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Structural Features of Three-dimensional Images

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From the view point of differential geometry, structure lines or surfaces of 3-D images and skeleton lines of surfaces are introduced in order to extract essential information. These are defined by the differential equations which are composed of geometrically invariant operators. Structural classification of surfaces can be made by means of division skeleton lines which bear convexity. This concept is based on an criterion intermediate between homeomorphism and motion transformation. Other properties of skeleton lines including global connectivity are also discussed. It is concluded that a hierarchical representation of 3-D images with these features is useful to not only image or surface recognition but also 3-D graphics techniques such as curve or surface generation.

INTRODUCTION

Although visual function of a human is very powerful for 2-D (two-dimensional) images, it is poor and hardly useful for 3-D (three-dimensional) images. Therefore, we need to help a computer to analyze a 3-D image pattern. There is a growing literature (Rosenfeld:1985) on 3-D images which are obtained by data collection techniques such as computed tomography and scanning electro microscopy etc.

We have investigated feature extraction method of images or surfaces by means of differential geometry, taking a 3-D image as a smooth scalar function in a 3-D space (Enomoto et al.:1979, Enomoto et al.:1982, Watanabe & Enomoto:1983).

OPERATIONS IN 3-D IMAGES

A 3-D image is generally considered to be a mapping ϕ: R^3->R ,where R is an Euclidian space. We deal with two kinds of 3-D images. One is a density $\phi(x_1,x_2,x_3)$ in a 3-D space such as image data collection of computed tomography, and the other is a surface $\phi(x_1,x_2,x_3)$=const. or $(x_1,x_2,x_3)=\mathbf{P}(u_1,u_2)$ such as a surface model used in the field of CAD.

An image processing in the ordinary sense of processing an image to obtain another is taken as an operator $O:I_1$->I_2, where

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I_1 and I_2 are subsets of a function space I of images. An important class of such operators is geometrically invariant, as described below.

[Definition 1] A geomerical operator is O_g which satisfies

$$I_2(\mathbf{x})=O_g(I_1(\mathbf{x}))=I_1(\mathbf{x}') \tag{1}$$

,where $\mathbf{x}$ and $\mathbf{x}'$ are 3-D vectors satisfying $\mathbf{x}'=R\mathbf{x}+\mathbf{d}$ for some 3x3 rotation matrix R and 3-D displacement vector $\mathbf{d}$.

[Definition 2] For a geometrical operator O_g, a geometrically invariant operator O_i is defined by

$$O_i\ O_g=O_g\ O_i \quad . \tag{2}$$

There are many kinds of geomerically invariant operators. From the view point of differential geometry, local operations including differential operators such as gradient and Laplacian play an important role to extract structural features of an image.

STRUCTURE LINES AND STRUCTURE SURFACES

We will extend structure lines (Enomoto & Katayama:1976) of 2-D images to 3-D version. For a 3-D image $\phi(x_1,x_2,x_3)$ which is class C^2, some of the 3-D structure lines and surfaces are defined as follows.

[Definition 3] A characteristic line of ϕ is a set of points where gradϕ $=\mathbf{0}$ or grad ϕ coincides with one of the principal directions on the collection of surfaces ϕ=const.

[Definition 4] An edge surface of ϕ is a set of points where the normal curvature for the direction of gradϕ is equal to zero.

[Definition 5] A division surface of ϕ is a set of points where the total curvature on the set of surfaces ϕ=const. is equal to zero.

Some structure lines and surfaces are illustrated in Fig.1. A characteristic line connects every point where the density ϕ varies extremely or loosely on the surface ϕ=const. An edge surface is a set of points of inflection in other words. It consists of some parts and some of them encloses extremal points of ϕ. A division surface partitions ϕ into some parts including at most one extremal point.

The following theorems give the equations of structure lines and surfaces, and those in xyz-coordinate system are indicated in APPENDIX.

[Theorem 1] A characteristic line is a set of points which satisfies

$$H\ \mathrm{grad}\phi = k\ \mathrm{grad}\phi \tag{3}$$

,where H is a Hessian of ϕ and k is a scalar.

[Theorem 2] An edge surface is a set of points which satisfies

$$^t\mathrm{grad}\phi\ H\ \mathrm{grad}\phi = 0 \tag{4}$$

,where $^t\mathbf{v}$ means an inverse vector of $\mathbf{v}$.

[Theorem 3] A division surface is a set of points which satisfies

$$^t\mathrm{grad}\phi_p\ H\ \mathrm{grad}\phi_p = 0 \tag{5}$$

,where gradϕ_p means a direction of principal curvature perpendicular to gradϕ on the surface ϕ =const.

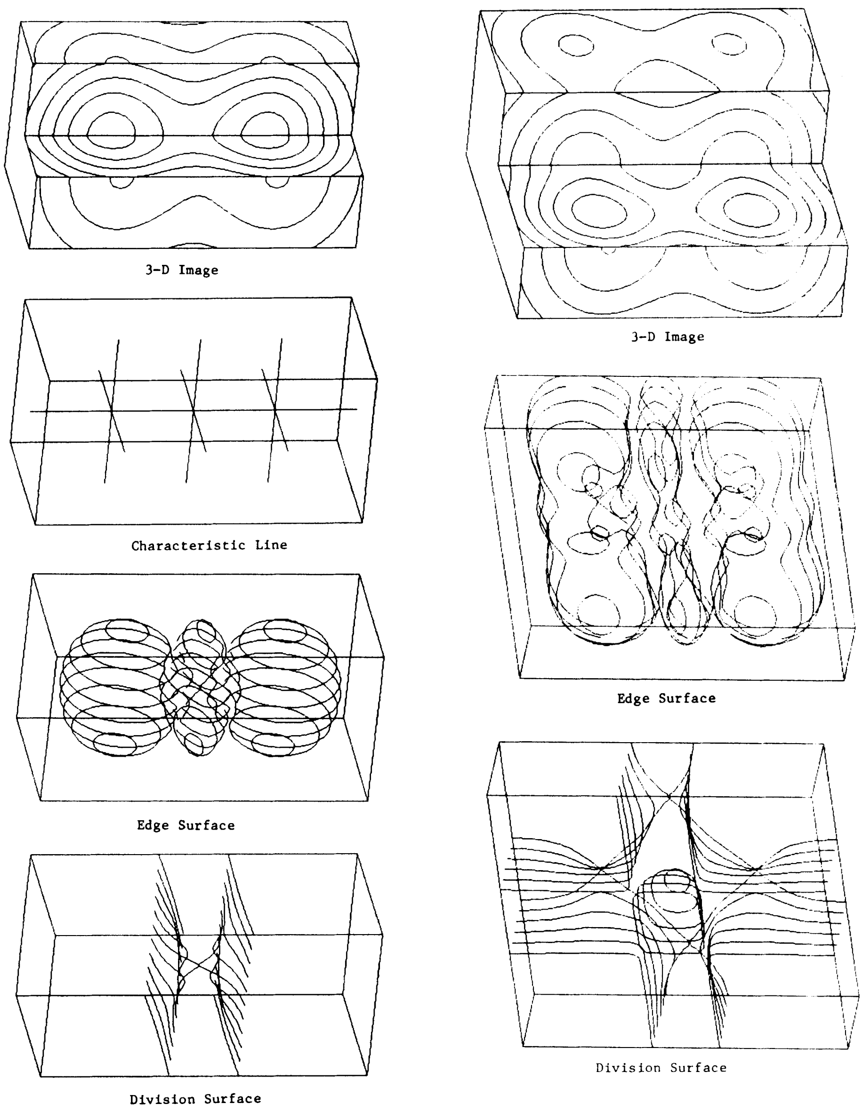
SKELETON LINES OF SURFACES AND THEIR PROPERTIES

A surface in a 3-D space is considered to be a mapping $S:(u_1,u_2)\text{->}(x_1,x_2,x_3)$, where u_1 and u_2 are two parameters of S. Now, differentiability of S will be assumed in order to define the following structural features of a surface.

[Definition 6] A K-characteristic skeleton line is a set of points where the total curvature is extremal along the line of curvature.

[Definition 7] A P-characteristic skeleton line is a set of points where the principal curvature is extremal along the line of curvature.

[Definition 8] A division skeleton line is a set of points where the total curvature is equal to zero along the line of curvature.



(a) Example 1. (b) Example 2.

Fig.1 Examples of structure lines and surfaces.

These lines are generically called skeleton lines, whose equations are shown in APPENDIX. Fig.2 illustrates some examples of skeleton lines.

It should be noticed that since there exists a pair of sets of lines of curvature for a surface, each skeleton line is defined in terms of both. Some interesting properties hold at the intersection of the skeleton lines defined for the different sets of lines of curvature. For example, the total curvature is extremal at the intersection of K-characteristic skeleton lines, and is equal to zero at the intersection of division skeleton line, which is called a flat point. Especially, neighbourhood at such an intersection has an important property as follows.

[Theorem 4] For the adjacent surface segments whose boundary is a division skeleton line, one is hyperbolic and the other is elliptic.

This theorem constrains the form of neighbourhood at the intersection of division skeleton lines as illustrated in Fig.3.

The following theorems are concerned with connectivity of skeleton lines.

[Theorem 5] For a closed surface, a K-characteristic skeleton line is connected with itself except an isolated singular point. And a similar fact holds for a P-characteristic skeleton line.

[Theorem 6] For a closed surface, a division skeleton line is connected with K- and P-characteristic skeleton lines.

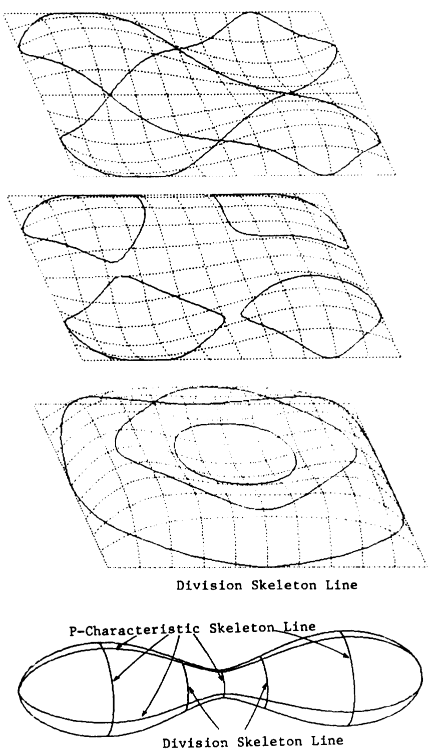
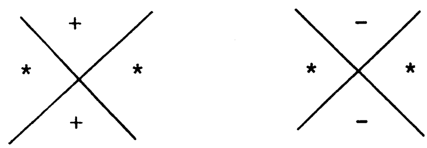
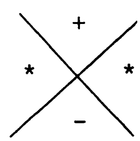


Fig.2 Skeleton lines.



(a)In case that the division skeleton lines of a different set intersect.



(b)In case that the division skeleton lines of the same set intersect.

Fig.3 Neighbourhood at the intersection of division skeleton lines. ´+´ and ´-´ indicate convex and concave elliptic surfaces, respectively. And ´*´ means a hyperbolic surface.

These theorems are proved by considering the variation of the total curvature K or the principal curvature P along a closed K- or P-characteristic skeleton line.

STRUCTURAL CLASSIFICATION OF SURFACES

It is known that closed surfaces are completely classified by their orientabilities and Euler characteristic numbers as shown in Tab.1 (Wallace 1968). It should be noted that a visible surface in the real world is homeomorphic to a sphere or a concatenation of n tori because they are orientable. The surfaces which have the same number of holes fall into the same category. This classification dose not reflect the structure of a surface and is too rough to recognize a surface pattern. On the other hand, the equivalence class in terms of motion transformation is defined by the classification with respect to the first and second fundamental quantities, which is too precise.

Hence, we propose a structural classification method, which is intermediate between both.

[Definition 9] The two surfaces S_1 and S_2 are isomorphic with respect to the operator O, if there exists a continuous mapping $f:S_1 \to S_2$, where $O(S_1)=O(S_2)$.

Especially, we will consider the operator h associated with the second fundamental quantity H as follows.

$$h(S)=\begin{cases} 1\ , & \det(H)>0 \\ 0\ , & \det(H)=0 \\ -1\ , & \det(H)<0 \end{cases} \qquad (6)$$

The set of points where det(H)=0 is a division skeleton line described above, which bears convexity. Therefore, it is found that the two surfaces which are isomorphic with respect to the operator h have the same structure in terms of convexity. It is easy to determine isomorphic with respect to h by using connecting structure of a division skeleton line and its connectivity with a characteristic skeleton line. Because the following theorem is proved, classification of surfaces whose criterion is intermediate between homeomorphism and motion is possible if we apply isomorphism with respect to h.

[Theorem 7] The class C^2 surfaces S_1 and S_2 are homeomorphic if they are isomorphic with respect to the operator h. And they are isomorphic with respect to h if they are transformed by motion each other.

It is found that the surfaces which are synthesized from a division skeleton line to hold convexity are isomorphic with respect to h. We can get a more precise classification by using another operator with more precise mode of a surface instead of h. For example, we may introduce an operator associated with a characteristic skeleton line for a class C^3 surface.

Table 1. Classification of closed surfaces.

Closed Surface	Euler Characteristic Number
Sphere	2
Concatenation of n Tori	2-2n
Concatenation of n Projective Planes	2-n

SYNTHESIS OF CURVES, SURFACES AND 3-D IMAGES

Various kinds of structural features of 3-D images or surfaces are have been discussed so far. It is found that characteristic surfaces in a 3-D image such as a division surface, characteristic curves of a surface such as a division skeleton line and characteristic points of a curve is hierachically defined. We will consider the problem of image or surface synthesis based on a hierarchical representation.

First of all, consider some characteristic points of a curve. It is known that its form is completely determined by its curvature and tortion. The curvature κ and tortion τ of a curve $\mathbf{P}(t)$ are represented by $\kappa = |\dot{P}\times\ddot{P}|/|\dot{P}|^3$ and $\tau = (\dot{P}\times\ddot{P})\cdot\dddot{P}/|\dot{P}\times\ddot{P}|^2$, where $\cdot$ means differential of $\mathbf{P}$ with respect to t. The first feature of κ or τ is a collection of points where their values are discontinuous. The second characteristic points are those where their values are equal to zero and the third ones take extremal values.

In order to synthesize a curve by using some characteristic points, it is necessary to give the positional vector $\mathbf{P}_0$, the unit tangent vector $\mathbf{t}_0$, the unit binormal vector $\mathbf{b}_0$ (or the unit principal normal vector $\mathbf{n}_0=\mathbf{b}_0\mathbf{x}\mathbf{t}_0$), curvature κ_0 and tortion τ_0. The problem is to synthesize a curve as in lower order or degree as possible when the boundary conditions are given.

Now we will make use of a Bernstein polynomial as an interpolation function, which is the basis function of a Bézier curve

$$P(t) = \sum_{i=0}^{m} {}_mC_i(1-t)^{m-i}\, t^i Q_i \tag{7}$$

,where m is the degree and $\mathbf{Q}_i$ (i=0,1,...,m) is the vertex of a characteristic polygon. Then, the boundary conditions on a characteristic point are

$$P_0 = Q_0 \tag{8}$$

$$t_0 = \frac{m}{\alpha_0}(Q_1-Q_0) \tag{9}$$

$$b_0 = \frac{m^2(m-1)}{\kappa_0\alpha_0^3}(Q_1-Q_0)\times(Q_2-2Q_1+Q_0) \tag{10}$$

$$\tau_0 = \frac{m(m-1)(m-2)}{\alpha_0} b_0\cdot(Q_3-3Q_2+3Q_1-Q_0) \tag{11}$$

Since similar conditions hold at the other end, the order m must not be less than 4. In case of m=4, $\mathbf{Q}_i$ is determined as follows.

$$Q_0 = P_0 \qquad Q_4 = P_4 \qquad Q_1 = P_0+\frac{\alpha_0}{4}t_0 \qquad Q_3 = P_4-\frac{\alpha_4}{4}t_4 \tag{12}$$

$$Q_2 = P_0+(\frac{\alpha_0}{4}+\beta_0)t_0+\frac{\kappa_0\alpha_0^2}{12}b_0\times t_0 \tag{13}$$

,where
$$\alpha_0 = \frac{6\{6(t_4\cdot b_0)((P_4-P_0)\cdot b_4)-\tau_4(P_4-P_0)\cdot b_0\}}{\tau_0\tau_4+36(t_0\cdot b_4)(t_4\cdot b_0)} \tag{14}$$

$$\alpha_4 = \frac{6\{6(t_0\cdot b_4)((P_4-P_0)\cdot b_0)-\tau_0(P_4-P_0)\cdot b_4\}}{\tau_0\tau_4+36(t_0\cdot b_4)(t_4\cdot b_0)} \tag{15}$$

$$\beta_0 = \frac{-1}{t_0\cdot b_4}\{\frac{\kappa_0\alpha_0^2}{12}(b_0\times t_0)\cdot b_4+\frac{\alpha_4\tau_4}{24}\} \tag{16}$$

Then, a surface is synthesized from characteristic curves such as division skeleton lines. It is necessary to give the tangent plane (or equivalently the first fundamental quantity) and the second fundamental quantity as boundary conditions on a curve $\mathbf{P}(t)$ for the purpose of synthesizing a surface because the form of a surface is completely defined by them. Suppose that $\mathbf{R}$ and $\mathbf{e}$ are a surface to be synthesized and its unit normal vector, respectively. Then, the tangent plane of $\mathbf{R}$ is determined by $\mathbf{e}$,or the unit tangent vector $\mathbf{t}$ and $\mathbf{a}=\mathbf{e}\times\mathbf{t}$. The boundary conditions with respect to the second fundamental quantity are constrained, specifying a characteristic curve such as a skeleton line. For example, if u and w are taken as the directions of $\mathbf{t}$ and $\mathbf{a}$, we might only give $\mathbf{R}_u$ in case of using a division skeleton line. The reason for it is that, taking $H=(h_{ij})$ as the second fundamental quantity of $\mathbf{R}$, $h_{22}=h_{12}^2/h_{11}$ is determined since $\det(H)=0$ on a division skeleton line, $h_{11}=\mathbf{R}_{uu}\cdot\mathbf{e}$ and $h_{12}=h_{21}=\mathbf{R}_{uw}\cdot\mathbf{e}$. Such conditions are transformed to those on a given cross section, and then a surface is synthesized on it in such a one-dimensional way that a set of curves are generated by means of the above method (Enomoto & Watanabe:1982).

Finally, a 3-D image is synthesized by the conditions at both ends on a set of lines transformed from the boundary conditions on a surface. The method interpolates a function $\phi(t)$ by using the Bernstein polynomial appeared earlier, where t is a normalized parameter on a line (Enomoto & Watanabe:1982).

$$\phi(t) = \sum_{i=0}^{m} {}_mC_i (1-t)^{m-i} t^i \psi_i \qquad (17)$$

When $\phi_0=\phi(0)$, $\phi_m=\phi(1)$, $\alpha_0=\phi'(0)$, $\alpha_m=\phi'(1)$, $\beta_0=\phi''(0)$, and $\beta_m=\phi''(1)$ are given, m must be equal to 5 for the unique solution. In this case,

$$\psi_0 = \phi_0 \qquad (18)$$

$$\psi_1 = \phi_0+\alpha_0/m \qquad (19)$$

$$\psi_2 = \phi_0+2\alpha_0/m+\beta_0/\{m(m-1)\} \qquad (20)$$

$$\psi_3 = \phi_m-2\alpha_m/m+\beta_m/\{m(m-1)\} \qquad (21)$$

$$\psi_4 = \phi_m-\alpha_m/m \qquad (22)$$

$$\psi_5 = \phi_m \qquad (23)$$

The surface and 3-D image synthesized in this way are shown in Figs.4 and 5, respectively.

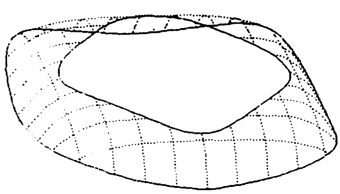


Fig.4 A synthesized surface.

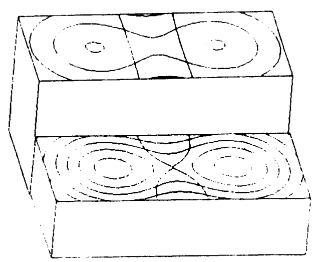


Fig.5 A synthesized 3-D image.

CONCLUDING REMARKS AND ACKNOWLEDGEMENT

The paper presented structural features and classification to extract essential structures of 3-D images and surfaces, and introduced a hierarchical representation of them. It is concluded that such a hierarchy makes a great contribution of an effective data compression and a higher description for pattern recognition and generation of a 3-D image or surface.

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APPENDIX

The equations of structure line and surfaces of a 3-D image $\phi(x,y,z)$ are as follows.

(1) Characteristic line:

$$\phi_y(\phi_{xx}\phi_x+\phi_{xy}\phi_y+\phi_{zx}\phi_z)=\phi_x(\phi_{xy}\phi_x+\phi_{yy}\phi_y+\phi_{yz}\phi_z),$$
$$\phi_z(\phi_{xy}\phi_x+\phi_{yy}\phi_y+\phi_{yz}\phi_z)=\phi_y(\phi_{zx}\phi_x+\phi_{yz}\phi_y+\phi_{zz}\phi_z) \quad \text{(A1)}$$

(2) Edge surface:

$$\phi_x^2\phi_{xx}+\phi_y^2\phi_{yy}+\phi_z^2\phi_{zz}+2\{\phi_x\phi_y\phi_{xy}+\phi_y\phi_z\phi_{yz}+\phi_z\phi_x\phi_{zx}\}=0 \quad \text{(A2)}$$

(3) Division surface:

$$\phi_x^2(\phi_{yy}\phi_{zz}-\phi_{yz}^2)+\phi_y^2(\phi_{zz}\phi_{xx}-\phi_{zx}^2)+\phi_z^2(\phi_{xx}\phi_{yy}-\phi_{xy}^2)$$
$$+2\{\phi_x\phi_y(\phi_{yz}\phi_{zx}-\phi_{xy}\phi_{zz})+\phi_y\phi_z(\phi_{zx}\phi_{xy}-\phi_{yz}\phi_{xx})$$
$$+\phi_z\phi_x(\phi_{xy}\phi_{yz}-\phi_{zx}\phi_{yy})\}=0 \quad \text{(A3)}$$

The equations of skeleton lines of a surface $S(u_1,u_2)$ are as follows, taking g_{ij} and H_{ij} $(i,j=1,2)$ as the first and second fundamental quantities respectively.

(1) P-characteristic skeleton line:

$$\frac{\partial P}{\partial u_1}du_1+\frac{\partial P}{\partial u_2}du_2 = 0 \quad \text{(A4)}$$

,where P is a solution of

$$(g_{11}g_{22}-g_{12}^2)P^2-(g_{11}H_{22}-2g_{12}H_{12}+g_{22}H_{11})P+(H_{11}H_{22}-H_{12}^2)=0$$

,and du_1 and du_2 are solutions of

$$(g_{11}H_{12}-g_{12}H_{11})(du_1)^2+(g_{11}H_{22}-g_{22}H_{11})du_1du_2$$
$$+(g_{12}H_{22}-g_{22}H_{12})(du_2)^2=0$$

(2) K-characteristic skeleton line:

$$\frac{\partial K}{\partial u_1}du_1+\frac{\partial K}{\partial u_2}du_2 = 0 \quad \text{(A5)}$$

,where $K=(H_{11}H_{22}-H_{12}^2)/(g_{11}g_{22}-g_{12}^2)$, and du_1 and du_2 are the same as (1).

(3) Division skeleton line:

$$H_{11}H_{22}-H_{12}^2=0 \quad \text{(A6)}$$

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