclude terms corresponding to the vacant parts. Mathematically, it is often difficult to carry out the summation under such a condition. We here propose a method to this end through introducing the conservation law.

In this case for $D_0 = d$, the whole space of the spin configurations can be occupied. On the other hand, let us imagine that there exist some conservation laws in the spin system. Some parts of the spin configuration, corresponding to voids in the pattern, are forbidden to be occupied, unless they fulfill the conditions. We consider here the simplest case, where the magnetization $\Sigma_i s_i$ is constrained to take on the value mn and the Hamiltonian is expressed by Eq. (44). Such patterns are shown in Fig. 7. Then we have to calculate the partition function

$$Z_n(q) = \sum_{\{s\}_n} \exp\left[-qH(\{s\}_n)\right] \Big/ \left\{ \sum_{\{s\}_n} \exp\left[-H(\{s\}_n)\right] \right\}^q, \tag{57}$$

where the sum is performed under the condition $\Sigma_i s_i / n = m$. Instead of practicing this, it is convenient to introduce the partition function for the "grand canonical ensemble" such as

$$\Xi_n(q, \eta) = \sum_{\{s\}_n} \exp\left[\eta \sum_i s_i \right] \exp\left[-qH(\{s\}_n)\right], \tag{58}$$

which yields easily

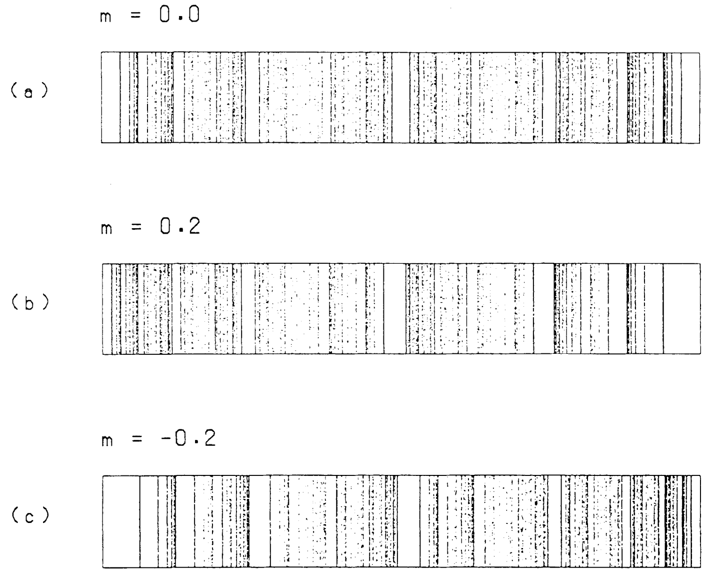

Fig. 7. One-dimensional multifractal patterns with a conserved quantity, $\Sigma_i s_i = nm$. Empty parts correspond to inhibited spin configurations. Each slit is 1/1024 in width. Note Figs. 7b and 7c are equivalent to each other.

$$\Xi_n(q, \eta) = \left[\mathrm{e}^{qJ}\cosh(\eta + qh) + \left\{ \mathrm{e}^{2qJ}\cosh^2(\eta + qh) - 2\sinh 2qJ \right\}^{1/2} \right]^n . \quad (59)$$

The familiar method of Legendre transformation leads to the wanted partition function $Z_n(q)$ as follows:

$$n^{-1}\ln Z_n(q) = -\eta(q, m)m + n^{-1}\ln\Xi_n(q, \eta(q, m)) \quad (60)$$

$$-q\left\{ -\eta(1, m)m + n^{-1}\ln\Xi(1, \eta(1, m)) \right\},$$

where $\eta(q, m)$ is an inverse function of

$$m = n^{-1}\partial \ln\Xi_n(q, \eta) / \partial\eta. \quad (61)$$

Substituting Eq. (60) into Eqs. (8), (9) and (12), we can obtain $\tau(q)$, $\alpha(q)$ and $f(q)$. We give here only $f(\alpha)$ as

$$f(\alpha)=\left[-(1/2J)\xi\ln\xi+(1/4J)\{\xi-2J(1-m)\}\ln|\xi-2J(1-m)|\right.$$

$$+(1/4J)\{\xi-2J(1+m)\}\ln|\xi-2J(1+m)| \quad (62)$$

$$\left.+(1/2)(1+m)\ln(1+m)+(1/2)(1-m)\ln(1-m)+\ln 2J\right]/\ln 2,$$

where

$$\xi=(\alpha-\alpha_0)\ln 2,\quad \left(0\le\xi\le 2J(1-|m|)\right), \quad (63)$$

$$\alpha_0=\left\{\ln\left[e^{-2J}+\left(1-m^2+m^2e^{-4J}\right)^{1/2}\right]\right.$$

$$-m\ln\left[\left\{me^{-2J}+\left(1-m^2+m^2e^{-4J}\right)^{1/2}\right\}(1+m)\right] \quad (64)$$

$$\left.-(1/2)(1+m)\ln(1+m)-(1/2)(1-m)\ln(1-m)\right\}/\ln 2.$$

As shown in Fig. 8, it is noted that $f(\alpha)$ is finite in the $\alpha\to\alpha_{max}$ ($q\to-\infty$) limit for $m\neq 0$ and an even function of m due to the left-right symmetry of Fig. 7. It is noted, moreover, that $f(\alpha)$ in Fig. 8 is a section of $f(\alpha,\psi)$ in Fig. 4 by $\psi=m$ plane.

Conservation law plays a basis role in studying the system in many fields of

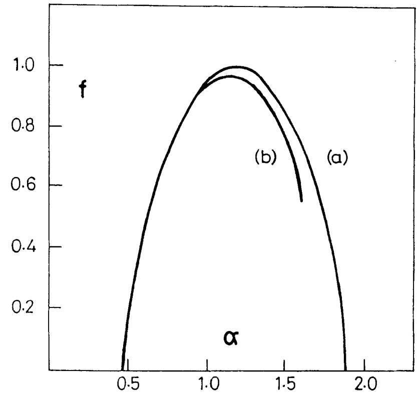

Fig. 8. Plot of f vs. α represented by Eq. (63) for the cases of (a) $m=0$ and (b) $m=\pm 0.2$.

physics. We expect it to play similar roles in the problem of characterizing patterns.

7. Discusslons and Summary

In the present paper, we have taken into account as simple as possible without loss of validity. In this section some generalizations are discussed.

7.1 Higher dimensions

We first divided a d-dimensional cube covering the pattern into k^d subcubes with a side length $1/k$. Such division continues for n steps. Each subcube with a side of k^{-n} is addressed by a sequence $(\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_n)$, where $\boldsymbol{s}_i$ denotes a vector with d components. Then we have a Hamiltonian of one-dimensional classical spin system with a degree of freedom d. It is noted here that, if the refinement process is markovian, the above assignment is equivalent to the Ising spins with k^d states. When the memory effect is necessary to be considered, however, there are some apparent differences between two assignments. The partition function for $1d$ classical spins can be easily calculated.

7.2 Statistics of random patterns

There are some randomness imposed on the patterns in nature. To analyze these patterns, we have to practice an average over the distribution of the coupling constant J and the external field h. Therefore, we get

$$\tau(q\ \ \eta) = -\lim_{n\to\infty} n^{-1} << \ln Z_n(q\ \ \eta\ \ \ J_i\ \ h_i) >> \tag{65}$$

where <<...>> represents the configuration average with respect to J_i and h_i.

7.3 Measures

We have discussed in this paper the case that measures in the multifractal patterns are nonuniform. The case that the division lengths vary from part to part can also be investigated along the same way. The difference appears only in the role of q replaced by τ (Katzen and Procaccia, 1987). A number of cases may not, however, permit to formulate them systematically if both p_i and l_i change together (Kohmoto, 1988). The latter case is forced to introduce two kinds of measures, which make analysis more complex.

In summary, we have demonstrated the possibility of the existence of hidden singularities in multifractal patterns based on the statistical mechanics formalism. This phenomenon should occur in widely generated cases, because the degeneracy of the free energy cannot be broken by the energy alone. Therefore, we have to introduce new pairs of thermodynamic variables to extract further the singularities in the multifractal patterns.

Moreover, we have defined the familiar but novel concepts in this field. One of them is the phase transition. If the memory effect in the refinement process continue weakly in infinite steps, we can expect to occur the phase transition in the multifractal patterns. Second concept is concerned with the conservation law, which has so far been recognized as the basic concept in physics. We expect that this plays an essential role in analyzing patterns.

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