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Selfsimilar Natures of Drainage Basins

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Abstract. Two branching systems, named Branching Systems I and II, were examined theoretically to clarify selfsimilar natures of drainage basins. These branching systems were characterized by value of the parameter ${}_{\eta}\varepsilon_{\psi}$ which denotes the average number of streams of order ψ entering into a stream of order η from the sides. Put $\varepsilon_{\phi} = {}_{\eta}\varepsilon_{\eta-\phi}$ and $K = \varepsilon_{\phi}/\varepsilon_{\phi-1}$ for $\phi \geq 2$. Then Branching System I is defined as the system which satisfies the condition that ε_1 and K are constant respectively for all possible ϕ. When $\varepsilon_2/\varepsilon_1 \neq \varepsilon_3/\varepsilon_2$, put $K' = \varepsilon_{\phi}/\varepsilon_{\phi-1}$ for $\phi \geq 3$. Then Branching System II is defined as the system in which ε_1, ε_2, and K' are constant respectively for all possible ϕ. Regularly selfsimilar drainage networks can be drawn following the definitions of both the systems. If streams are assumed to meander selfsimilarly in them, the fractal dimension D_s of a stream is given by the stream length ratio R_L, the basin area ratio R_A, and the fractal dimension D_b of the drainage baisin. Namely $D_s = D_b \log R_L / \log R_A$, where

$$R_A = \left[2 + \varepsilon_1 + K + \sqrt{(2 + \varepsilon_1 + K)^2 - 8K}\right] / 2$$

for Branching System I and

$$R_A = \left[2 + \varepsilon_1 + K' + \sqrt{(2 + \varepsilon_1 + K')^2 - 4(2K' + \varepsilon_1 K' - \varepsilon_2)}\right] / 2$$

for Branching System II. The drainage network which belongs to either of these systems has the fractal dimension of D_b for any value of D_s which satisfies the

inequality $1 \leq D_s < D_b$. It is reasonable to consider that randomness affects more or less confluences of stream channels in actual drainage basins. Branching System I is appreciated most applicable to actual drainage basins among some devisable branching systems including Branching System II because it permits smooth interposition of randomness on the confluences of stream channels.

1. Introduction

The laws of drainage composition have been thought for a long time to be expressed by geometrical progressions constituted with parameters called the bifurcation ratio, basin area ratio, stream length ratio, etc. (Horton, 1945; Schumm, 1956; Morisawa, 1962). If these geometrical progressions are valid, the parameters should express the properties of a drainage basin which are independent from its size.

Such properties of drainage basins are noted at present in the field of fractal geometry because those were regarded as to incarnate the selfsimilar natures of actual substances (Mandelbrot, 1977, 1983). Several selfsimilar branching systems have been proposed to simulate drainage basins (Mandelbrot, 1977, 1983). Some of them satisfy Horton's laws of basin areas and stream lengths, but do not satisfy his law of stream numbers given by the geometrical progression. On the other hand, Tokunaga (1978) has proposed the drainage basin model in which the laws of basin areas and stream lengths are given by the geometrical progressions but the law of stream numbers takes another form.

This paper is written to give a geometrical basis to interpretation of the laws of drainage composition using the concept of selfsimilarity in fractal geometry. Then the drainage basin model proposed by Tokunaga (1978) was examined in relation to regularly selfsimilar forms. Some branching systems which seem not to be realistic as a model for actual drainage basins were also examined to get deeper understanding of selfsimilar natures of drainage basins.

2. Selfsimilar Drainage Basin

There is a clear evidence that Horton (1945) had the idea which would be developed to the concept of selfsimilarity in fractal geometry. He tried to illustrate the development of a drainage network in Fig. 1. We find a gern of the concept of selfsimilarity in this figure. Two subbasins on both sides of the mainstream in Fig. 1-(C) can be regarded as geometrically similar with the basin in Fig. 1-(B). We can see further development of the basin in Fig. 1-(D) following the rule to draw a finite Peano island (Mandelbrot, 1977, 1983) shown in Fig. 2. Horton (1945) tried to explain the hydrophysical basis of his laws of stream numbers and stream lengths by the illustrations in Fig. 1. The drainage network in the finite Peano island, however, does not satisfy Horton's law of stream numbers. This will be shown later together with the explanation on the characterization of drainage networks by

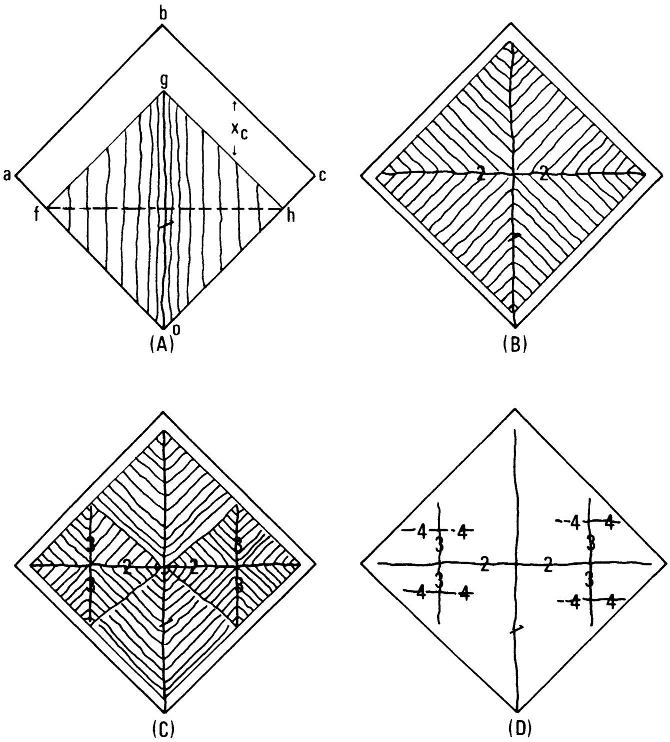

Fig. 1. Development of a drainage net in a stream basin schematically illustrated by Horton (1945). After Horton in Geological Society of America Bulletin, vol. 56, 340 page.

Tokunaga (1978).

Some branching systems were related to another types of selfsimilar forms. A network composed of ordered streams is drawn together with a peninsula composed of triadic Koch curves, famous fractal (Mandelbrot, 1977, 1983), in Fig. 3-(I). Strahler's ordering method is available for this network when a trifurcating point was regarded as to be composed of two bifurcating points which were dislocated from each other at an infinitesimal distance. Each ordered stream has one to one correspondence to a peninsula of the appropriate size. The drainage basin of order κ in the finite Peano island is obtained by transforming the stream network and the outline of the peninsula in Fig. 3-(I) continuously. Then the outline becomes the drainage divides.

The drainage basin of order κ in Fig. 4, which was named a drainage basin with sink holes here, is also obtained by transforming the stream network and outline of the peninsula in Fig. 3-(II) continuously. The outline is composed of two self similar curves and each curve may be called a pentadic Koch curve. All drainage basins in

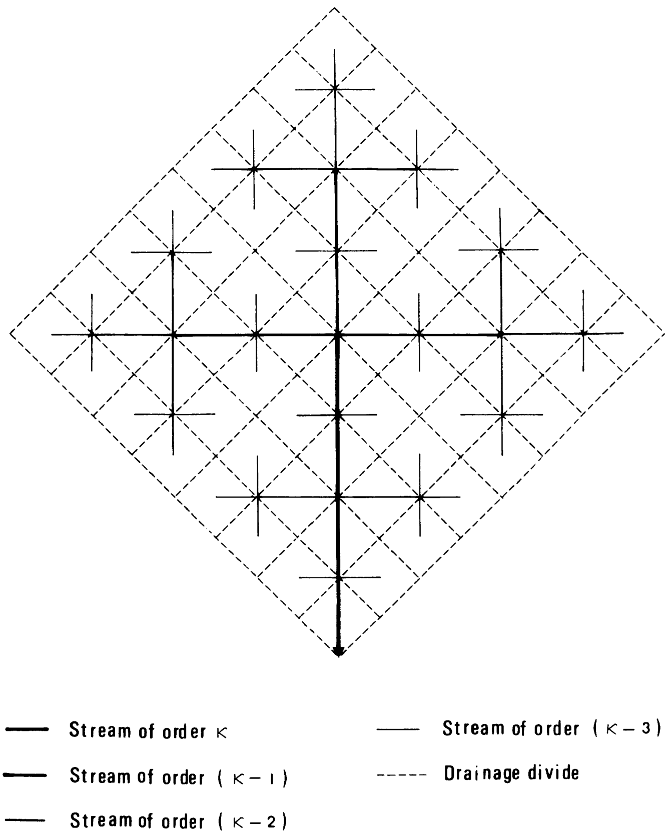

Fig. 2. A drainage basin in a finite Peano island drawn by referring Mandelbrot (1977). In the basin, $\varepsilon_1 = 1$ and $K = 2$.

Figs. 2 and 4 satisfy the condition of selfsimilarity (Mandelbrot, 1977, 1983). The basin area ratio is 4 in Figs. 2 and 7 in the largest drainage basin in Fig. 4. The stream length ratio is 2 in Figs. 2 and 3 in the largest drainage basin in Fig. 4. These two drainage basin models provide a hint on classifying branching systems and it will be shown in the next section.

3. Law of Stream Numbers and Classification of Selfsimilar Branching Systems

Tokunaga (1966, 1978) has proposed a parameter different from Horton-Strahler's bifurcation ratio. That is the average number ${}_{\eta}\varepsilon_{\psi}$ of streams of order ψ which enter into a stream of order η from the sides in a drainage basin. Actual

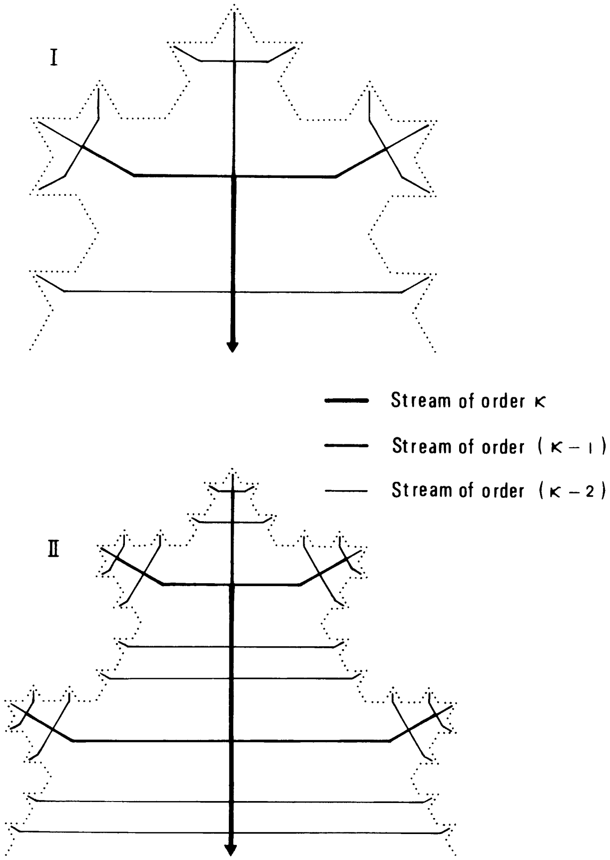

Fig. 3. (I) Triadic Koch curves and a stream system, (II) Pentadic Koch curves and another stream system.

drainage basins satisfy approximately the following relation (Tokunaga, 1966, 1978; Onda and Tokunaga, 1987).

$$\begin{aligned} {}_{\kappa}\varepsilon_{\kappa-1} &= {}_{\kappa-1}\varepsilon_{\kappa-2} = \cdots = {}_{\lambda+1}\varepsilon_{\lambda} \\ {}_{\kappa}\varepsilon_{\kappa-2} &= {}_{\kappa-1}\varepsilon_{\kappa-3} = \cdots = {}_{\lambda+2}\varepsilon_{\lambda} \\ &\cdots \\ &\cdots \\ {}_{\kappa}\varepsilon_{\kappa-\phi} &= {}_{\kappa-1}\varepsilon_{\kappa-\phi-1} = \cdots = {}_{\lambda+\phi}\varepsilon_{\lambda} \end{aligned} \tag{1}$$

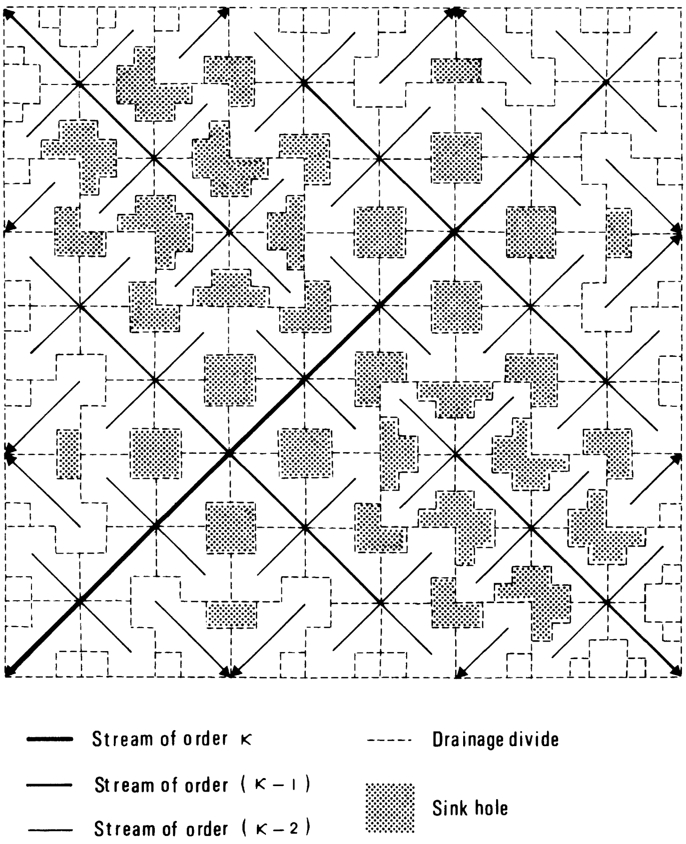

Fig. 4. Drainage basins with sink holes. In the largest basin, $\varepsilon_1 = 3$, $\varepsilon_2 = 8$, and $K' = 3$. In the basin which has the outlet at the top left-hand corner or at the bottom right-hand one, $\varepsilon_1 = 5$, $\varepsilon_2 = 18$, and $K' = 4$.

where κ is the highest stream order and λ is the lowest one in a basin on topographic maps or aerial photos of a given scale. In Fig. 5 a value of order is increased by Strahler's method, but the highest and lowest orders are given by variables following the method by Tokunaga (1978). Equation (1) provides a subsequent parameter: $\varepsilon_\phi = {}_\eta\varepsilon_{\eta-\phi}$. Actual drainage basins satisfy the following equation with regard to this parameter (Tokunaga, 1966, 1978; Onda and Tokunaga, 1987).

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_3}{\varepsilon_2} = \cdots = \frac{\varepsilon_\phi}{\varepsilon_{\phi-1}} = \cdots \tag{2}$$

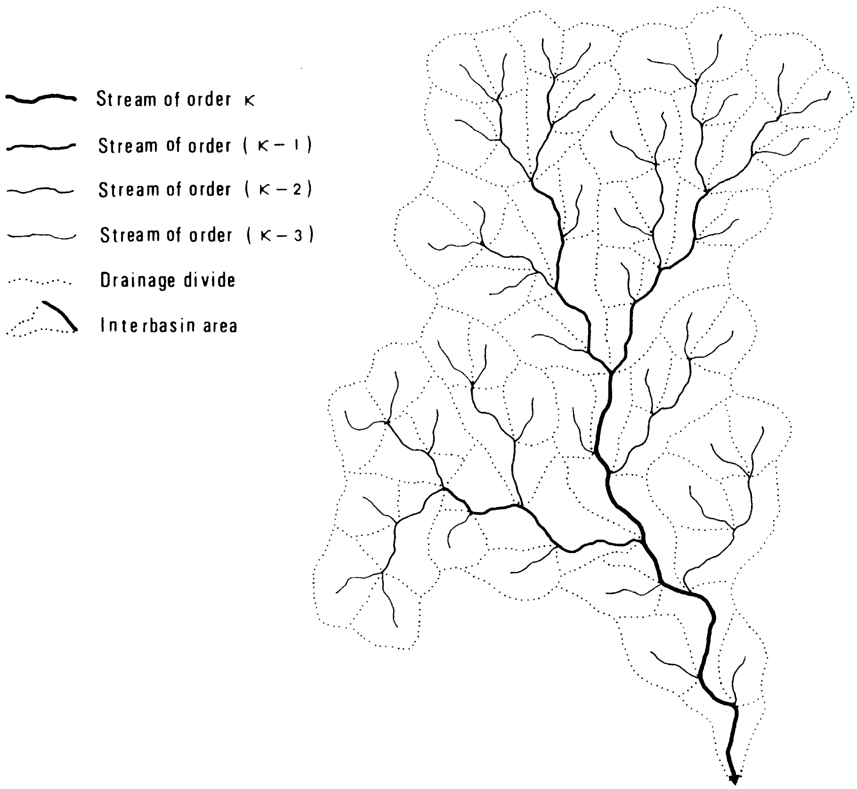

Fig. 5. Hypothetical drainage basin. A case that $\varepsilon_1 = 1$ and $K = 2$.

This equation also provides a subsequent parameter: $K = \varepsilon_{\eta-\psi}/\varepsilon_{\eta-\psi-1} = \varepsilon_{\phi}/\varepsilon_{\phi-1}$.

Tokunaga (1966, 1978) derived the equation, which expresses the law of stream numbers using ε_1 and K, from Eqs. (1) and (2). Then the law of stream numbers is given by an alternating equation. The average number ${}_{\eta}\mu_{\psi}$ of streams of order ψ in a basin of order η is

$$ {}_{\eta}\mu_{\psi} = \frac{2+\varepsilon_1 - P}{Q-P} Q^{\eta-\psi} + \frac{2+\varepsilon_1 - Q}{P-Q} P^{\eta-\psi} \tag{3} $$

where

$$ P = \left[2+\varepsilon_1 + K - \sqrt{(2+\varepsilon_1+K)^2 - 8K} \right] / 2, $$

$$Q = \left[2 + \varepsilon_1 + K + \sqrt{(2 + \varepsilon_1 + K)^2 - 8K} \right] / 2,$$

${}_{\eta}\mu_{\eta} = 1$, and ${}_{\eta}\mu_{\eta-1} = 2 + \varepsilon_1$. The mathematical pocedure to derive Eq. (3) from Eqs. (1) and (2) is shown in Appendix I. The basin in Fig. 5 is drawn rather ideally because in each subbasin ${}_{\eta}\varepsilon_{\psi}$ takes the value same with that of the whole basin for a given value of $(\eta - \psi)$. However ${}_{\eta}\varepsilon_{\psi}$ may take a value different in respective subbasins in an actual drainage basin. The drainage basin model defined to satisfy Eqs. (1) and (2) is called the cyclic system (Tokunaga, 1978). The law of stream number is also expressed by a continued fraction (Tokunaga, 1972). The continued fraction is given as follows:

$$\frac{{}_{\eta}\mu_{\psi}}{{}_{\eta}\mu_{\psi+1}} = 2 + \varepsilon_1 + K - \cfrac{2K}{2 + \varepsilon_1 + K - \cfrac{2K}{\cdots \cdots 2 + \varepsilon_1 + K - \cfrac{2K}{\cfrac{{}_{\eta}\mu_{\eta-1}}{{}_{\eta}\mu_{\eta}}}}} \tag{4}$$

The mathematical procedure to derive Eq. (4) from Eqs. (1) and (2) is also shown in Appendix I.

The drainage basin in Fig. 2 satisfies Eqs. (1) and (2) with $\varepsilon_1 = 1$, $K = 2$, $P = 1$, and $Q = 4$. The equations which state the relation of stream number to order in the drainage basin are obtained by substituting these values into Eqs. (3) and (4). Namely,

$${}_{\eta}\mu_{\psi} = \frac{2}{3} 4^{\eta-\psi} + \frac{1}{3} \tag{5}$$

and

$$\frac{{}_{\eta}\mu_{\psi}}{{}_{\eta}\mu_{\psi+1}} = 5 - \cfrac{4}{5 - \cfrac{4}{\cdots \atop {\cdots \atop 5 - \cfrac{4}{\cfrac{3}{1}}}}} \tag{6}$$

These equations show that the law of stream numbers of a drainage basin in the finite Peano island can not be expressed by a geometrical progression and the drainage basin is only one example among the branching systems which were characterized by Eqs. (1) and (2).

The selfsimilar drainage basins in Fig. 4 satisfies Eq. (1) but does not Eq. (2). In the largest basin, $\varepsilon_1 = 3$, $\varepsilon_2 = 8$, $\varepsilon_3 = 24$, $\varepsilon_4 = 72$, $\cdots$. Therefore $\varepsilon_2/\varepsilon_1 = 8/3$ and $\varepsilon_3/\varepsilon_2 = \varepsilon_4/\varepsilon_3 = \cdots = 3$. Then the following relation is induced instead of Eq. (2) for such a branching system.

$$\frac{\varepsilon_2}{\varepsilon_1} \neq \frac{\varepsilon_3}{\varepsilon_2} = \cdots = \frac{\varepsilon_\phi}{\varepsilon_{\phi-1}} = \cdots \tag{7}$$

where $\phi \geq 3$. Here put $K' = \varepsilon_\phi/\varepsilon_{\phi-1}$. Then the equation which expresses the law of stream numbers is obtained. That is written as follows:

$$_{\eta}\mu_{\psi} = \frac{2 + \varepsilon_1 - P'}{Q' - P'} Q'^{\,\eta-\psi} + \frac{2 + \varepsilon_1 - Q'}{P' - Q'} P'^{\,\eta-\psi} \tag{8}$$

where

$$P' = \left[2 + \varepsilon_1 + K' - \sqrt{(2 + \varepsilon_1 + K')^2 - 4(2K' + \varepsilon_1 K' - \varepsilon_2)} \right] / 2,$$

$$Q' = \left[2 + \varepsilon_1 + K' + \sqrt{(2 + \varepsilon_1 + K')^2 - 4(2K' + \varepsilon_1 K' - \varepsilon_2)} \right] / 2,$$

${}_{\eta}\mu_{\eta} = 1$, and ${}_{\eta}\mu_{\eta-1} = 2 + \varepsilon_1$. The mathematical procedure to obtain Eq. (8) is shown in Appendix II. The law of stream numbers is also given in the form of continued fraction as follows:

$$\frac{{}_{\eta}\mu_{\psi}}{{}_{\eta}\mu_{\psi+1}} = 2+\varepsilon_1+K' - \cfrac{2K'+\varepsilon_1K'-\varepsilon_2}{2+\varepsilon_1+K' - \cfrac{2K'+\varepsilon_1K'-\varepsilon_2}{\cdots \atop {\cdots \atop 2+\varepsilon_1+K' - \cfrac{2K'+\varepsilon_1K'-\varepsilon_2}{\cfrac{{}_{\eta}\mu_{\eta-1}}{{}_{\eta}\mu_{\eta}}}}}}} \tag{9}$$

The mathematical procedure to obtain Eq. (9) is also shown in Appendix II. The above equation indicates together with Eq. (4) that structures of this kind of selfsimilar figures are expressed imaginably by continued fractions. For the drainage basin of order κ in Fig. 4, $\varepsilon_1 = 3$, $\varepsilon_2 = 8$, and $K' = 3$. Consequently, $P' = 1$ and $Q' = 7$. Therefore, the equations which state the relation of stream number to order in the drainage basin are given as follows:

$${}_{\eta}\mu_{\psi} = \frac{2}{3}7^{\eta-\psi} + \frac{1}{3} \tag{10}$$

$$\frac{{}_{\eta}\mu_{\psi}}{{}_{\eta}\mu_{\psi+1}} = 8 - \cfrac{7}{8 - \cfrac{7}{\cdots \atop {\cdots \atop 8 - \cfrac{7}{\cfrac{5}{1}}}}} \tag{11}$$

For the drainage basin which has its outlet at the top left-hand corner or the bottom right-hand one, $\varepsilon_1 = 5$, $\varepsilon_2 = 18$, and $K' = 4$. Therefore $P' = 1$ and $Q' = 10$.

The drainage basin illustrated in Fig. 6 is obtained by regulating the form of the fractal called plane-filling recursive bronchi (Manderbrot, 1983). Then this basin can be called regulated plane-filling recursive bronchi. A drainage divide drawn beside a stream in Fig. 6 should be considered to approach the stream at an

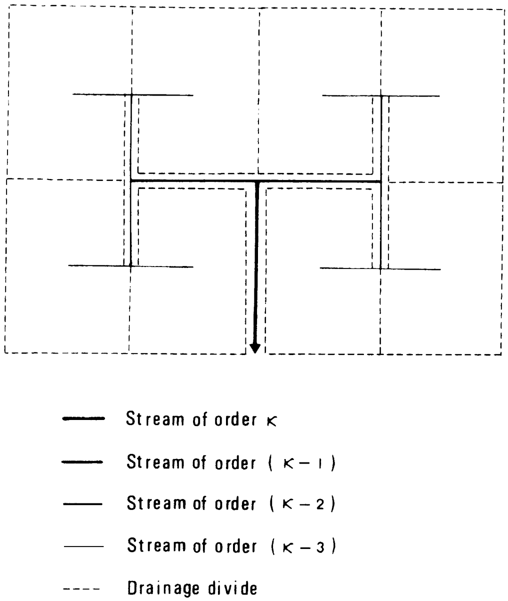

Fig. 6. Regulated plane-filling recursive bronchi drawn by referring Madelbrot (1983) ($\varepsilon_1 = 0$ and K is indeterminate).

infinitesimal distance in reality. This drainage basin satisfies Eqs. (1) and (2). Then $\varepsilon_1 = 0$ and K is indeterminate. On the other hand, the famous fractal called a fudgeflake (Mandelbrot, 1983), satisfies Eqs. (1) and (7), for which $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $K' = 2$, $P' = 1$, and $Q' = 3$.

The regularly selfsimilar forms presented here show that Eqs. (1), (2), and (7) were employed for a classification of branching systems. The branching system which satisfies Eqs. (1) and (2) was named Branching System I and that which satisfies Eqs. (1) and (7) Branching System II.

4. Laws of Basin Areas and Stream Lengths

The drainage basins illustrated in Figs. 2, 4, and 6 were regularly constructed. Therefore we can readily know the laws of basin areas and stream lengths of those drainage basins seeing the figures. Mathematical deductions are, however, needed to introduce such laws for drainage basins characterized only by Eqs. (1) and (2) or Eqs. (1) and (7).

With the assumption that a drainage basin was divided into infinitesimal subbasins and interbasin areas (cf. Fig. 5) in the ultimate, Tokunaga (1975, 1978) derived the law of basin areas from Eqs. (1) and (2). The law was expressed by the progression of common ratio Q as follows:

$$A_\eta = Q^{\eta-\psi} A_\psi \tag{12}$$

where A_η is the average area of basins of order η and A_ψ is that of order ψ. Consequently Q is the basin area ratio. The mathematical procedure to derive Eq. (12) from Eqs. (1) and (2) is shown in Appendix III. As shown in Figs. 2 and 6, $Q = 4$ for a drainage basin in the finite Peano island and $Q = 2$ for the regulated plane-filling recursive bronchi.

The law of basin areas for Branching System II was expressed by the following equation.

$$A_\eta = Q'^{\eta-\psi} A_\psi \tag{13}$$

where Q' is the basin area ratio. The mathematical procedure to obtain the above equation is shown in Appendix IV. As shown in Fig. 4, $Q' = 7$ for the largest basin and $Q' = 10$ for the second largest basins in the figure. The fudgeflake satisfies Eq. (13) with $Q' = 3$. This is understood by seeing the illustration by Mandelbrot (1983).

Here, let D_b be the fractal dimension of a drainage basin. Then one can readily know that $D_b = \log 7/\log 3$ in the largest basin and $D_b = \log 10/\log 4$ in the second largest basins in Fig. 4, while $D_b = 2$ in Figs. 2 and 6, and in the fudgeflake, according to its definition by Mandelbrot (1977, 1983). In Figs. 2, 4, and 6, each ordered stream is given by a straight line, and therefore, has the one dimensional measure. Seeing the figures, one can readily understand that Horton's law of stream lengths is satisfied and the stream length ratio is given by Q^{1/D_b} or Q'^{1/D_b}. This implies that the law of stream lengths can be easily deduced for the drainage networks composed of streams expressed by straight lines. However, such networks are not real as models for natural landforms. Mandelbrot (1977, 1983) has proposed to approximate a river course by a wiggly selfsimilar line. A drainage network composed of wiggly selfsimilar streams is of a general character in the meaning that their dimension is given by a variable. This will be shown hereafter. Some parameters are needed to obtain the equation which expresses the law of stream lengths for such a network.

Here, let denote the distance in the straight line between the uppermost point and the lowest one of the stream of order η by l_η and that of order ξ by l_ξ. Then the following relation must be satisfied in a selfsimilar drainage basin.

$$\cdots = l_\eta^{D_b} / A_\eta = l_{\eta-1}^{D_b} / A_{\eta-1} = \cdots = l_\xi^{D_b} / A_\xi = \cdots \tag{14}$$

This equation states merely the dimensionless relation between the two quantities, namely, length and area of geometrical similar basins. For the basins with reasonable irregularities, l_η, l_ξ, etc. are given by the average values.

For Branching System I, the following equation is derived from Eqs. (12) and (14).

$$\cdots = \frac{l_\eta}{l_{\eta-1}} = \frac{l_{\eta-1}}{l_{\eta-2}} = \cdots = \frac{l_{\xi+1}}{l_\xi} = \cdots = Q^{1/D_b} \tag{15}$$

For Branching System II, the following equation is derived from Eqs. (13) and (14).

$$\cdots = \frac{l_\eta}{l_{\eta-1}} = \frac{l_{\eta-1}}{l_{\eta-2}} = \cdots = \frac{l_{\xi+1}}{l_\xi} = \cdots = Q'^{1/D_b} \tag{16}$$

The drainage basins in Figs. 2 and 6 satisfy Eq. (15) and those in Fig. 4 and the fudgeflake satisfy Eq. (16).

Here, let denote the number of steps of dividers of opening l_ξ along the stream of order η by $N_{\eta,\xi}$, where $\eta > \xi$ and it is assumed that $N_{\eta,\xi}$ can take non-integer values. Then the length $L_{\eta,\xi}$ of the stream of order η measured by the dividers is $N_{\eta,\xi} l_\xi$. Let assume the following relations between streams of various orders:

$$\begin{aligned} \cdots &= N_{\eta+1,\xi+2} = N_{\eta,\xi+1} = N_{\eta-1,\xi} = \cdots, \\ \cdots &= N_{\eta+2,\xi+2} = N_{\eta+1,\xi+1} = N_{\eta,\xi} = \cdots, \\ \cdots &= N_{\eta+3,\xi+2} = N_{\eta+2,\xi+1} = N_{\eta+1,\xi} = \cdots. \end{aligned} \tag{17}$$

Then the fractal dimension D_s of an ordered stream is given by the following equation according to its definition by Mandelbrot (1977, 1983).

$$D_s = \log\left(N_{\eta,\xi} / N_{\eta,\xi+1}\right) / \log\left(l_{\xi+1} / l_\xi\right) \tag{18}$$

From Eqs. (15), (17), and (18), the following equation is obtained for Branching System I.

$$\frac{L_{\eta,\xi}}{L_{\eta-1,\xi}} = \frac{N_{\eta,\xi}}{N_{\eta-1,\xi}} \frac{l_\xi}{l_\xi} = \frac{N_{\eta,\xi}}{N_{\eta-1,\xi}} = Q^{D_s/D_b} \tag{19}$$

From (16), (17), and (18), the following equation is obtained for Branching

System II.

$$\frac{L_{\eta,\xi}}{L_{\eta-1,\xi}} = \frac{N_{\eta,\xi}\ l_\xi}{N_{\eta-1,\xi}\ l_\xi} = \frac{N_{\eta,\xi}}{N_{\eta-1,\xi}} = Q'^{D_s/D_b} \tag{20}$$

The procedures to derive Eqs. (19) and (20) are shown in Appendix V.

The above equations mean that the stream length ratio is Q^{D_s/D_b} or Q'^{D_s/D_b} when lengths of streams of various orders are measured by dividers of a certain length opening. Therefore, they also provide the equations which express the law of stream lengths for Branching Systems I and II. The equations are written as follows:

$$L_{\eta,\xi} = L_{\psi,\xi} Q^{D_s(\eta-\psi)/D_b} \tag{21}$$

$$L_{\eta,\xi} = L_{\psi,\xi} Q'^{D_s(\eta-\psi)/D_b} \tag{22}$$

For $D_s = 1$, validity of these equations is readily confirmed by examining Figs. 2, 4, and 6.

Let denote the stream length ratio $L_{\eta,\xi}/L_{\eta-1,\xi}$ by R_L, and the basin area ratio by R_A, then Eqs. (19) and (20) are rewritten as follows:

$$D_s = D_b \log R_L / \log Q = D_b \log R_L / \log R_A \tag{23}$$

$$D_s = D_b \log R_L / \log Q' = D_b \log R_L / \log R_A \tag{24}$$

These equations show that the fractal dimension of a stream is given by the stream length ratio and the basin area ratio, and the fractal dimension of the drainage basin for Branching Systems I and II. The figure of a fudgeflake shows that $R_L = 2$ and $Q' = R_A = 3$ in it (Mandelbrot, 1983). Substituting these values and $D_b = 2$ into Eqs. (24) yields $D_s = \log 4/\log 3$. This value coincides with that obtained by Mandelbrot (1983). Equations (23) and (24) are also valid for the one dimensional stream channels in Figs. 2, 4, and 6.

5. Fractal Dimension of a Drainage Network

Tarboton *et al.* (1988) showed that the fractal dimension of actual drainage (or channel) networks is near 2. Their theoretical explanation is, however, made based on Horton's laws of stream numbers and stream lengths. The problem on the fractal dimension of a drainage network is clearly explained by using Branching Systems introduced in this paper. Let denote the total length of streams constituting a network of order η which satisfies the definition of Branching System I by L_t, then L_t is given by the following equation.

$$L_t = \sum_{\psi=\xi}^{\eta} L_{\psi,\xi}\ {}_{\eta}\mu_{\psi}$$

where $L_{\xi,\xi} = l_\xi$. From Eq. (21), $L_{\psi,\xi} = L_{\xi,\xi} Q^{D_s(\psi-\xi)/D_b}$. Substitution of Eq. (3) and this equation into the above equation leads to

$$L_t = C_1 l_\xi Q^{\eta-\xi} \cdot \frac{1-\left(Q^{D_s/D_b-1}\right)^{\eta-\xi+1}}{1-Q^{D_s/D_b-1}} + C_2 l_\xi P^{\eta-\xi} \cdot \frac{1-\left(Q^{D_s/D_b}/P\right)^{\eta-\xi+1}}{1-\left(Q^{D_s/D_b}/P\right)} \quad (25)$$

where $C_1 = (2+\varepsilon_1-P)/(Q-P)$ and $C_2 = (2+\varepsilon_1-Q)/(P-Q)$. Here put $N(l_\xi) = L_t/l_\xi$, then the D_b-dimensional measure M of the drainage network is given by

$$M = \lim_{(\eta-\xi)\to\infty} N(l_\xi) l_\xi^{D_b} = \lim_{(\eta-\xi)\to\infty} L_t l_\xi^{D_b-1}$$

From Eq. (15), $l_\xi^{D_b} = l_\eta^{D_b} Q^{-(\eta-\xi)}$. Substitution of this equation and Eq. (25) into the above equation leads to

$$M = \lim_{(\eta-\xi)\to\infty} l_\eta^{D_b}\left[C_1 \cdot \frac{1-\left(Q^{D_s/D_b-1}\right)^{\eta-\xi+1}}{1-Q^{D_s/D_b-1}} + C_2 \cdot \frac{(P/Q)^{\eta-\xi} - \left(Q^{D_s/D_b-1}\right)^{\eta-\xi}\left(Q^{D_s/D_b}/P\right)}{1-\left(Q^{D_s/D_b}/P\right)}\right]$$

Here, $Q^{D_s/D_b-1} < 1$ for $1 \le D_s < D_b$ and $Q > P$. Therefore,

$$\lim_{(\eta-\xi)\to\infty}\left(Q^{D_s/D_b-1}\right)^{\eta-\xi}=0,\qquad \lim_{(\eta-\xi)\to\infty}(P/Q)^{\eta-\xi}=0$$

Consequently,

$$M=\frac{C_1 l_\eta^{D_b}}{1-Q^{D_s/D_b-1}} \tag{26}$$

When l_η is finite, M takes finite values. This means that the dimension of the drainage network which satisfies exactly Eqs. (1) and (2) is D_b. The D_b-dimensional measure of a drainage network of Branching System II is obtained by substituting Q' into Q and $C'_1 = (2 + \varepsilon_1 - P')/(Q' - P')$ into C_1 in Eq. (26). This means that the dimension of the drainage network which satisfies exactly Eqs. (1) and (7) is also D_b.

6. Selfsimilarity of Actual Drainage Basins

The selfsimilar drainage basins illustrated in Figs. 2, 4, 6, and the fudgeflake differ from actual drainage basins in some points. All the subbasins in them are drawn similarly with the respective main basins. On the other hand actual drainage basins take more or less irregular shapes for random components acting on them. In Fig. 6 and the fudgeflake, $\varepsilon_1 = 0$, but such a situation seldom happens in actual drainage basins because confluences of stream channels are affected also by randomness. Actual stream channels bifurcate and wiggle in general. This means that the stream channels in Figs. 2 and 4, which ever trifurcate and were expressed by straight lines, were divorced from reality.

There are some disagreements between the selfsimilar drainage basin models drawn regularly and actual drainage basins as shown above. These models, however, help us with better understanding of the structure of drainage basins. Recursivity is a key concept to understand the structure of selfsimilar figures. Equations (3) and (8), which express the law of stream numbers of Branching Systems I and II respectively, were derived from the corresponding recurrence equations as shown Appendices I and II. Equations (12) and (13) show recursive relations of areas of ordered basins. Equations (21) and (22) also show recursive relations between lengths of ordered streams. Consequently Branching Systems I and II can be called recursive systems. The drainage basin which satisfies Eqs. (3), (12), and (21) approximately with ε_1 and K given by the average values is regarded to be statistically recursive and selfsimilar belonging to Branching System I.

It seems possible to define the many types of recursive and selfsimilar branching systems by equations besides the ones mentioned already. For example, the branching system defined by Eq. (1) and the following relation is imaginable and it will probably satisfy the condition of recursivity as well as selfsimilarity.

$$\frac{\varepsilon_2}{\varepsilon_1} \neq \frac{\varepsilon_3}{\varepsilon_2} \neq \frac{\varepsilon_4}{\varepsilon_3} = \frac{\varepsilon_5}{\varepsilon_4} = \cdots = \frac{\varepsilon_\phi}{\varepsilon_{\phi-1}} = \cdots$$

Here it should be, however, noted that the law of stream numbers given by Eq. (3) was derived empirically at first and only Branching System I has been verified in actual drainage basins (Tokunaga, 1966, 1978; Onda and Tokunaga, 1987).

There is, however, still a discrepancy even between Branching System I and actual drainage basins. The subbasins of the lowest order in an actual drainage basin have finite sizes while Branching System I was built on the assumption that a drainage basin can be ultimately divided into infinitesimal subbasins and interbasin areas. This postulation is common to the drainage basin models which were built on the basis of fractal geometry. The discrepancy mentioned above was one which lies between an ideal gas and real gases. Larger is the difference between the order of an actual drainage basin and the lowest order of subbasins in it, more favourable is the postulation to it.

Equation (3) shows that the plots of logarithm of stream number to order of Branching System I exibit upconcavity except the case of $K = 0$ at the part of higher orders and then $\log Q$ means the gradient of asymptote which the plots approach at the part of lower orders (Tokunaga, 1966, 1972, 1978). The bifurcation ratio which was obtained by applying a straight line to the semilogarithmic plots should be related to the basin area ratio in the manner that the bifurcation ratio takes the value somewhat smaller than the basin area ratio Q. The stream length ratio Q^{D_s/D_b} is also as a matter of course related to the basin area ratio Q.

Shimano (1978) studied the relations between the parameters of Horton's laws, viz. the bifurcation ratio, the basin area ratio, etc., of 180 drainage basins in the Japanese Islands. Then the streams were ordered by Strahler's method on topographic maps of 1:50,000. Areas of the basins range from 14 km^2 to 165 km^2 and most of them were located in mountainous regions. The orders of almost all basins were expected to exceed 3 judging from their sizes and locations. The values of the parameters were obtained from the best fit regressions of straight lines applied to semilogarithmic plots. The study shows that the basin area ratio R_A is related to the bifurcation ratio R_B with the equation; $R_A = 0.966R_B + 0.430$, and then the correlation coefficient is 0.876. The study also shows that the stream length ratio is related to the basin area ratio with the correlation coefficient of 0.709. The value of basin area ratio exceeds that of bifurcation ratio in 161 basins. The equation and the values of correlation coefficient demonstrate the applicability of Branching System I to actual drainage basins.

Branching System I is unique in describing the mean state of drainage basins constructed by random confluences of stream channels. That is explained by using the terms; topologically distinct channel networks, topologically random channel networks, infinite topologically random channel networks, etc. defined by Shreve (1966, 1967, 1969). The mean state of infinite topologycally random channel networks satisfies Eqs. (1) and (2) with $\varepsilon_1 = 1$ and $K = 2$. Therefore, the law of stream numbers for the population of infinite topologically random channel networks is obtained by substituting $\varepsilon_1 = 1, P = 1$, and $Q = 4$ into Eq. (3) (Shreve, 1969; Tokunaga, 1978). The law of basin areas for the population is obtained by substituting $Q = 4$ into Eq. (12) (Tokunaga, 1978). It has been shown that finite topologically random channel networks with considerablly large number of first order streams satisfy approximately Eqs. (1) and (2) (Tokunaga, 1972). In this way randomness is smoothly introduced into Branching System I without disturbing its selfsimilarity and recursivity.

On the other hand, Branching System II rejects to introduce randomness in the manner same with the case of Branching System I by the relation that $\varepsilon_2/\varepsilon_1 \neq \varepsilon_3/\varepsilon_2$, though it possesses the nature of selfsimilarity and recursivity.

Onda and Tokunaga (1987) calculated the values of ε_1, K, and Q in 11 actual drainage basins. The result shows that the value of Q exceeds 4 in 10 basins. The measurement of basin area ratio by Shimano (1978) shows that it is below 4 only in 22 out of the 180 basins which make the average value of 4.52. These values mean that stream confluences do not happen completely at random but were affected by non-random force, because the expected value for Q or the basin area ratio in infinite topologically random channel networks was 4 (Tokunaga, 1978).

Here it should be noted that the value of Q or the basin area ratio is not so far from 4 in actual drainage basins. This fact indicates that randomness should be evaluated as one of the important factors which controls the confluence of stream channels. Therefore, Branching System I seems to be favourably applicable to actual drainage basins in a homogeneous environment, in that it permits the interposition of randomness in the confluence of stream channels.

Branching System II has not yet been tested against data obtained in actual drainage basins. Therefore, there are no confident bases to affirm or deny its applicability. Consideration on the role of randomness in the stream channel confluences in Branching System I, however, suggests that Brahcing System II or more complicated systems will not be widely applied to actual drainage basins.

The fractal dimension of a drainage basin without "sink holes" is 2. Therefore, that of the drainage network in it should be 2. This was endorsed by the values obtained in actual drainage basins by Tarboton *et al.* (1988).

REFERENCES

Horton, R. E. (1945) Erosional development of streams and their drainage basins: hydrophysical approach to quantitative geomorphology: Bull. Geol. Soc. Am., 56, 275–370.

Mandelbrot, B. B. (1977) Fractals-form, chance, and dimension: Freeman, San Francisco. 365 p.

Mandelbrot, B. B. (1983) The fractal geometry of nature: Freeman, San Francisco, 468 p.

Morisawa, M. E. (1962) Quantitative geomorphology of some watersheds in Appalatian Plateau: Bull.

Geol. Soc. Am., 73, 1025–1046.
Onda, Y. and Tokunaga, E. (1987) Laws of the composition of divide-segment system and drainage basin morphology: Geogr. Rev. Japan, 60 (Ser. A). 593–612. (in Japanese with English summary).
Schumm, S. A. (1956) Evolution of drainage systems and slopes in badlands at Perth Amboy, New Jersy: Bull. Geol. Soc. Am., 67, 597–674.
Shimano, Y. (1978) On the characteristics of the channel networks in Japanese drainage basins: Geogr. Rev. Japan, 51, 776–784. (in Japanese with English summary).
Shreve, R. L. (1966) Statistical law of stream numbers: Jour. Geol., 74, 17–37.
Shreve, R. L. (1967) Infinite topologically random channel networks: Jour. Geol., 75, 178–186.
Shreve, R. L. (1969) Stream lengths and basin areas in topologically random channel networks: Jour. Geol., 77, 379–414.
Tarboton, D. G., Bras, R. L., and Rodriguez-Inturbe, I. (1988) The factal nature of river networks: Wat. Resour. Res., 24, 1317–1322.
Tokunaga, E. (1966) The composition of drainage network in Toyohira River Basin and valuation of Horton's first law: Geophys. Bull. Hokkaido Univ., 15, 1–19. (in Japanese with English summary).
Tokunaga, E. (1972) Topologically random channel networks and some geomorphological laws: Georg. Rep. Tokyo Metrop. Univ. 6/7, 39–49.
Tokunaga, E. (1975) Laws of drainage composition in idealized drainage basins: Geogr. Rev. Japan, 48, 351–364. (in Japanese with English summary).
Tokunaga, E. (1978) Consideration on the composition of drainage networks and their evolution: Georgr. Rep. Tokyo Metrop. Univ., 13. 1–27.

Appendix I

The following equation is derived from Eqs. (1) and (2) by using *K*.

$$ {}_{\kappa}\mu_{\lambda} = 2\,{}_{\kappa}\mu_{\lambda+1} + \sum_{\eta=\lambda+1}^{\kappa} \varepsilon_1 K^{\eta-\lambda-1}\,{}_{\kappa}\mu_{\eta} \qquad (\mathrm{A-1-1}) $$

A shift of the index ($\lambda \to \lambda + 1$) yields

$$ {}_{\kappa}\mu_{\lambda+1} = 2\,{}_{\kappa}\mu_{\lambda+2} + \sum_{\eta=\lambda+2}^{\kappa} \varepsilon_1 K^{\eta-\lambda-2}\,{}_{\kappa}\mu_{\eta} \qquad (\mathrm{A-1-2}) $$

where ${}_{\kappa}\mu_{\kappa} = 1$ and ${}_{\kappa}\mu_{k-1} = 2 + \varepsilon_1$. Subtracting Eq. (A-1-2) $\times K$ from Eq. (A-1-1) yields

$$ {}_{\kappa}\mu_{\lambda} = (2 + \varepsilon_1 + K)\,{}_{\kappa}\mu_{\lambda+1} - 2K\,{}_{\kappa}\mu_{\lambda+2} \qquad (\mathrm{A-1-3}) $$

This equation is rewritten as follows:

$$\frac{{}_{\kappa}\mu_{\lambda}}{{}_{\kappa}\mu_{\lambda+1}} = 2 + \varepsilon_1 + K - \frac{2K}{\dfrac{{}_{\kappa}\mu_{\lambda+1}}{{}_{\kappa}\mu_{\lambda+2}}}$$

The above equation holds for $\lambda \leq \kappa - 2$. Put $\kappa = \eta$ and $\lambda = \psi$ in the equation. Then expansion of the consequential equation up to $\psi = \eta - 2$ leads to Eq. (4). By substituting $(\lambda + \eta - 1)$ into λ in Eq. (A-1-3), the following equation is obtained.

$${}_{\kappa}\mu_{\lambda+\eta-1} = (2 + \varepsilon_1 + K)\,{}_{\kappa}\mu_{\lambda+\eta} - 2K\,{}_{\kappa}\mu_{\lambda+\eta+1}$$

This equation holds for $1 \leq \eta \leq \kappa - \lambda - 1$ and is rewritten by using P and Q as follows:

$${}_{\kappa}\mu_{\lambda+\eta-1} - P\,{}_{\kappa}\mu_{\lambda+\eta} = Q\,{}_{\kappa}\mu_{\lambda+\eta} - Q\,P\,{}_{\kappa}\mu_{\lambda+\eta+1} \qquad (\text{A}-1-4)$$

Equation (A-1-4) $\times\ Q^{\eta-1}$ is

$$Q^{\eta-1}\,{}_{\kappa}\mu_{\lambda+\eta-1} - Q^{\eta-1}P\,{}_{\kappa}\mu_{\lambda+\eta} = Q^{\eta}\,{}_{\kappa}\mu_{\lambda+\eta} - Q^{\eta}P\,{}_{\kappa}\mu_{\lambda+\eta+1} \qquad (\text{A}-1-5)$$

The product of Eq. (A-1-5) for $\eta = 1, 2, \cdots, \kappa - \lambda - 1$ is

$$\prod_{\eta=1}^{\kappa-\lambda-1}\left(Q^{\eta-1}\,{}_{\kappa}\mu_{\lambda+\eta-1} - Q^{\eta-1}P\,{}_{\kappa}\mu_{\lambda+\eta}\right)$$
$$= \prod_{\eta=1}^{\kappa-\lambda-1}\left(Q^{\eta}\,{}_{\kappa}\mu_{\lambda+\eta} - Q^{\eta}P\,{}_{\kappa}\mu_{\lambda+\eta+1}\right) \qquad (\text{A}-1-6)$$

The following equation is derived by eliminating the equal terms in the both sides in Eq. (A-1-6) and putting ${}_{\kappa}\mu_{\kappa} = 1$ and ${}_{\kappa}\mu_{\kappa-1} = 2 + \varepsilon_1$.

$${}_{\kappa}\mu_{\lambda} - P\,{}_{\kappa}\mu_{\lambda+1} = Q^{\kappa-\lambda-1}(2 + \varepsilon_1) - Q^{\kappa-\lambda-1}P \qquad (\text{A}-1-7)$$

This relation holds for $\lambda \leq \kappa - 1$. By substituting $(\lambda + \eta - 1)$ into λ in Eq. (A-1-7) and multiplying consequential equation by $P^{\eta-1}$, the following equation is obtained.

$$P^{\eta-1}\,{}_{\kappa}\mu_{\lambda+\eta-1} - P^{\eta}\,{}_{\kappa}\mu_{\lambda+\eta}$$
$$= Q^{\kappa-\lambda-\eta}P^{\eta-1}(2 + \varepsilon_1) - Q^{\kappa-\lambda-\eta}P^{\eta} \qquad (\text{A}-1-8)$$

The sum of Eq. (A-1-8) for $\eta = 1, 2, \ldots, (\kappa - \lambda - 1)$ is

$$ {}_{\kappa}\mu_{\lambda} - P^{\kappa-\lambda-1}(2+\varepsilon_1) = \sum_{\eta=1}^{\kappa-\lambda-1} \left[Q^{\kappa-\lambda-\eta} P^{\eta-1}(2+\varepsilon_1) - Q^{\kappa-\lambda-\eta} P^{\eta} \right] $$

Then,

$$ {}_{\kappa}\mu_{\lambda} = Q^{\kappa-\lambda-1} \left[(2+\varepsilon_1) \frac{1-(P/Q)^{\kappa-\lambda-1}}{1-P/Q} - P \frac{1-(P/Q)^{\kappa-\lambda-1}}{1-P/Q} \right] + P^{\kappa-\lambda-1}(2+\varepsilon_1) \qquad (\mathrm{A}-1-9) $$

Put $\kappa = \eta$ and $\lambda = \psi$ in Eq. (A-1-9). Then Eq. (3) is obtained by rearranging the consequential equation.

Appendix II

The following equation is derived from Eqs. (1) and (7) by using K'.

$$ {}_{\kappa}\mu_{\lambda} = (2+\varepsilon_1)_{\kappa}\mu_{\lambda+1} + \sum_{\eta=\lambda+2}^{\kappa} \varepsilon_2 K'^{\,\eta-\lambda-2}{}_{\kappa}\mu_{\eta} \qquad (\mathrm{A}-2-1) $$

A shift the index ($\lambda \rightarrow \lambda + 1$) in the above equation leads to

$$ {}_{\kappa}\mu_{\lambda+1} = (2+\varepsilon_1)_{\kappa}\mu_{\lambda+2} + \sum_{\eta=\lambda+3}^{\kappa} \varepsilon_2 K'^{\,\eta-\lambda-3}{}_{\kappa}\mu_{\eta} \qquad (\mathrm{A}-2-2) $$

where ${}_{\kappa}\mu_{\kappa} = 1$, and ${}_{\kappa}\mu_{\kappa-1} = 2 + \varepsilon_1$. Subtracting Eq. (A-2-2) $\times K'$ from Eq. (A-2-1) and rearranging the consequential equation yield

$$ {}_{\kappa}\mu_{\lambda} = (2+\varepsilon_1+K')_{\kappa}\mu_{\lambda+1} - (2K'+\varepsilon_1 K' - \varepsilon_2)_{\kappa}\mu_{\lambda+2} \qquad (\mathrm{A}-2-3) $$

Equation (A-2-3) holds for $\lambda \le \kappa - 2$. Put $\kappa = \eta$ and $\lambda = \psi$ in the equation. Then ${}_{\eta}\mu_{\psi}/{}_{\eta}\mu_{\psi+1}$ can be expanded into the continued fraction given by Eq. (9).

Substituting $(\lambda + \eta - 1)$ into λ in Eq. (A-2-3) yields

$${}_{\kappa}\mu_{\lambda+\eta-1} = \left(2 + \varepsilon_1 + K'\right)_{\kappa}\mu_{\lambda+\eta} - \left(2K' + \varepsilon_1 K' - \varepsilon_2\right)_{\kappa}\mu_{\lambda+\eta+1}$$

This equation is rewritten by using P' and Q' as follows:

$${}_{\kappa}\mu_{\lambda+\eta-1} - P'_{\kappa}\mu_{\lambda+\eta} = Q'_{\kappa}\mu_{\lambda+\eta} - Q'P'_{\kappa}\mu_{\lambda+\eta+1}$$

This equation holds for $\eta = 1, 2, \cdots, \kappa - \lambda - 1$ and takes the form same with Eq. (A-1-4). Then Eq. (8) is obtained through the procedure same with that to derive Eq. (3) from Eq. (A-1-4).

Appendix III

Let ξ be the lowest basin order in the basin of order κ and assume that the basin of order κ is divided into infinitesimal subbasins and interbasin areas in the ultimate according to Eqs. (1) and (2). When $(\kappa - \xi) \rightarrow \infty$, the following equation is derived from Eqs. (1) and (2) by using K.

$$A_{\kappa} = 2A_{\kappa-1} + \varepsilon_1 \sum_{\eta=\xi}^{\kappa-1} K^{\kappa-\eta-1} A_{\eta} \qquad (\mathrm{A-3-1})$$

A shift of the index ($\kappa \rightarrow \kappa - 1$) leads to

$$A_{\kappa-1} = 2A_{\kappa-2} + \varepsilon_1 \sum_{\eta=\xi}^{\kappa-2} K^{\kappa-\eta-2} A_{\eta} \qquad (\mathrm{A-3-2})$$

Subtracting Eq. (A-3-2) $\times$ K from Eq. (A-3-1) and rearranging the consequential equation yield

$$A_{\kappa} = \left(2 + \varepsilon_1 + K\right)A_{\kappa-1} - 2KA_{\kappa-2} \qquad (\mathrm{A-3-3})$$

Equation (A-3-3) is generalized by substituting η into κ.

$$A_{\eta} = \left(2 + \varepsilon_1 + K\right)A_{\eta-1} - 2KA_{\eta-2} \qquad (\mathrm{A-3-4})$$

Let suppose subbasins of order λ in the basin of order κ, where $\kappa \geq \lambda + 2$. Then the equation which expresses the relation among A_{κ}, $A_{\lambda+1}$, and A_{λ} is derived from Eq. (A-3-4) in the way similar with that to obtain Eq. (3). The equation is given as follows:

$$A_\kappa = \frac{A_{\lambda+1} - PA_\lambda}{Q-P} Q^{\kappa-\lambda} + \frac{A_{\lambda+1} - QA_\lambda}{P-Q} P^{\kappa-\lambda} \qquad (A-3-5)$$

The same relation can be assumed among A_λ, $A_{\xi+1}$, and A_ξ by supposing basins of orders lower than ξ. Namely,

$$A_\lambda = Q^{\lambda-\xi}\left[\frac{A_{\xi+1} - PA_\xi}{Q-P} + \frac{A_{\xi+1} - QA_\xi}{P-Q}\left(\frac{P}{Q}\right)^{\lambda-\xi}\right]$$

A shift of the index ($\lambda \to \lambda + 1$) leads to

$$A_{\lambda+1} = Q^{\lambda-\xi+1}\left[\frac{A_{\xi+1} - PA_\xi}{Q-P} + \frac{A_{\xi+1} - QA_\xi}{P-Q}\left(\frac{P}{Q}\right)^{\lambda-\xi+1}\right]$$

Here, $Q > P$. Therefore,

$$\lim_{(\lambda-\xi)\to\infty}\left(A_{\lambda+1} / A_\lambda\right) = Q$$

Substituting $A_{\lambda+1} = QA_\lambda$ into Eq. (A-3-5) yields

$$A_\kappa = Q^{\kappa-\lambda} A_\lambda \qquad (A-3-6)$$

Equation (12) is obtained by putting $\kappa = \eta$ and $\lambda = \psi$ in Eq. (A-3-6).

Appendix IV

Let ξ be the lowest basin order in the basin of order κ and assume that the basin of order κ is divided into infinitesimal subbasins and interbasin areas in the ultimate according to Eqs. (1) and (7). When $(\kappa - \xi) \to \infty$, the following equation is derived from Eqs. (1) and (7) by using K'.

$$A_\kappa = (2+\varepsilon_1)A_{\kappa-1} + \varepsilon_2 \sum_{\eta=\xi}^{\kappa-2} K'^{\kappa-\eta-2} A_\eta \qquad (\mathrm{A-4-1})$$

A shift of the index ($\kappa \to \kappa - 1$) leads to

$$A_{\kappa-1} = (2+\varepsilon_1)A_{\kappa-2} + \varepsilon_2 \sum_{\eta=\xi}^{\kappa-3} K'^{\kappa-\eta-3} A_\eta \qquad (\mathrm{A-4-2})$$

Subtracting Eq. (A-4-2) × K' from Eq. (A-4-1) and rearranging the consequential equation yield

$$A_\kappa = (2+\varepsilon_1+K')A_{\kappa-1} - (2K'+\varepsilon_1 K' - \varepsilon_2)A_{\kappa-2} \qquad (\mathrm{A-4-3})$$

Equation (13) is derived from Eq. (A-4-3) in the way same with that to obtain Eq. (12).

Appendix V

The following equation is derived from Eq. (18).

$$N_{\eta,\xi} / N_{\eta,\xi+1} = \left(l_{\xi+1} / l_\xi\right)^{D_s} \qquad (\mathrm{A-5-1})$$

Substituting $l_{\xi+1}/l_\xi = Q^{1/D_b}$ in Eq. (15) into the above equation yields

$$N_{\eta,\xi} / N_{\eta,\xi+1} = Q^{D_s/D_b} \qquad (\mathrm{A-5-2})$$

Equation (19) is obtained by substituting $N_{\eta,\xi+1} = N_{\eta-1,\xi}$ in Eq. (17) into Eq. (A-5-2).

The following equation is derived from Eqs. (16) and (A-5-1).

$$N_{\eta,\xi} / N_{\eta,\xi+1} = Q'^{D_s/D_b}$$

Substituting $N_{\eta,\xi+1} = N_{\eta-1,\xi}$ into the above equation yields Eq. (20).