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Application of the Variation Principle to Geographic Pattern Analysis

Masashige HIRANO

Department of Geography, Fac. Lit., Osaka City University, Sugimoto 3-3, Sumiyoshi-ku, Osaka 558, Japan

Abstract. The variation principle is very much useful to analyse the origin of some geographic patterns rationally as the stationary curve or the optimum path. Total time, cost, force, etc. are often given by a functional which is the integral along a course defined by some function mathematically, and the function minimizing or maximizing the integral gives the stationary curve. This concept supplies an important basis to understand linear pattern and the background distribution of a scalar, concerned, as shown here by several examples.

Introduction; variation calculus

For instance, the route connecting two points fastest under a given condition is a kind of the optimum path (Fig. 1). For this problem, the velocity is given as a function of location generally, and we have $v = v(x, y)$ in cartesian co-ordinate. The time required along the route l; $y(x)$ on a plane is given by the integral,

$$T = \int_l \frac{ds}{v(x,y)} = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{v(x,y)} dx, \tag{1}$$

and the optimum path is given by the stationary curve minimizing the integral. The Eq. (1) is re-writen more generally as

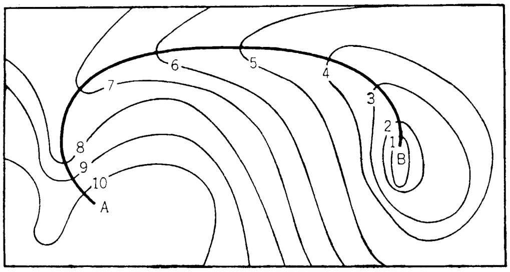

Fig. 1. The optimum path for walk across the marsh in which the velocity changes locally due to ground condition. Numerals by iso-chrone give the time required to access there from the starting point B. After Bunge (1962).

$$I = \int_{x_0}^{x_1} F(x, y, y')dx, \tag{1'}$$

and called the functional. The function $y(x)$ minimizing the integral (1') satisfies the Euler's equation,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{2}$$

Especially when F does not depend on x explicitly, Eq. (2) has the first integral given by

$$F - y'\frac{\partial F}{\partial y'} = C_1, \tag{2'}$$

where C_1 is a constant of integration. Further integration of Eq. (2') brings another constant of integration C_2. The constants are determined by the boundary conditions,

$$\left.\begin{array}{l} y = y_0 \ \text{ at } \ x = x_0 \\ y = y_1 \ \text{ at } \ x = x_1 \end{array}\right\}$$

which imply that the curve passes the two specified points.

It is possible generally to obtain the optimum path connecting two points on a given surface by this method. The author discussed the application of this analysis to some geographic patterns briefly (Hirano, 1978) and wishes here to refer to this again with some additional results obtained thereafter.

1. Geodesics

It is shown immediately by application of this method that the shortest path (geodesic) on a plane is given by a straight line and that the one on a sphere by a great circle. The geodesic on a plane minimizes the functional,

$$L_1 = \int_l ds = \int_{x_0}^{x_1} \sqrt{1+y'^2}\, dx. \tag{3a}$$

Application of Eq. (2′) to $F=\sqrt{1+y'^2}$ brings $y'=\pm C_*$ with $C_*=\sqrt{C_1^{-2}-1}$. By further integration, we have

$$y = \pm C_* x + C_2,$$

finally, which defines a straight line as the shortest path connecting two points.

Likewise, we have for a sphere the integral,

$$L_2 = \int_{\theta_0}^{\theta_1} \sqrt{1+\phi'^2 \sin^2\theta}\, d\theta \tag{3b}$$

to be minimized, where ϕ is longitude and θ is given by $\theta = 90° - \omega$ for latitude ω. Following Eq. (2′) we have

$$\frac{\phi' \sin^2\theta}{\sqrt{1+\phi'^2}\sin\theta} = C_1$$

and it is known that $\phi' = 0$ for $C_1 = 0$. This gives constant ϕ which means the meridian, a great circle. An airline route covering a long distance follows the great circle. Of course a slight modification of it is resulted to save fuel following air pressure patterns as discussed by Waranz (1961) for the flight course across the North Atlantic.

2. Law of Traffic Path Refraction

Consider a path for transportation across two distinct areas such as sea and land. Cost for transportation differs from the areas, and the refraction of the path occurs at the contact boundary of the areas, because that straight line is the optimum path in the respective area and that there is a point on the boundary that minimizes the sum of the costs along the straight paths (Fig. 2). This phenomenon is well known as the law of the traffic path refraction since Stackelberg (1938). The water path for a point facing the shore from an inland city gets long-shore course as long as possible to form the critical angle with the inland path as the result of this refraction.

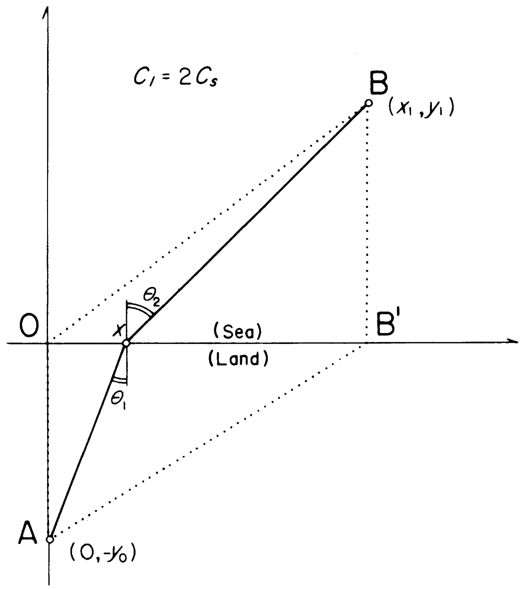

Fig. 2. The optimum path for the minimum sum of the transportational cost across the two distinct areas such as land and sea. The path is refracted at the boundary (shore). After Hirano (1978).

3. Near Shore Railway

Consider the case to plan a railway route near shoreline. It is approximately the case that the cost of construction for railway is proportional to the reciprocal of a linear function of the distance from the shore. Because the closer to the shore line the area, the more densely populated is. The cost is given totally by

$$C = \int_{x_0}^{x_1} \frac{kw}{a + by} \sqrt{1 + y'^2}\, dx, \tag{4}$$

where w is the width of the zone needed for railway construction. The path minimizing the total cost is given by an arc. Especially when $a = 0$, the center of the circle is on the shore at $y = 0$ (Fig. 3). The location of the three railways, JR, Hankyu-line, and Shinkansen, connecting Osaka and Kobe shows a distinct pattern as given in Fig. 4. Their locations relative to the coast line and to the residential area can be explained by this way. The exceptional case of the Hanshin-line almost along the shore connecting the cities among Osaka and Kobe has a quite different process of location of the line. Namely, request and support of the areas through which the line runs brought the present pattern in spite of its later age of construction than JR.

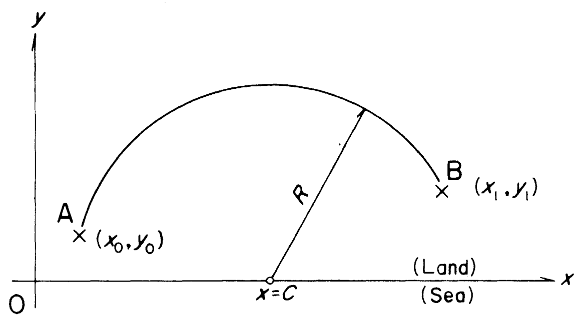

Fig. 3. The optimum path minimizing the total cost for construction of the railway connecting two cities near shore. The cost is proportional to the reciprocal of the distance from the shore. After Hirano (1978).

4. Bypass Problem

Land cost in urban area is proportional to the reciprocal of the distance from the center of the city, generally. Large part of the cost required at road construction comes from that for getting the land. Here we assume that the cost is proportional to the reciprocal of the linear function of the distance r form the city center. The functional giving the total cost is thus

$$C = \int_{0}^{\theta_1} \frac{kw}{a + br} \sqrt{r^2 + r'^2}\, d\theta, \tag{5}$$

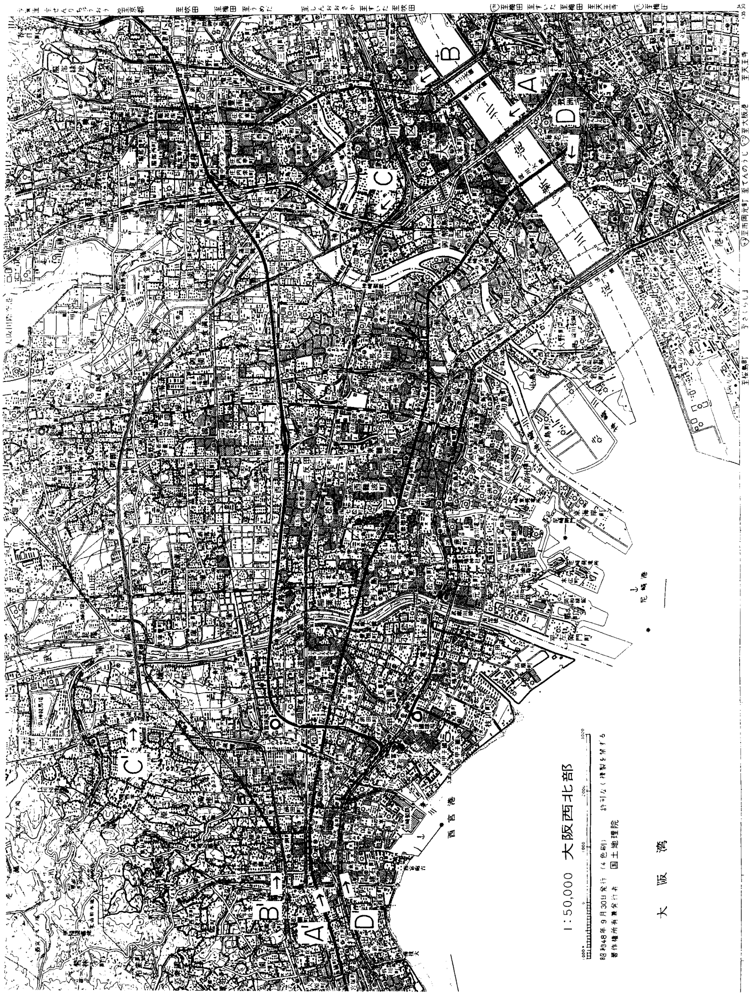

Fig. 4. Geographic pattern of the railways, JR (A-A'), Hankyu-line (B-B'), and Shinkansen (C-C') in ascending order, connecting Osaka and Kobe showing the more bulging off the shore following the later age of construction. The exceptional case of the Hanshin-line (D-D') has a quite different situation at planning. After Osaka-seihokubu quadrangle of 1/50000 by Geogr. Surv. Inst. Japan.

when one of the terminal points is chosen to be at $\theta = 0$. Especially for the case of $a = 0$, we have a logarithmic (equiangular) spiral

$$\frac{r}{r_0} = \left(\frac{r_1}{r_0}\right)^{\theta/\theta_1} \tag{5'}$$

as the optimum path (Fig. 5). A good example of this case is shown by the Hamamatsu bypass for the route 1; Tokaido (Fig. 6). The generality of this case is evident from the distributional mode of the land cost approximated by Eq. (5) except for the region just near the center of the city.

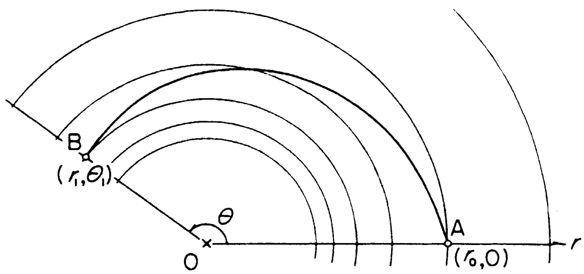

Fig. 5. The optimum path minimizing the total cost for construction of bypass route avoiding the center of a city where land cost decreases outward in proportion to the reciprocal distance from the center. After Hirano (1978).

It is remarked in near shore railway or in bypass problem that density of such institutions to be avoided as school, hospital, historical monument, etc. is a function of distance analogous to the land cost, and thus the land cost itself is unnecessarily proportional to the reciprocal of the distance directly. It is also emphasized here that mathematically same problem as those shown by Eq. (4) or (5) appears at propagation of seismic wave in subsurface area as clearly shown by Miarage phenomenon.

5. Transversality Condition and Port-Orientated Railway

The boundary conditions often modify the geographic patterns we observe. Especially the transversality condition is important to understand the access route to a specified boundary like coast line from a given point of inland area. If the route wishes to connect the shore and inland city by the shortest line, the route gets perpendicular to the shore. This is the simplest case of the transversality condition

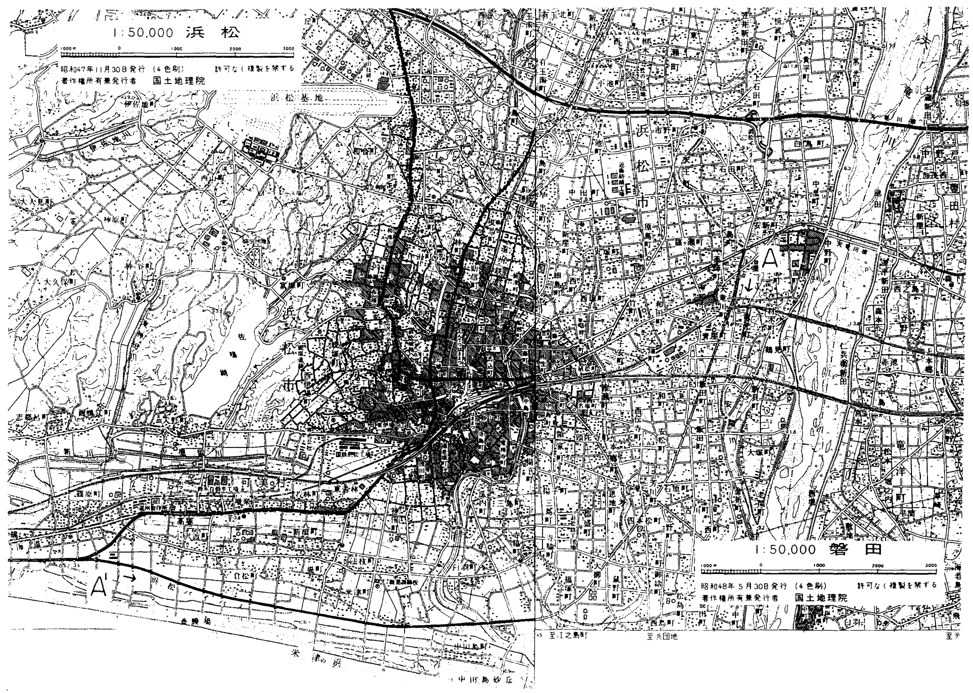

Fig. 6. Example of the Hamamatsu bypass avoiding the center of the city suggesting such spatial distribution of construction cost as employed in model analysis. After Hamamatsu and Iwata quadrangles of 1/50000 by Geogr. Surv. Inst. Japan.

which means that one of the terminal of the stationary curve is located on a given curve. Denoting the boundary by $y = g(x)$, the transversality condition is given by

$$F + (g' - y')\frac{\partial F}{\partial y'} = 0, \quad \text{at} \quad x = x_1. \tag{6}$$

Especially for $F = \sqrt{1 + y'^2}$, Eq. (6) reduces to

$$g' y' = -1 \quad \text{at} \quad x = x_1 \tag{6'}$$

which means orthogonal relationship of y, the stationary curve, and g, the boundary. We can find often this type of problems in geography, and the case of port-orientated railway route is a good example as shown in Fig. 7.

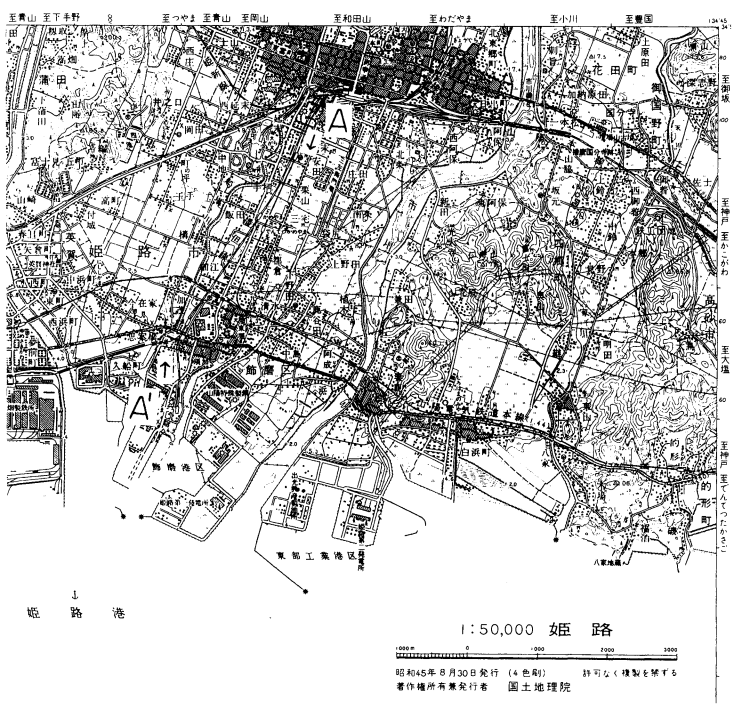

Fig. 7. Port-directed railway accessing perpendicular to the coast. The example of the optimum path satisfying the transversality condition in variation calculus. After Himeji quadrangle of 1/50000 by Geogr. Surv. Inst. Japan.

If a geographic field or a spatial distribution of a scalar was described by contour lines, the orthogonal curves to contour lines form a group of stationary curves, satisfying the transversality condition. For instance, consider the optimum paths for person trip corresponding to a given isochrone map as given in Fig. 1, the trace of trip is everywhere perpendicular to the isochrone lines. The line of maximum inclination for a given elevational contour map is also normal to the contour lines and defines the characteristics of the given topographic relief.

6. Natural Condition and River Bed Profile

The nature of longitudinal profile of rivers is often discussed in geomorphology. It is possible to obtain it after the variation principle as the brachistchrone, the fastest line along which the mass glides down in gravitational field. Thus, the profile is the stationary curve of the functional,

$$T = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y}} dx \tag{7}$$

as $v = \sqrt{2g(h_0 - h)} = k\sqrt{y}$. It is well known that the profile is cycloid as no friction acts on river bed in this case.

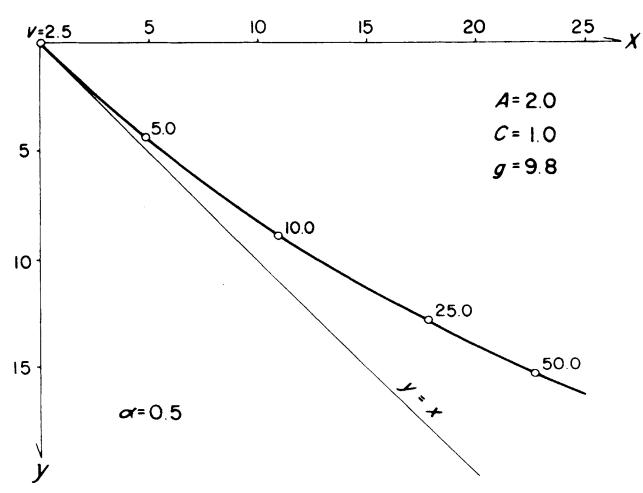

Fig. 8. Longitudinal profile of river bed as a brachistchrone obtained for the case in which the bottom resistance is proportional to the square of flow velocity. The result of numerical solution for given values of constants has been shown after Hirano (1978).

If the friction proportional to square powers of velocity works on the bottom, we get a concave profile analogous to cycroid or exponential curve as shown in Fig. 8. The natural condition,

$$\left(\frac{\partial F}{\partial y'}\right) = 0, \quad \text{at} \quad x = x_1 \tag{8}$$

has to be satisfied generally at the terminal point where the river water approaches, because no restriction on the velocity at the end is specified in this case. The result given in Fig. 8 was obtained by numerical integration under this condition.

7. Mountain Climbing Route

Climbing route guiding us to the mountain peak goes straightly at the gentle foot, and shows zig-zag pattern on the steep slope. This particular pattern can be explained as the optimum path minimizing the distance (geodesics) or the total or excess energy required for climbing. Approximating the mountain slope by a cone, it is found that geodesics on it directing the summit is a radial line passing the center of the mountain, and that the optimum path for the energy is logarithmic spiral on the steep slope (Fig. 9). Geodesics for a gentle slope is the optimum path even concerning the energy required at climbing. Mt. Fuji climbing route is a good example of this case as discussed by Hirano (1983), because the mountain is well approximated by a cone and that the climbing route has established over long time for climbers without mechanical support.

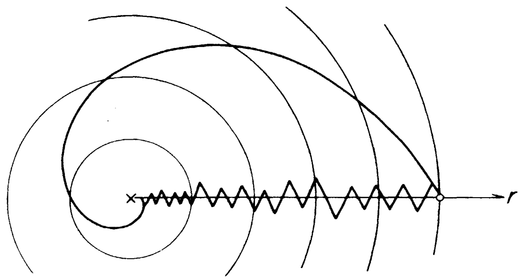

Fig. 9. Mountain climbing route as the optimum path minimizing the energy required for climbing. The mountain with the top at the center is approximated by a cone shown by concentric contour lines and the route given by logarithmic spiral has been tipped back to give a zig-zag pattern in order to be directed the mountain top. After Hirano (1978).

8. Glacial Valley Profile

It is well known that glacial valley shows very much common and beautiful cross profile often referred to the U-shaped profile. It has been discussed that profile can be approximated by parabola or by catenary. If the contact length of ice and rock at the gracier bottom is constant, it is shown by variation calculus that the profile giving the extreme value to the bottom friction proportional to the pressure due to ice weight is a catenary (Fig. 10) as shown by Hirano (1981). The curve defines the maximum total friction along the contact plane, and this means the most effective profile for bottom erosion and for glacier ice storage. It's clear that both situation are necessary and important for development of the U-shaped glacier valley by scoring or pracking. The result gives way to analyse the developmental process of mountain glacier as discussed by Hirano and Aniya (1988, 1989). The optimum profile for glacier ice transportation is discussed in a same way as for the landslide being discussed next to maximize the cross area under the constraint condition of the constant bottom resistance.

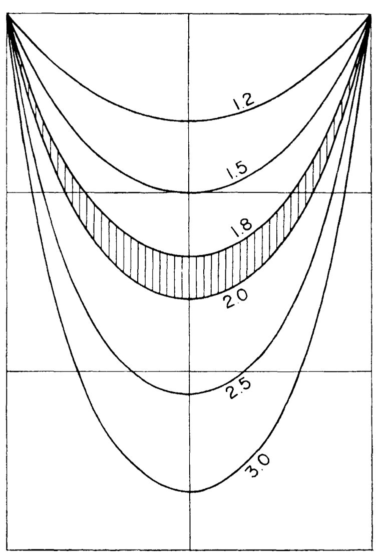

Fig. 10. Catenary as the stationary curve maximizing the bottom friction on a given contact length of glacier valley cross profile. Variety of curves are resulted in corresponding to the boundary conditions and to the contact length (the constraint condition). Numerals given the ratio of the contact length to the valley width. After Hirano (1981).

9. Cross Profile of Landslide Block

Cohesion and friction works on the bottom of landslide block to resist the driving force due to gravitational acceleration. Stability analysis of landslide block is referring to the balance of the forces working in the longitudinal profile. However, some of the landslide shows elongated plan shape and arcuate cross profile. For this kind of landslide, it is often difficult to approximate the sliding plane by a simple arc in the longitudinal profile, and cross profile analysis (transverse two-dimensional analysis) can bring another kind of stability criterion. In addition, we are free from the moment of rotation in this case and the problem is much simplified. If cohesion works exclusively, it is shown by variation calculus that the most unstable cross profile maximizing the cross area of the block is hemi-circle. If the friction constitutes of the portions proportional to depth and of constant part identical to cohesion, the most unstable cross profile is elliptic and the flatteness of it is determined by the ratio of the identical cohesion to maximum bottom friction. Experimental result (Fig. 11) by Hirano and Ishii (1989) supports the above conclusion.

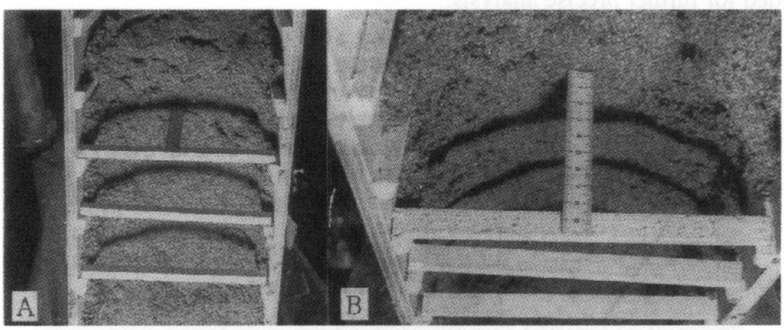

Fig. 11. Example of cross profile of landslide developed in wet sand. Shadows of cross bars give the outline of the shape. The cross profile is a kind of optimum one that make the slide block most unstable having the cross area maximum. After Hirano and Ishii (1989).

10. Iso-Perimetric Problem

In the cases of U-shaped glacial valley and land slide block, the constraint condition defining the contact length or the resistant force was introduced. The condition is given by the integral

$$I = \int_{x_0}^{x_1} G(x, y, y')dx. \tag{9}$$

This has been specified for the same range of x at the integral (3) that defines the functional to be minimized or maximized. Euler's equation in this case is applied to $F + \lambda G$ instead of F, where λ is the Lagrange multiplier. This is called the iso-perimetric problem in variation calculus. We can find out a lot of examples to which this type of constraint condition is attached in the nature. The fundamental solution is a circle for two-dimensional problem and a sphere for three dimensional one. They enclose the maximum area or volume for a given perimeter or surface area.

Recently, we had an exciting example of this problem, and the solution was marked beautifully on the land surface. Namely, Fig. 12 shows a collapse developed over the subsurface cave where Oya stone were curried before. The plan shape of the fallen block shows an almost perfect circle, and this is the stationary (most unstable) shape under the situation that cohesion and/or friction supporting the soil layer along a given circumference is specified, though detailed discussion on the locality of supporting columns or on the exact shape of subsurface cave will be needed for further precise analysis.

Fig. 12. Collapse of the covering earth layer over the cave made by curring the Oya-stone in Tochigi Prefecture. Excitingly typical solution of iso-perimetric problem in variation calculus. The photo by Asahi Shimbun Publishing Company just after the occurrence on February 10, 1989.

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