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The Motion of a Vortex Sheet in an Ideal Fluid

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Abstract. A review is given on a Lagrangian representation of the motion of a vortex sheet in an ideal fluid and on similarity solutions of the motions of vortex sheets in two- and three-dimensions. A comment is also made on similarity solutions of the three-dimensional motion of a line vortex.

1. Introduction

Vortex flows in nature and technology exhibit various fascinating flow patterns. For example, the appearance of spiral flow pattern, which is one of the most familiar flow patterns, is usually associated with the existence of vortices in the flows.

In many flows, vortex often exists only in a very thin region outside of which the flow is irrotational. The region may be thin in just one (two) of its dimensions, and then it is usually called a vortex sheet (line vortex). Recently, a simple Lagrangian representation of the three dimensional motion of an infinitely thin vortex sheet in an ideal fluid has been derived (Caflisch, 1989; Kaneda, 1989b, 1990a). This representation may be regarded as a generalization of well-known Birkhoff's equation for the motion of a vortex sheet in a two-dimensional flow to that in a three-dimensional flow.

In this paper, a brief review is given in Section 2 on the representation and on some conservation properties of the dynamics of the vortex sheet. The conservation properties have a close relation to the basic principle, such as the homogeneity in space and time, of the vortex dynamics. Clarifying them may be useful for checking the accuracy of numerical computation of the motion, and also for getting some idea on the motion without our being involved in algebraic or computational detail.

In Section 3, a review is given on similarity solutions of the motion of a vortex sheet. Since it is quite intractable to solve generally the governing equation of the vortex motion, it is interesting at the present stage of our knowledge to consider similarity solutions, for which the equations to be solved can be much simplified. Such solutions often exhibit simple and beautiful spatial structures and are expected to shed some light on our understanding of vortex dynamics.

The search for similarity solutions need not be restricted to the motion of a vortex sheet. A comment is made in Section 4 on possible similarity solutions of the motion of an isolated line vortex in an ideal fluid.

2. Lagrangian Representation of the Motion a Vortex Sheet

2.1 Equations of motion

Let S be a smooth and infinitely thin vortex sheet in an ideal fluid of unit density that obeys

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p,$$

$$\mathrm{div}\boldsymbol{u} = 0,$$

where $\boldsymbol{u}$ and p are the velocity and the pressure of the fluid, respectively. If the flow field is irrotational outside S, then there exists a velocity potential $\phi = \phi(\boldsymbol{r}, t)$ such that $\boldsymbol{u} = \nabla\phi$ outside S, where $\boldsymbol{r}$ denotes the position vector. Let the position vector $\boldsymbol{r}$ of a point on S at time t be represented by by two parameters, say λ_1 and λ_2, as

$$\boldsymbol{r}(t) = \boldsymbol{R}(\lambda_1, \lambda_2, t),$$

and let the jump $\Phi(\lambda_1, \lambda_2)$ of the velocity potential across S at a certain initial time, say $t = 0$, be defined by

$$\Phi(\lambda_1, \lambda_2) \equiv \phi^+(\boldsymbol{R}(\lambda_1, \lambda_2, 0), 0) - \phi^-(\boldsymbol{R}(\lambda_1, \lambda_2, 0), 0),$$

where

$$\phi^{\pm}(\boldsymbol{r}, t) \equiv \lim_{\delta \to +0} \phi(\boldsymbol{r} \pm \delta \boldsymbol{N}, t), \quad \boldsymbol{N} \equiv \frac{\partial \boldsymbol{R}}{\partial \lambda_1} \times \frac{\partial \boldsymbol{R}}{\partial \lambda_2}.$$

Then it is shown (Caflisch, 1989; Kaneda, 1989b, 1990a) that the motion of S is given by

$$\frac{\partial \boldsymbol{R}(\lambda_1,\lambda_2,t)}{\partial t} = -\frac{1}{4\pi}\mathrm{p.v.}\iint \frac{\boldsymbol{X}\times \boldsymbol{W}(\lambda'_1,\lambda'_2,t)}{|\boldsymbol{X}|^3} d\lambda'_1\, d\lambda'_2 + \boldsymbol{u}_H, \tag{1}$$

where

$$\boldsymbol{X} \equiv \boldsymbol{R}(\lambda_1,\lambda_2,t) - \boldsymbol{R}(\lambda'_1,\lambda'_2,t),$$

$$\boldsymbol{W}(\lambda_1,\lambda_2,t) = \Phi_1(\lambda_1,\lambda_2)\boldsymbol{R}_2(\lambda_1,\lambda_2,t) - \Phi_2(\lambda_1,\lambda_2)\boldsymbol{R}_1(\lambda_1,\lambda_2,t), \tag{2}$$

the subscripts 1 and 2 attached to Φ and $\boldsymbol{R}$ denote the differentiations with respect to λ_1 and λ_2, respectively, and the symbol p.v. denotes that the principal value of the integral is to be taken. The vector $\boldsymbol{u}_H$ in Eq. (1) represents the contribution from the irrotational flow field that is to be determined by appropriate boundary conditions and the connectedness of the flow domain. For the sake of simplicity. we assume $\boldsymbol{u}_H$ to be zero in the following. We also assume that Φ may be parametrized as $\Phi_2 = 0$, i.e., Φ_1 is a function of only λ_1, say $\gamma(\lambda_1)$. Then we may write Eq. (2) as

$$\boldsymbol{W}(\lambda_1,\lambda_2,t) = \gamma(\lambda_1)\boldsymbol{R}_2(\lambda_1,\lambda_2,t). \tag{3}$$

The derivation of Eq. (1) shows that the motion of an infinitely thin vortex sheet can be determined irrespectively of the detailed internal structure of the vortex sheet or how the limit of zero thickness of the sheet is approached. Such a possibility of the determination of vortex motion irrespectively of the detailed internal structure of concentrated vortex region is not trivial because, in the asymptotic limit of zero thickness of the concentrated region, the vorticity is nonzero only where the velocity is discontinuous, and in this sense the product of the vorticity and velocity is mathematically similar to the product of the delta function and Heaviside's function. As is well-known, it is difficult to well define such a product and as a matter of fact the equation of the motion of the vorticity contains, in the Eulerian representation, the product of the velocity and vorticity.

2.2 Two-dimensional motion—Birkhoff's equation

When a two dimensional flow field $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \boldsymbol{u}(x,y,t) = u(x,y,t)\mathrm{e}_x + v(x,y,t)\mathrm{e}_y,$$

and $\boldsymbol{R}$ and Φ are given by

$$\boldsymbol{R}(\lambda_1, \lambda_2, t) = \left(x(\lambda_1, t), y(\lambda_1, t), \lambda_2\right), \;\; \Phi(\lambda_1, \lambda_2) = \Phi(\lambda_1),$$

then

$$\boldsymbol{R}_2 = \mathrm{e}_z, \;\; \Phi_1 = d\Phi(\lambda_1) / d\lambda_1 \equiv \gamma(\lambda_1), \;\; \Phi_2 = 0,$$

where (x, y, z) is a Cartesian coordinate system, $(\mathrm{e}_x, \mathrm{e}_y, \mathrm{e}_z)$ are the unit vectors in the directions of the x, y, z axes, respectively.

It can be shown by integrating Eq. (1) with respect to λ'_2 that

$$\frac{\partial \boldsymbol{r}(\lambda, t)}{\partial t} = \frac{1}{2\pi} \mathrm{p.v.} \int \frac{(x - x')\mathrm{e}_y - (y - y')\mathrm{e}_x}{(x - x')^2 + (y - y')^2} \gamma(\lambda') d\lambda', \tag{4}$$

where

$$\boldsymbol{r}(\lambda, t) \equiv x\mathrm{e}_x + y\mathrm{e}_y, \;\; (x, y) \equiv \left(x(\lambda, t), y(\lambda, t)\right) \text{ and } (x', y') \equiv \left(x(\lambda', t), y(\lambda', t)\right).$$

It is not difficult to derive celebrated Birkhoff's equation

$$\frac{\partial Z^*(\lambda, t)}{\partial t} = \frac{1}{2\pi i} \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{\gamma(\lambda') d\lambda'}{Z(\lambda, t) - Z(\lambda', t)},$$

from Eq. (4), where $Z(\lambda, t) = x(\lambda, t) + iy(\lambda, t)$, and the star denotes the complex conjugate.

2.3 Integral invariants of the motion

As is well known, the conservations of the total energy, momentum, and angular momentum in the dynamics of Hamiltonian systems are respectively derived from the homogeneity in time and space, and the isotropy in space. The similar is shown to be also true in the vortex dynamics given by Eqs. (1) and (3); it has the following integral invariants T, $\boldsymbol{P}$ and $\boldsymbol{Q}$ under appropriate conditions;

$$T \equiv \int (\boldsymbol{W} \cdot \boldsymbol{W}') G d\lambda_1 d\lambda_2 d\lambda'_1 d\lambda'_2 = \int (\boldsymbol{R}_2 \cdot \boldsymbol{R}'_2) G dA dA',$$

$$\boldsymbol{P} \equiv \int \boldsymbol{R} \times \boldsymbol{W} d\lambda_1 d\lambda_2 = \int \boldsymbol{R} \times \boldsymbol{R}_2 dA,$$

$$\boldsymbol{Q} \equiv \int \boldsymbol{R} \times (\boldsymbol{R} \times \boldsymbol{W}) d\lambda_1 d\lambda_2 = \int \boldsymbol{R} \times (\boldsymbol{R} \times \boldsymbol{R}_2) dA,$$

where

$$G \equiv G(\boldsymbol{R}, \boldsymbol{R}') \equiv \frac{1}{4\pi} \frac{1}{|\boldsymbol{R} - \boldsymbol{R}'|},$$

$$\boldsymbol{R} \equiv \boldsymbol{R}(\lambda_1, \lambda_2, t), \ \ \boldsymbol{R}' \equiv \boldsymbol{R}(\lambda'_1, \lambda'_2, t),$$

$$dA \equiv \gamma(\lambda_1) d\lambda_1 d\lambda_2, \ \ dA' \equiv \gamma(\lambda'_1) d\lambda'_1 \, d\lambda'_2 .$$

The invariants T, $\boldsymbol{P}$ and $\boldsymbol{Q}$ can be related to the total energy, momentum and angular momentum of the fluid. Their conservations are associated with the homogeneity in time, homogeneity in space and isotropy, respectively. As a consequence of the invariance of T under a scale transformation such that

$$T = \frac{1}{\alpha} \int \int \left[(\alpha \boldsymbol{R})_2 \cdot (\alpha \boldsymbol{R}')_2 \right] G(\alpha \boldsymbol{R}, \alpha \boldsymbol{R}') dA dA',$$

is constant independent of α, the dynamics given by (1) also satisfies

$$\int \dot{\boldsymbol{R}} \cdot (\boldsymbol{R} \times \boldsymbol{R}_2) dA = T/2.$$

The reader may refer to Kaneda (1990b) for the justification of the above mentioned properties.

3. Similarity Solutions

3.1 Two-dimensional similarity solutions

Extensive studies have been made both numerically and analytically on the motion of a vortex sheet in two-dimensional flow, (cf. e.g. Krasny (1986) and references cited therein). It is now known that a spontaneous singularity may appear on an initially smooth infinitely thin vortex sheet.

As regards similarity solutions, at least two classes of solutions are known; one is a class in which the vortex sheet takes the form of an algebraic spiral, and the other is a class in which the sheet takes the form an exponential spiral near its center.

A solution belonging to the former class was first studied by Kaden (1931), and many studies have been made both analytically and numerically on this class of solutions.

Solutions belonging to the latter class were first found by Prandtl (1922) who showed that the motion of a cut in the plane of a complex variable associated with a simple complex function of the complex variable (say, z),

$$f(z) = \text{const} \times t^{\mu} z^{\gamma},$$

can represent a motion of a vortex sheet that takes the form of a single-branched exponential spiral, where t is the time and μ, γ are complex constants. His solutions were then generalized by Alexander (1971) to the cases of a multi-branched exponential spiral vortex sheet possessing a certain central symmetry.

They were further generalized by Kaneda (1989a) who showed that a cut in the z plane associated with a function of the form

$$f(z) = f_1(z) + f_2(z)$$

can represent the motion of a double branched exponential vortex sheet for which the symmetry assumed in Alexander's solution is not necessarily imposed, where

$$f_n(z) \equiv A_n(t)\exp[i\phi_n(t)]z^{\gamma} = A_n(t)\exp[\gamma\zeta + i\phi_n(t)], \quad (n = 1,2),$$

$$\gamma \equiv \alpha + i\beta, \quad \zeta \equiv \ln(z) = \ln r + i\theta,$$

A_n, ϕ_n are real functions of time t; α, β are time-independent real constants satisfying

$$\alpha^2 + \beta^2 = 2\alpha.$$

The shape of the vortex sheet is then given by a double-branched exponential spiral;

$$r = \exp[(\lambda_n - \alpha\theta - \phi_n(t)) / \beta]$$

where λ_n is a solution of λ satisfying

$$e^{-2\pi\beta}\sin(\lambda + 2\pi\alpha) = \sin(\lambda).$$

The evolutions of the amplitudes A_1, A_2 and the "pseudo-phases" ϕ_1, ϕ_2 are to satisfy a certain set of nonlinear equations. The explicit form of these equations may be found in Kaneda (1989a) and is not reproduced here. Two simple special solutions satisfying the equations are shown to give exact expressions of time-dependent gravity-free free-surface flow. Such solutions are interesting, particularly because exact time-dependent solutions of equations for a fluid with a free-surface are quite rare.

In regard to the relation of the spiral shape given by the above similarity solutions to the one found by numerical simulations, the reader may refer to the discussions in Kaneda (1989a) and references cited therein.

3.2 Three-dimensional similarity solutions

The vortex dynamics in three dimensional flows is fundamentally different from that in two-dimensional flows because there exists the effect of vortex stretching in three dimensional flows, and the three dimensionality of the flow plays an important role in many cases. In spite of these facts, very little seems to be known about the motion of a vortex sheet in three-dimensional flows, as compared to the motion in two-dimensional flows. This is presumably because of the difficulty in solving the problems of the three dimensional motion. It is therefore, interesting to see whether Eq. (1) has any similarity solution, i.e., whether it is possible to simplify Eq. (1) or to reduce the number of the independent variables of Eq. (1) by assuming a certain similarity form.

Let us try here the following similarity form (Kaneda, 1990b);

$$\boldsymbol{R} = t^m \boldsymbol{r}(\xi, \lambda),$$

where

$$\xi \equiv \Gamma t^n, \quad d\Gamma \equiv \gamma(\lambda_1) d\lambda_1, \quad \text{and} \quad \lambda \equiv \lambda_2.$$

It is easy to show that if m and n satisfy

$$2m + n = 1, \tag{5}$$

then substituting the similarity form into Eq. (1) with Eq. (3) yields

$$\left[m + n\xi \frac{\partial}{\partial \xi}\right] \boldsymbol{r}(\xi, \lambda) = -\frac{1}{4\pi} \int\int_S \frac{\boldsymbol{x} \times \boldsymbol{w}(\xi', \lambda')}{|\boldsymbol{x}|^3} d\xi' d\lambda', \tag{6}$$

where

$$\boldsymbol{x} \equiv \boldsymbol{r}(\xi, \lambda) - \boldsymbol{r}(\xi', \lambda'), \quad \boldsymbol{w}(\xi, \lambda) = \frac{\partial \boldsymbol{r}(\xi, \lambda)}{\partial \lambda}.$$

Thus the number of the independent variables may be reduced from three in Eq. (1) to two in Eq. (6) by assuming a suitable similarity form. As for such similarity solutions in two-dimensions, see e.g. Pullin and Phillips (1981) and references cited therein. The solutions satisfying Eq. (6) in three-dimensions remain to be explored.

It may be worthwhile to note that if Eq. (5) holds, then Eq. (1) is invariant under the following scale transformation;

$$\Gamma \to \alpha^{-n}\Gamma, \quad \boldsymbol{R} \to \alpha^{m}\boldsymbol{R}, \quad t \to \alpha t,$$

where α is an arbitrary real constant. This means that if

$$\boldsymbol{R} = \boldsymbol{R}_0\left(\Gamma, \lambda_2, t\right),$$

is a solution of Eq. (1) and if $2m + n = 1$, then

$$\boldsymbol{R} = \alpha^{m}\boldsymbol{R}_0\left(\alpha^{n}\Gamma, \lambda_2, t/\alpha\right),$$

is another solution of Eq. (1) for any real α.

4. Similarity Solution of the Motion of a Line Vortex

The motion of a very thin isolated line vortex in an unbounded ideal fluid by its own induction without stretching is often approximated by the so-called localized induction approximation derived by Arms (1962; from a private communication to Hama) and used by Hama (1962), which may be written as

$$\frac{\partial \boldsymbol{r}}{\partial t} = G\kappa\boldsymbol{b}, \tag{7}$$

where $\boldsymbol{r}=\boldsymbol{r}(s,t)$ is the position vector on the line vortex, t the time, s the length measured along the filament, κ the curvature, $\boldsymbol{b}$ the unit vector in the direction of the binormal and G is a coefficient determined by the structure e.g., the radius, the circulation, etc., of the line vortex. We assume here that G may be regarded to be a constant .

Hasimoto (1972) showed that Eq. (7) can be transformed, after a suitable choice of units of time and length, to a non-linear Shrödinger equation

$$\frac{1}{i}\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial s^2} + \frac{1}{2}\left(|\psi|^2 + A\right)\psi, \tag{8}$$

where ψ is the complex variable

$$\psi = \kappa \exp\left(i\int_0^s \tau ds\right), \tag{9}$$

τ the torsion of the line vortex, and A is a function of t, which can be eliminated by the introduction of the new variable

$$\Psi = \psi \exp\left[-\frac{1}{2} i \int_0^t A(t) dt \right].$$

We may therefore take A in Eq. (8) to be zero without loss of generality. He also showed that Eq. (8) yields a soliton like solution describing the propagation of a loop or a hump of a helical motion along the line vortex with a constant velocity 2τ.

We consider here whether Eq. (7) or Eq. (8) can have any similarity solution, i.e., whether it is possible to reduce the number of the independent variables of Eq. (8) and simplify Eq. (8) by assuming a certain similarity form of the solution. Let us try here the following form;

$$\psi = s^m f(\xi), \quad (\xi \equiv t^n s).$$

Substituting this into Eq. (8) with $A = 0$ yields

$$-ins^{m+(1/n)} \xi^{1-(1/n)} f' = s^{m-2}\left[m(m-1)f + 2m\xi f' + \xi^2 f'' \right] + \frac{1}{2} s^{3m} |f|^2 f, \tag{10}$$

where the prime $'$ denotes the differentiation with respect to ξ. Equation (10) suggests

$$m + \frac{1}{n} = m - 2 = 3m,$$

i.e.,

$$m = -1, \quad n = -\frac{1}{2},$$

and the form

$$\psi = \frac{1}{s} f\left(\frac{s}{\sqrt{t}} \right) = \frac{1}{\sqrt{t}} g\left(\frac{s}{\sqrt{t}} \right), \tag{11}$$

where $g(\xi) \equiv f(\xi)/\xi$ and $\xi = s/\sqrt{t}$.

Substituting Eq. (11) into Eq. (8) with A = 0 yields

$$i(\xi g)' = 2g'' + |g|^2 g. \tag{12}$$

If we put

$$g = \rho \exp(i\theta), \quad \theta \equiv \int^{\xi} \phi(\xi) d\xi,$$

then the real and imaginary parts of Eq. (12) divided by exp($i\theta$) yield

$$\rho'' = \left(\phi^2 - \frac{1}{2}\xi\phi\right)\rho - \frac{1}{2}\rho^3, \tag{13}$$

and

$$(\xi\rho)' = 4\rho'\phi + 2\rho\phi', \tag{14}$$

i.e.,

$$\phi(\xi) = \frac{1}{2\rho^2}\int^{\xi} \rho \frac{d(\xi\rho)}{d\xi} d\xi.$$

Thus, by assuming the similarity form Eq. (11), the partial differential Eq. (8) may be reduced to one ordinary differential Eq. (12) for one complex function g or to two ordinary differential Eqs. (13) and (14) for two real functions $\rho = \rho(\xi)$ and $\phi = \phi(\xi)$. Once g is known by solving these equations, then ψ as well as the curvature κ and the torsion τ can be known from Eqs. (9) and (11).

If $\rho \equiv |g|$ is so small that the second term on the right-hand side of Eq. (12) may be neglected, then Eq. (12) is reduced to

$$i(\xi g)' = 2g'',$$

the general solution of which is given by

$$g = \exp\left(\frac{i}{4}\xi^2\right)\left\{C_1\int_0^{\xi}\exp\left(-\frac{i}{4}\xi^2\right)d\xi + C_2\right\},$$

where C_1 and C_2 are constants.

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