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Compatibility Conditions for Granular Assembly

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We deal with compatibility conditions of a granular assembly from three different viewpoints. First we consider a granular assembly by replacing into a graph. It is found that the compatibility condition for the relative displacement is expressed in a simple form by use of the fundamental matrices of the replaced graph and the dislocation vector appears in a similar equation for the modified relative displacement. It is explained that this dislocation vector does work against the void force. Secondly we consider a special compatibility condition for a granular assembly which states that no particle can overlap each other after deformation. It is seen that this condition results the dilatancy of the assembly. Lastly we consider the condition for some particles in contact. New conditions are shown as extensions of Soddy's rule.

INTRODUCTION

We consider compatibility conditions of a granular assembly from three different viewpoints. The compatibility condition for granular assemblies is regarded to be somewhat different and complicated compared with that in continua. First we consider the compaibility condition of relative displacement by using the graph theory (Satake.1985). Introducing new quantities, the void force and the dislocation vector, which are defined for each void of the granular assembly,it is found that the dislocation vector plays an important role in the analysis of the compatibility condition of the modified relative displacement. It is also shown that the dislocation vector does work against the void force and the void force is considered to be a quantity which corresponds to the stress function in continua. Secondly we consider a special compatibility condition of granular assembly stating that no particle can overlap during deformation. From this condition we obtain an inequality that represents the phenomenon of dilatancy, which is one of the significant characteristics of granular materials. Lastly we deals with Soddy's rule and its extensions. It is considered that such an analysis may be necessary for the consideration of the global compatibility of a packing of granular materials.

For simplicity, we mainly consider the 2-dimensional case, and the grains are assumed to be circular and completely rigid.

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DEFINITION OF VOID FORCE

We consider an assembly of grains by replacing to an oriented graph as is shown in Fig.1. We can easily find a correspondence between the replaced graph and the assembly, and it is seen that point, branch and loop of the graph correspond to particle, contact point and void of the assembly respectively. For the replaced graph we introduce the *incidence matrix* D_{ij} and the *loop matrix* L_{kj} defined as

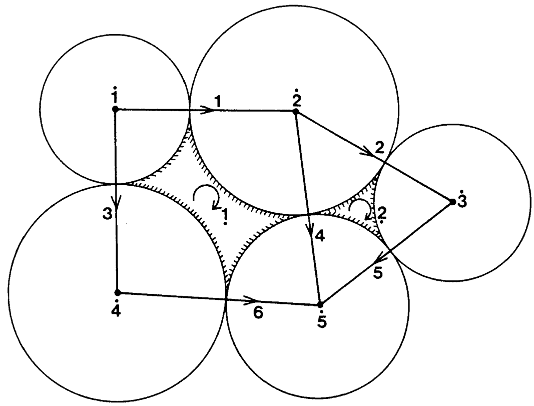

Fig.1 Replaced graph

$$D_{ij} = \begin{cases} 1 & \text{if branch } j \text{ is incident at point } i \text{ and is oriented away from point } i, \\ -1 & \text{if branch } j \text{ is incident at point } i \text{ and is oriented toward point } i, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

$$L_{kj} = \begin{cases} 1 & \text{if branch } j \text{ is in loop } k \text{ and the orientation of the branch and that of the loop coincide,} \\ -1 & \text{if branch } j \text{ is in loop } k \text{ and the orientation of the branch and that of the loop do not coincide,} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

where i, j and k are indices corresponding to points, branches and loops respectively. The orientation of loop is fixed as clockwise in this paper. For D_{ij} and L_{kj}, we have well-known identities

$$D_{ij}\,L_{jk} = 0, \qquad L_{kj}\,D_{ji} = 0 \tag{3}$$

where the symbol of summation for j is omitted, and the similar convention for summation will be applied in this paper.

The equilibrium equation of contact force $\underset{\sim}{S}_j$ for a grain i is written, by virtue of D_{ij}, as

$$D_{ij}\,\underset{\sim}{S}_j + \underset{\sim}{F}_i = 0 \tag{4}$$

where $\underset{\sim}{F}_i$ denotes the body force of the grain i. It is noted here that the contact force (or couple) is defined by the one applied to the grain in positive contact, where positive or negative contact means that the corresponding element of the incidence matrix is 1 or -1 respectively. If $\underset{\sim}{F}_i$ is absent, we have

$$D_{ij}\,\underset{\sim}{S}_j = 0 \tag{5}$$

Introducing a force $\underset{\sim}{V}_k$, which is defined for each void of the assembly, we can put $\underset{\sim}{S}_j$ as

$$\underset{\sim}{S}_j = - L_{jk}\underset{\sim}{V}_k \tag{6}$$

Substituting Eq.6 and using the identity Eq.3, we have Eq.5 automatically. $\underset{\sim}{V}_k$ is called the *void force*. It is easy to see that the relation between contact force $\underset{\sim}{S}_j$ and void force $\underset{\sim}{V}_k$ in Eq.6 is quite similar to that between stress and stress function in continuum mechanics.

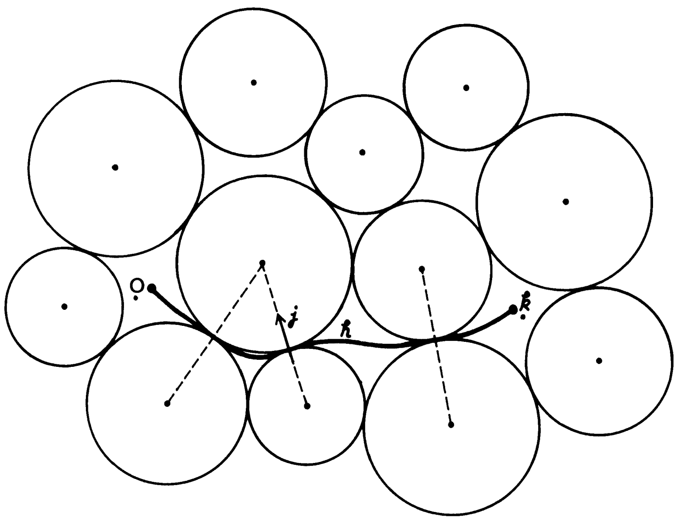

Fig.2 Cut *h*

The definition of void force $\underset{\sim}{V}_k$ is given by the form

$$\underset{\sim}{V}_k = \underset{\sim}{V}_0 + C_{hj}\, \underset{\sim}{S}_j \tag{7}$$

where $\underset{\sim}{V}_0$ denotes the void force of a reference void 0, and C_{hj} is the *cut matrix* of a cut *h* which starts from void 0 and ends at void *k*, as is shown in Fig.2. The cut matrix C_{hj} is defined as

$$C_{hj} = \begin{cases} 1 & \text{the direction of branch } j \text{ coincides with that obtained by turning the direction of cut } h \text{ for } \pi/2 \text{ clockwise,} \\ -1 & \text{the direction of branch } j \text{ coincides with that obtained by turning the direction of cut } h \text{ for } \pi/2 \text{ counter-clockwise,} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

FIRST COMPATIBILITY CONDITION AND DISLOCATION VECTOR

Letting $\underset{\sim}{u}_i$ and $\underset{\sim}{w}_i$ be the displacement and the rotation of a grain *i* respectively, we can write the *relative displacement* $\Delta\underset{\sim}{u}_j$ and the *modified relative displacement* $\overset{*}{\Delta}\underset{\sim}{u}_j$ respectively as

$$\Delta\underset{\sim}{u}_j = - D_{ji}\, \underset{\sim}{u}_i \tag{9}$$

$$\overset{\star}{\Delta}\underset{\sim}{u}_j = \Delta \underset{\sim}{u}_j + w_j \hat{\underset{\sim}{\ell}}_j \tag{10}$$

where j indicates the corresponding branch of the contact point (Satake.1978). $\underset{\sim}{\ell}_j$ denotes the branch vector with given orientation and ^ denotes the conjugate vector defined by

$$\hat{\underset{\sim}{v}} = (v_2, -v_1) \tag{11}$$

where v_1 and v_2 are components of a vector $\underset{\sim}{v}$. w_j denotes the *branch rotation* defined by

$$w_j = \frac{1}{\ell_j} D_{ji} r_i w_i \tag{12}$$

where ℓ_j is the length of branch j, and r_i and w_i are the radius and the rotation of grain i respectively. The modified relative displacement is considered to denote the relative displacement of a contact point of two particles after deformation, and it is noted that the relative displacement is also to be defined with respect to the grain in positive contact.

For a loop k, we have

$$L_{kj} \Delta \underset{\sim}{u}_j = - L_{kj} D_{ji} \underset{\sim}{u}_i = 0 \tag{13}$$

$$L_{kj} \overset{\star}{\Delta}\underset{\sim}{u}_j = L_{kj} \Delta \underset{\sim}{u}_j + L_{kj} w_j \hat{\underset{\sim}{\ell}}_j = -\underset{\sim}{d}_k \tag{14}$$

where

$$\underset{\sim}{d}_k = -L_{kj} w_j \hat{\underset{\sim}{\ell}}_j \tag{15}$$

Eq.13 is considered as the compatibility condition for the relative displacement $\Delta \underset{\sim}{u}_j$. In the case of the modified relative displacement $\overset{\star}{\Delta}\underset{\sim}{u}_j$, we have a residual term $-\underset{\sim}{d}_k$, as is shown in Eq.14. $\underset{\sim}{d}_k$ is a vector needed to close the polygon composed of modified relative displacements of contact points around a void k (see Fig.3), and is called the *dislocation vector* of void k. If the rotation of every particle is same value w, we have

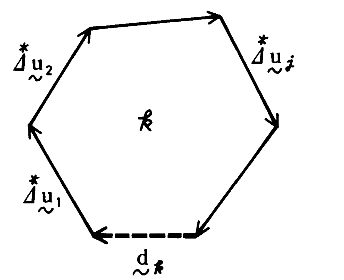

Fig.3 Dislocation vector of loop k

$$\underset{\sim}{d}_k = - w L_{kj} \hat{\underset{\sim}{\ell}}_j = 0 \tag{16}$$

To find the role of the void force, we consider the internal work of a granular assembly. Neglecting contact couples, the internal work W is written as

$$\begin{aligned} W &= \sum_R \underset{\sim}{S}_j \cdot \overset{\star}{\Delta}\underset{\sim}{u}_j \\ &= \sum_R (- L_{jk} \underset{\sim}{V}_k) \cdot \overset{\star}{\Delta}\underset{\sim}{u}_j \\ &= - \sum_R \underset{\sim}{V}_k \cdot (L_{kj} \overset{\star}{\Delta}\underset{\sim}{u}_j) + \sum_{\partial R} \underset{\sim}{v}_j \cdot \overset{\star}{\Delta}\underset{\sim}{u}_j \end{aligned}$$

$$= \sum_{R} \underset{\sim}{V}_k \cdot \underset{\sim}{d}_k + \sum_{\partial R} \underset{\sim}{V}_j \cdot \overset{\star}{\Delta} \underset{\sim}{u}_j \qquad (17)$$

where $\underset{\sim}{V}_j$ denotes the *boundary force* which is defined in the similar manner to the void force for every divided region surrounding the assembly (See Fig.4). It is recognized from Eq.17 that the void force does work against the corresponding dislocation vector. Further it is easy to see that Eq.17 is similar to the equation

$$W = \int_R \underset{\sim}{\sigma} \cdot\cdot \underset{\sim}{\varepsilon}\, dV = \int_R \underset{\sim}{\chi} \cdot\cdot \underset{\sim}{\eta}\, dV + \oint_{\partial R} \underset{\sim}{T} \cdot \underset{\sim}{u}\, dS \qquad (18)$$

in the generalized continuum mechanics, where $\underset{\sim}{\sigma}$, $\underset{\sim}{\varepsilon}$, $\underset{\sim}{\chi}$, $\underset{\sim}{\eta}$, $\underset{\sim}{T}$, and $\underset{\sim}{u}$ denote stress, strain, stress function, incompatibility, external force and displacement respectively, and dV and dS are volume and areal elements respectively (Satake.1983). This analogy again indicates that the void force is an analogous quantity to the stress function in continua.

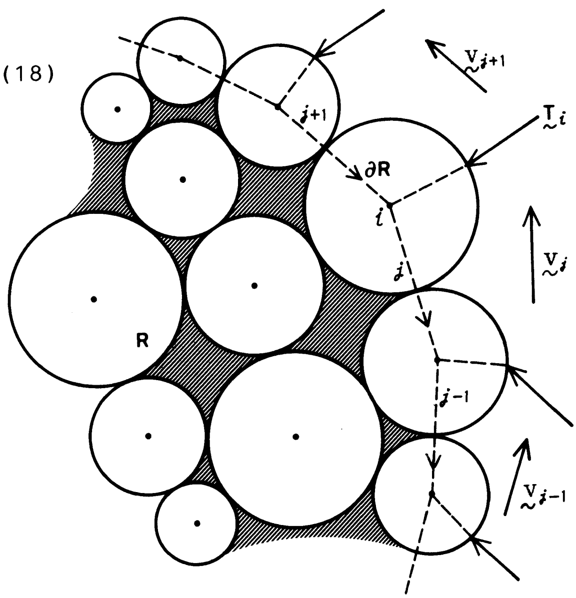

Fig.4 Boundary force

SECOND COMPATIBILITY CONDITION

It is easily recognized that an additional compatibility condition, which states that no two particles can overlap after deformation, becomes necessary in granular assemblies. Regarding Fig.5, this second compatibility condition will be expressed as

$$\underset{\sim}{n}_j \cdot \overset{\star}{\Delta} \underset{\sim}{u}_j \geq 0 \qquad (19)$$

where $\underset{\sim}{n}_j$ is the branch direction, which is the

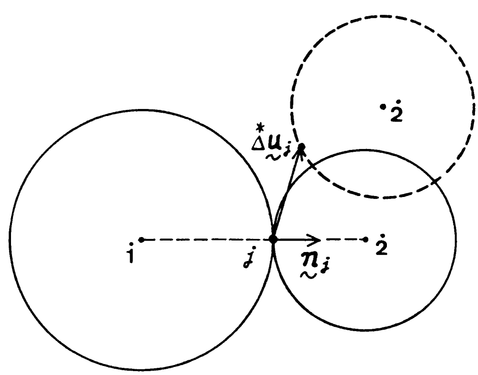

Fig.5 Second compatibility condition

same to the outward normal at the contact point with respect to the grain in positive contact. Introducing the displacement gradient tensor $\underset{\sim}{\gamma}$, which is defined as

$$\underset{\sim}{\gamma} = \frac{2}{\sum_R \ell_j} \sum_R \underset{\sim}{n}_j \overset{*}{\Delta} \underset{\sim}{u}_j \tag{20}$$

we have

$$\mathrm{tr}\, \underset{\sim}{\gamma} = \frac{2}{\sum_R \ell_j} \sum_R \underset{\sim}{n}_j \cdot \overset{*}{\Delta} \underset{\sim}{u}_j \geq 0 \tag{21}$$

where R is a *mesodomain*, i.e. a circular region which consists of a sufficient number of particles to apply statistical analysis and is yet small enough to be able to consider an average value for local measures, such as stress and strain. It is seen that Eq. 21 shows the phenomenon of dilatancy which is one of the significant characteristics of granular materials.

EXTENSION OF SODDY'S RULE

If we consider 4 mutually-tangent spheres with radii a, b, c, d in a 3-dimensional space, the volume of the tetrahedron defined by the centers of these spheres is given by

$$V = \frac{1}{3\,\alpha\beta\gamma\delta} \sqrt{(\alpha+\beta+\gamma+\delta)^2 - 2(\alpha^2+\beta^2+\gamma^2+\delta^2)} \tag{22}$$

where $\alpha, \beta, \gamma, \delta$, are reciprocals of a, b, c, d respectively. If

$$V = 0 \tag{23}$$

the centers of 4 mutually-tangent spheres lie on a plane, and it is seen that Eq.23 is reduced to Soddy's rule, a relation of radii of 4 mutually-tangent circles (Soddy, 1936), expressed as,

$$(\alpha+\beta+\gamma+\delta)^2 = 2(\alpha^2+\beta^2+\gamma^2+\delta^2) \tag{24}$$

or

$$\delta = \alpha + \beta + \gamma + 2\sqrt{\alpha\beta + \beta\gamma + \gamma\alpha} \tag{25}$$

In general, the volume of an n-dimensional simplex defined by the centers of (n+1) mutually-tangent super-spheres is given by

$$V = K \frac{(\alpha, \beta, \gamma, \ldots)}{\alpha \beta \gamma \ldots} \tag{26}$$

where $\alpha, \beta, \gamma, \ldots$ are reciprocals of the radii of (n+1) super-spheres, K is a constant coefficient and

$$(\alpha, \beta, \gamma, \ldots)^2 = (\alpha + \beta + \gamma + \ldots)^2 - (n-1)(\alpha^2 + \beta^2 + \gamma^2 + \ldots) \tag{27}$$

Thus the condition V=0, i.e.

$$(\alpha, \beta, \gamma, \ldots) = 0 \qquad (28)$$

is considered to be a compatibility condition for radii of (n+1) mutually-tangent superspheres in an (n-1)-dimensional space.

To extend Soddy's rule in 2-dimensions, we consider (n+1) circles in contact as is shown in Fig.6. It is easy to show that the compatibility condition for radii r_i of these circles is written as

$$\sum_{r=1}^{n} \sin^{-1}\sqrt{\frac{r_{i+1}}{(r_0+r_i)(r_0+r_{i+1})}} = \pi, \qquad (r_{n+1} \equiv r_1) \quad (29)$$

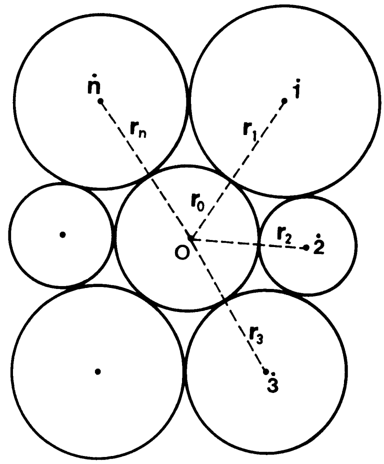

Fig.6 Circles surrounding a circle in contact

For n=3, we obtain Soddy's rule, and for n=4, we obtain

$$\begin{aligned} &16\rho_0^4 - 8\rho_0^2(\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_4 + \rho_4\rho_1 + 2\rho_1\rho_3 + 2\rho_2\rho_4) \\ &\quad - 16\rho_0(\rho_1\rho_2\rho_3 + \rho_1\rho_2\rho_4 + \rho_1\rho_3\rho_4 + \rho_2\rho_3\rho_4) \\ &\quad - 16\rho_1\rho_2\rho_3\rho_4 + (\rho_1 - \rho_3)^2(\rho_2 - \rho_4)^2 = 0 \end{aligned} \qquad (30)$$

where ρ_i denotes the reciprocal of r_i.

If we consider another extension of Soddy's rule in the *fractal* manner, as is shown in Fig.7, it is easy to show

$$\begin{aligned} \alpha' &= \alpha + 4\beta + 4\gamma + 4\sqrt{\alpha\beta + \beta\gamma + \gamma\alpha} \\ &= 2(\beta + \gamma + \delta) - \alpha \end{aligned} \qquad (31)$$

and similar equations for β' and γ', where α', β', γ' and $\alpha, \beta, \gamma, \delta$ denotes the reciprocals of radii of the circles A', B', C' and A, B, C, D in Fig.7, respectively. Thus we have

$$\alpha' + \beta' + \gamma' = 3(\alpha + \beta + \gamma) + 6\delta \qquad (32)$$

Quite similarly, we obtain

$$\begin{aligned} \alpha'' &= \alpha + 4\beta + 4\gamma - 4\sqrt{\alpha\beta + \beta\gamma + \gamma\alpha} \\ &= 2(\beta + \gamma + \delta') - \alpha \end{aligned} \qquad (33)$$

and similar equations for β'' and γ'', where $\alpha'', \beta'', \gamma''$ and δ' denotes the reciprocals of radii of the circles A", B", C" and D' in Fig.7 respectively.

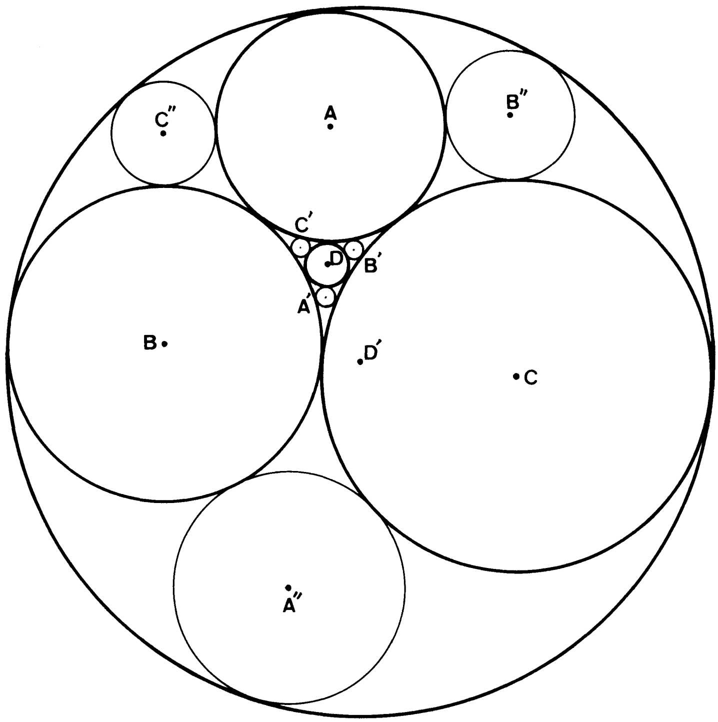

Fig. 7 Fractal Soddy's contact

Thus we have

$$\alpha'' + \beta'' + \gamma'' = 3(\alpha + \beta + \gamma) + 6\delta' \tag{34}$$

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4-3

Q: Is the friction force included in the vector? (H. Tsubota)

A: Yes. The contact force is the resultant force of compressive force and friction force.

Q: What is the physical meaning behind the void force? (J. Beddow)

A: The void force is an imaginary force, but is considered to be corresponded to the stress function in continuum. Because the contact force, which satisfies the equilibrium equation: $D_{ij}\, \underset{\sim}{S}_j = 0$

is expressed as $\underset{\sim}{S}_j = -L_{jk}\, \underset{\sim}{V}_k$

by using void force $\underset{\sim}{V}_k$, where D_{ij} and L_{jk} are the incidence and loop matricies respectively. This relation is very similar to that of stress and stress function, where the stress which satisfies the equilibrium equation : $\underset{\sim}{\nabla} + \underset{\sim}{\sigma} = 0$
is expressed as $\underset{\sim}{\sigma} = \underset{\sim}{\nabla}\, \underset{\sim}{\nabla} \times \times \underset{\sim}{\chi}$
by using the stress function tensor $\underset{\approx}{\chi}$ when $\underset{\sim}{\nabla}$ is the vectorial differential operator.