Science on Form: Proceedings of the First International Symposium for Science on Form,
General Editor: S. Ishizaka, Editors: Y. Kato, R. Takaki, and J. Toriwaki, 121–128.

On a Distance between Two Curves

Kazuo Kishimoto

Faculty of Engineering, Hiroshima University, Saijo-cho, Higashi-Hiroshima 724, Japan

Keywords: Distance, Integral Geometry, Jordan Curves

ABSTRACT

An L_1-like distance is defined between two Jordan curves in N-dimensional Euclidean space whose end points coincide. This distance equals to the area of the domain surrounded by these two curves if N = 2, and if two curves are simple, and have no intersection points. In the case of higher dimension, it gives a lower bound of the areas of the curved surfaces whose boundary is the union of these curves. It is also shown that it satisfies the axioms of distance.

1. Introduction

A "distance" betweeen two Jordan curves C_1 and C_2 defines how well these two lines approximate each other. In particular, if a "distance" is zero, the two lines are understood as the "same" line. Because approximation and identification of two lines are most fundamental procedures in the analysis of space curved lines, the definition and the measurement of a distance play important roles in this analysis.

If we were to have a unique parametrizations of these curves C_1 and C_2, we can use all the well established results of conventional functional analysis. We can use, for example, L_p measure defined by

$$d(C_1, C_2) = \left(\int_0^1 |x_1(s) - x_2(s)|^p \, dt \right)^{1/p}.$$

However, when we want to analyse a figure of a Jordan curve given as a point set, such as coast lines, lines drawn by xy-plotters or so on, so long as the author knows, we have no plausible parametrizations, which makes the situation much more difficult.

So long as the present author knows, there exist only two definitions of distance between two Jordan curves with no unique pre-determined parametrizations: that by Frechet(See, e.g., Iyanaga and Kawada, 1977.) and that by Steinhaus(1929)(See also Santalo, 1976, p.38.).

The distance by Steinhaus is defined by

$$d(C_1, C_2) = (1/2)\int |N_1(G) - N_2(G)|dG,$$

where $N_i(G)(i = 1, 2)$ are the number of intersection points between the curve C_i and the variable line G, and the integration is taken over all the straight lines on the plane. This measure is defined only for rectifiable curved lines on two-dimensional plane.

The Frechet distance is defined as

$$\inf_{s(t)} \sup_{t} |x_1(t) - x_2(s(t))|.$$

Here, the infinum is taken over all the possible parametrization $s = s(t)$. Frechet distance is, in a sense, a generalization of maximum norm in the conventional functional analysis.

As in the case of functional analysis, one sometimes wants an L_p-like measure which is essentially expressed as a sum of local volume, but the plausible definition of such a distance seems to be difficult. For example, the readers are recommended to check that the inf sup approach in the Frechet distance no more works in this case. That is,

$$\inf_{s(t)} \sup_{t} |x_1(t) - x_2(s(t))|^p$$

is reduced to the infimum of $|x_1(t) - x_2(s)|^p$ over two independent parameters s and t, which no more satisfies the axioms of distance.

This paper tries to solve this lack by proposing a distance which is a generalization of L_1 metric. The distance in this paper, in the case of $N = 2$, coincides with the area of the domain which is enclosed by the two curves. In three or higher dimensional cases, it defines a certain lower bound of the area of curved surface whose boundary is the given two Jordan curves.

Pohl(1968) considered a quantity similarly obtained but slightly different from ours, and derived a formula which relates it to another quantity. His quantity does not satisfy the axioms of distance, but it might suggest a possible existence of a similar formula for our distance, though I cannot show in this paper.

In the following, in section 2, we give the definition, and then in section 3, we describe some of its properties. Some other discussions will be made in section 4.

2. The definition

2. 1 The Definition of $d(C_1, C_2)$

A Jordan curve C is defined by a continuous image $\{x(t):x(t) = (x^1(t), x^2(t), \ldots, x^N(t)), 0 \leq t \leq 1\}$ from a unit interval [0, 1] into an N-dimensional Euclidean space. If $x(0) = x(1)$, C is called a closed Jordan curve.

Our approach is based on a new definition of an enclosing "area" A(C) of a closed Jordan curve C, which will be given in the following subsections. If C is a simple plane curve, A(C) coincides with the conventional area of the domain surrounded by C. If C is a higher-dimensional simple curve, A(C) gives a lower bound of areas of two-dimensional curved surfaces whose boundary is C.

The definition of our distance $d(C_1, C_2)$ is given as follows. Suppose that two Jordan curves $C_1 = \{x_1(t) | 0 \leq t \leq 1\}$, $C_2 = \{x_2(t) | 0 \leq t \leq 1\}$ are given in an N-dimensional Euclidean space. The pair C_1 and C_2 of Jordan curves define a closed Jordan curve $C = C_1 \cup C_2$. Thus, we define our distance $d(C_1, C_2)$ by A(C).

2. 2 A(C) in Two-Dimensional Case

We remark here that, for any point x_0 on a two-dimensional plane, a closed Jordan curve defines a winding number $w(C, x_0)$ of x_0 with respect to the Jordan curve C:

$$w(C, x_0) = \int_0^1 d \arg x(t)x_0.$$

We use its absolute value as the weight of integration over the whole plane. Then, the distance A(C) between C_1 and C_2 is defined by

$$A(C) = \int_{\text{whole plane}} |w(C, x)| dxdy.$$

It is easy to see that A(C) equals to the area of the domain enclosed by C if C is simple.

2. 3 Three or higher-dimensional cases

Let C be a closed Jordan curves C = { x(t) : $0 \leq t \leq 1$ } given in an N-dimensional Euclidean space. Let h be a (N - 2)-dimensional hyperplane which passes the orgin of the space. We project the curved line C and the hyperplane h onto the two-dimensional plane k which passes the origin, and is perpendicular to h. On k, we have a closed Jordan curve C' from C and a single point x_0' from h, from which their winding number $w(C'; x_0')$ is defined.

By considering all the (N - 2)-dimensional plane h' which are parallel to h, we obtain w(C'; x') for all the points x' on k. Integrating thus obtained w(C'; x') over the whole plane k, we can define the "projected area" D(C; h) = A(C') of C with respect to the plane k or equivalently with respect to the plane h.

We regard h as a point in the Grassmann manifold $E_{N-2,N}$, and integrate D(C; h) with respect to h over $E_{N-2,N}$. Denoting the invariant measure on the Grassmann manifold by $dE_{N-2,N}$, we define A(C) for N-dimensional closed Jordan curves as follows:

$$A(C) = \int_{\text{whole manifold}} \Gamma(N) D(C;\ h)/(2\pi)^{N-2}\, dE_{N-2,N}.$$

We here remark that, from the definition, A(C) is invariant under rotations and translations of C. We also notice that the constant factor in this integration is determined so that the A(C) is equal to the area of the domain enclosed by C if C is a simple Jordan curve contained in a two-dimensional plane in the N-dimensional space. (See also Lemma 1 in the next section.)

In actual computation of A(C) for arbtrarily given C, we must take to the numerical integration. In the lower dimensional cases, such as N = 2 or 3, this integration causes no trouble.

3. Some properties of the distance

3. 1 Axioms of distance

Our distance satisfies the axioms of distance.

Theorem 1. Let C_1, C_2 and C_3 be Jordan curves in N-dimensional Euclidean space. Then

(a) The following relations holds:

(a.1) $d(C_1, C_1) = 0$;

(a.2) if $d(C_1, C_2) = 0$, then $d(C_2, C_1) = 0$;

(a.3) if $d(C_1, C_2) = 0$ and $d(C_2, C_3) = 0$, then $d(C_1, C_3) = 0$.

That is, the relation $d(C_i, C_j) = 0$ defines an equivalence class.

(b) $d(C1, C2) = d(C2, C1)$;

(c) $d(C1, C3) \leq d(C1, C2) + d(C2, C3)$.

(Proof) The propositions (a) and (b) are evident, so we show (c). Let C_{12}, C_{23}, C_{13} be the closed Jordan curves which are the unions of C_1 and C_2, C_2 and C_3, and C_1 and C_3, respectively. It is well known that, for any x in the plane k (for k, see the former section),

$$w(C_{13}, x) = w(C_{12}, x) + w(C_{23}, x).$$

We take the absolute value of this equality. From the triangular inequality, we have

$$|w(C_{13}, x)| \leq |w(C_{12}, x)| + |w(C_{23}, x)|.$$

By taking the integral of this equality, we have

$$d(C_1 , C_3) \leq d(C_1, C_2) + d(C_2, C_3).$$

3. 2 Lower bound property

We need the following Lemmas:

Lemma 1.(See e.g. Santalo, 1976. Chaps. 12 and 13.) Let F be an arbitrary two-dimensional curved surface in an N-dimensional Euclidean space. The area S(F) of F is described by

$$S(F) = \int_{\text{whole maifold}} \Gamma(N) H(C;\ h) (2\pi)^{N-2} dE_{N-2,N},$$

where H(C; h) is given by

$$H(C;\ h) = \int_{x \text{ in } k} I(F,\ h) dx,$$

and I(F, h) is the number of intersections of h with F.

Suppose that there exists a two-dimensional curved surface F whose boundary is C. Then, F must be homeomorphic to a two dimenional unit disk D_2, so that there is homeomorphic, and hence continuous mapping from D_2 to F. Let Pr.F be the projection of F into the two-dimensional plane k. Then, we can define a continuous mapping f from D_2 to Pr.F. Let deg(f, D_2, x) be the degree of f at x relative to D_2. Then we have:

Lemma 2.(See e.g. Lloyd, 1978, Chaps.1 and 2.) Let I(F, h) be the number of intersection between a two-dimensional curved surface F and a (N - 2)-dimensional hyperplane h.

$$|\deg(f,\ D_2,\ x)| \leq I(F,\ h).$$

Noticing that the winding number is equal to the degree, the following theorem is a direct consequence of the definition of D(C) and Lemmas 1 and 2.

Theorem 2. Let F be a two-dimensional curved surface whose boundary is a closed Jordan curve C in an N-dimensional space. Then, the area S(F) of this curved surface satisfies

$$S(F) \geq D(C).$$

4. Discussions

This paper has proposed a L_1-like distance between two Jordan curves in N-dimensional Euclidean space whose points coincide. Possible extension to a general L_p-like distance is still in question.

This work has its origin in the author's trial to compare the "fitness of approximation" of the approximation algorithms on graphic displays(Kishimoto, to appear), where this L_1-like measure for two-dimensional case is used without any further investigation. The present work gives a theoretical foundation for it.

Pohl(1968) investigated a certain integral formula related to the quantity defined in a similar manner as ours. Seeking for a similar formula for our case might be a possible future works.

Finally we must notice some anomalous features of this distance, which is also observed in the conventional L_1 measure. Our distance is defined based on the "area" of the domain enclosed by a closed curve. Because the "area" of long figure may be very small if it is thin. Thus, the distance between the two curves in Fig. 1 is very small even though, intuitively, it should not be.

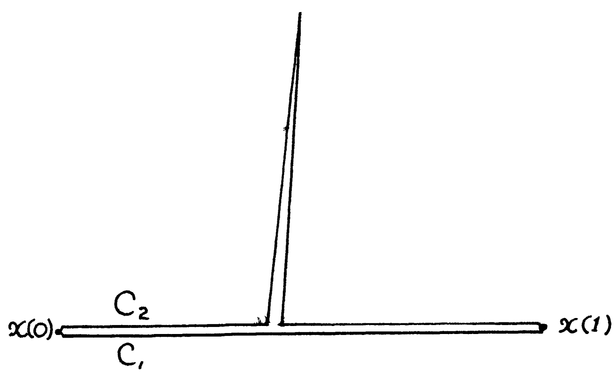

Figure 1. An anomalous example of $d(C_1, C_2)$

Acknowledgment

The auther wishes to express his gratitude to Prof. Onaga for his warm encourgement. He is also indebted to Prof. Miles for informing him of the relevant works by Pohl.

This work is partially supported by Grant-in-aid for the Encourgement of the Young Scientists by the Ministry of Education, Science and Culture of Japan, to which the author also wishes to express his gratitude.

REFERENCE

Iyanaga, S. and Kawada, Y.(Eds.)(1977): Encyclopedic Dictionary of Mathematics, Cambridge, Massachusetts: MIT Press.

Kishimoto, K., Onaga, K. and Yamamoto, K.(to appear): Theoretical Error Assessments of Curved Line Digitization Schemes on Graphic Displays, To appear in Computer Vision, Graphics and Image Processing

Lloyd, N.G.(1978): Degree Theory, London: Cambridge University Press.

Pohl, W.F.(1968): Some integralformulas for space curves and their generalization, Amer. J. Math., 90(1968), 1321-1345.

Santalo, L.A.(1976): Integral Geometry and Geometric Probability, Reading, Massachusetts: Addison-Wesley.

Steinhaus, H.(1929): Sur la portee pratique et theorique de quelques theoremes sur la mesure des ensembles de droites, C. R. 1er Congr. Mathematiciens des Pays Slaves, Warszawa, 348-354./