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A Correspondence between Line Drawings of Polyhedrons and Plane Skeletal Structures*

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This paper presents a new correspondence between the rigidity of rod-and-joint structures and the reconstructibility of polyhedrons from line drawings. Diagrams composed of straight line segments can be interpreted as two-dimensional skeletal structures (i.e., structures consisting of rigid rods and rotatable joints) on one hand, and as projections of polyhedrons (i.e., solid objects bounded by planar faces) on the other hand. It has long been known that there is a nontrivial correspondence between the two ways of interpretation, where the correspondence is 'metric' in the sense that it is established through diagrams drawn on a plane. In this paper, a similar but 'nonmetric' correspondence between the two systems is established. It is shown that a certain subclass of rigid structures and a certain subclass of pictures of polyhedrons are characterized by the same set of underlying graphs.

INTRODUCTION

Diagrams composed of straight line segments can, on one hand, be thought of as two-dimensional skeletal structures (i.e., structures composed of rods and rotatable joints) and, on the other hand, be thought of as projections of three-dimensional polyhedral objects (i.e., solid objects bounded by planar surfaces). An intimate relation between the two systems has been known. That is, if a diagram is a projection of some polyhedron, then the associated skeletal structure has a nontrivial stress that is in the state of equilibrium. This relation has long been used in mechanisms for graphical calculus of stresses in structures (Maxwell (1864) and Cremona (1890)), and also been used in scene analysis for the reconstruction of polyhedrons from diagrams (Mackworth (1973), Huffman (1977), and Whiteley (1979)).

The converse of the relation, on the contrary, does not hold; a diagram representing a skeletal structure with a nontrivial stress is not necessarily a projection of a polyhedron. However, Whiteley and Crapo (1977) proved that the converse holds if the diagram is planar in a graph-theoretical sense (see also Whiteley (1982)).

The above correspondence between the two systems is a 'metric' one; the correspondence is established through diagrams drawn on a plane, that is, diagrams whose vertex positions are given as pairs of real numbers. In the present paper, on the other hand, we shall show that a similar but 'nonmetric' correspondence can also be established between the two systems.

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INCIDENCE STRUCTURES AND RECONSTRUCTIBILITY

An *incidence structure* is a triple $I = (V,F,R)$, where $V = \{v_1,\ldots,v_n\}$ and $F = \{f_1,\ldots,f_m\}$ are mutually disjoint finite sets and $R \subsetneq V\times F$. The elements of V are called *vertices*, and those of F *faces*. A *picture* of I, denoted by $I(p)$, is the incidence structure I together with a $2n$-dimensional real vector $p = (x_1,y_1,\ldots,x_n,y_n) \in R^{2n}$. We refer to the points (x_i,y_i) $(i = 1,\ldots,n)$ as the vertices of $I(p)$.

A *realization* of $I(p)$ is an $(n+3m)$-dimensional real vector $w = (z_1,\ldots,z_n,a_1,b_1,c_1,\ldots,a_m,b_m,c_m) \in R^{n+3m}$ such that

$$a_jx_i + b_jy_i + z_i + c_j = 0 \tag{1}$$

for any (v_i,f_j) in R. The realization can be regarded as the collection of n points and m faces in the three-dimensional space, where the points are represented by Cartesian coordinates (x_i,y_i,z_i) and the planes by the equations

$$a_jx + b_jy + z + c_j = 0, \tag{2}$$

such that the orthographic projection of the points to the x-y plane coincides with the vertices of $I(p)$ and that, for any (v_i,f_j) in R, the ith point is on the jth plane. The realization w is said to be *nondegenerated* if $(a_i,b_i,c_i) \neq (a_j,b_j,c_j)$ for any $1 \leqq i < j \leqq m$. We say that I is *reconstructible* from $I(p)$ if $I(p)$ has a nondegenerated realization.

Suppose that a picture $I(p)$ is given. Then x_1, y_1, ..., x_n, y_n are known, and hence Eq. 1 is linear in z_i, a_j, b_j, and c_j. Collecting Eq. 1 for all elements in R, we get the system of linear equations

$$A(I,p)w^t = 0, \tag{3}$$

where $A(I,p)$ is a constant matrix of size $|R|\times(n+3m)$, and t denotes transposition. A vector w is a realization of $I(p)$ if and only if it is a solution to Eq. 3. The realizations of $I(p)$ form a linear subspace of R^{n+3m}, and its dimension is $n+3m-\text{rank}(A(I,p))$. This dimension is called the *degree of freedom* of $I(p)$.

The point $p = (x_1,y_1,\ldots,x_n,y_n)$ is *in generic position* if x_1, y_1, ..., x_n, y_n are algebraically independent transcendental numbers over the rational field. From the definition of algebraic independence, any polynomial of x_1, y_1, ..., x_n, y_n with rational coefficients is 0 if and only if it is identically equal to 0 when x_1, y_1, ..., x_n, y_n are considered as indeterminate symbols. Hence, if p is in generic position, the degree of freedom and the reconstructibility of $I(p)$ do not depend on p, but on I only. An incidence structure is *generically reconstructible* if $I(p)$ is reconstructible for p in generic position. The degree of freedom of $I(p)$ for p in generic position is called the *generic degree of freedom* of I.

We define a function μ_I on 2^F by, for any $X \subsetneq F$,

$$\mu_I(X) = |V(X)| + 3|X| - |R(X)| - 4,$$

where $V(X) = \{v \mid v \in V, (\{v\}\times X)\cap R \neq \emptyset\}$ and $R(X) = (V\times X)\cap R$.

The next proposition is one variation of the main theorem on generic reconstructibility by Sugihara (1979, 1984) and Whiteley (1984).

Proposition 1. An incidence structure $I = (V,F,R)$ is generically reconstructible and its generic degree of freedom is 4, if and only if
(1A) $\mu_I(F) = 0$, and
(1B) $\mu_I(X) \geqq 0$ for any subset X of F such that $|X| \geqq 2$.

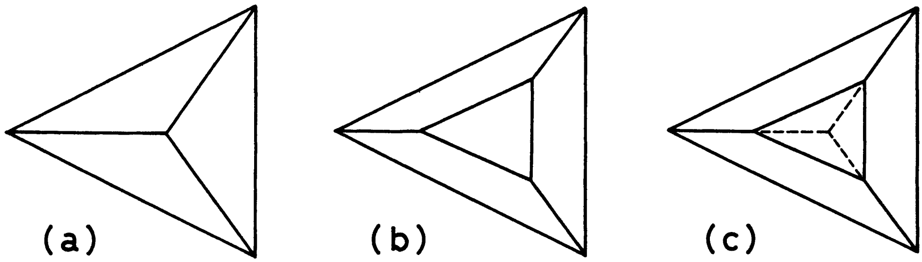

Fig. 1. Pictures of polyhedrons: the incidence structure obtained from a tetrahedron in (a) is generically reconstructible, whereas the incidence structure obtained from the truncated pyramid in (b) is not.

Example 1. A typical class of incidence structures is obtained from polyhedrons; an incidence structure $I = (V,F,R)$ is obtained from a polyhedron P when we consider V and F as the set of vertices and that of faces, respectively, of P, and R be the set of vertex-face pairs such that the former lies on the latter. Fig. 1 shows some pictures of polyhedrons. The polyhedron shown in (a) is a tetrahedron, which has 4 vertices and 4 faces (where in addition to the three visible faces we also count the rear invisible face). The incidence structure of this object satisfies the conditions (1A) and (1B). We can see that the picture (a) represents a tetrahedron, and this fact is not disturbed even if the vertices are displaced on the picture plane. Moreover, the tetrahedron is uniquely fixed in the space if we specify the three-dimensional positions of all the 4 vertices; it has 4 degrees of freedom. This is what Proposition 1 states.

The object represented by the picture in (b) is a truncated pyramid, which has 6 vertices and 5 faces (including the rear invisible triangular face). It had 18 incidence pairs, because there are 2 triangular faces and 3 quadrilateral faces. Thus, we get $\mu_I(F) = |V|+3|F|-|R|-4 = 6+3\times 5-18-4 = -1$; the condition (1A) is not fulfilled. This picture represents a truncated pyramid only when the three side edges have a common point of intersection (when they are extended) as indicated in (c), and hence the incidence structure is not generically reconstructible.

PLANE SKELETAL STRUCTURES

This section is a brief review of some results obtained by Laman (1970), Asimow and Roth (1979), and Lovasz and Yemini (1982), which are necessary for our discussion.

Let $G = (V,E)$ denote an undirected graph having node set $V = \{v_1, \dots, v_n\}$ and arc set E without loops or multiple arcs. A *plane skeletal structure*, denoted by $G(p)$, is the graph G together with a $2n$-dimensional vector $p = (x_1, y_1, \dots, x_n, y_n) \in R^{2n}$. We refer to the points (x_i, y_i) as the nodes of $G(p)$, and the line segments connecting (x_i, y_i) with (x_j, y_j) for $\{v_i, v_j\} \in E$ as the arcs of $G(p)$.

An *infinitesimal displacement* of $G(p)$ is a $2n$-dimensional real vector $\mathbf{d} = (\dot{x}_1, \dot{y}_1, \dots, \dot{x}_n, \dot{y}_n) \in R^{2n}$ such that

$$(x_i - x_j)(\dot{x}_i - \dot{x}_j) + (y_i - y_j)(\dot{y}_i - \dot{y}_j) = 0 \tag{4}$$

for any $\{v_i, v_j\} \in E$. The infinitesimal displacement can be regarded as the assignment of velocities $(\dot{x}_i, \dot{y}_i)$ to the vertices v_i such that no edge is stretched or compressed. Gathering the equations of the form (4) for all edges in E, we get a system of linear equations

$$B(G,p)\mathbf{d}^t = 0, \qquad (5)$$

where $B(G,p)$ is a constant matrix of size $|E| \times 2n$. A vector $\mathbf{d}$ is an infinitesimal displacement if and only if it satisfies Eq. 5. The infinitesimal displacements of $G(p)$ form a linear space and the rigid motions of $G(p)$ in R^2 yield a three-dimensional subspace of this linear space. $G(p)$ is called *infinitesimally rigid* (or *rigid* in short) if the infinitesimal displacements of $G(p)$ form a three-dimensional linear space, and *infinitesimally flexible* otherwise.

Whether $G(p)$ is rigid depends on both the underlying graph G and the point $p \in R^{2n}$. However, if p is in generic position, the rigidity of $G(p)$ depends on G only. The graph G is said to be *generically rigid* if $G(p)$ is rigid for p in generic position, and *generically flexible* otherwise.

We define a function μ_G on 2^V by, for any $X \subseteq V$,

$$\mu_G(X) = 2|X| - |E(X)| - 3,$$

where $E(X)$ is the set of arcs connecting nodes in X.

The next proposition is a direct consequence of Laman's theorem (1970).

Proposition 2. A graph $G = (V,E)$ is generically rigid, remains generically rigid if any one arc is deleted, but becomes generically flexible if any two arcs are deleted, if and only if
(2A) $\mu_G(V) = -1$, and
(2B) $\mu_G(X) \geqq 0$ for any proper subset X of V such that $|X| \geqq 2$.

Example 2. Fig. 2 gives three plane skeletal structures. All of them are obviously rigid. Their strength against the break of rods, however, is different. The structure in (a) becomes flexible if any one rod is broken, whereas the structures in (b) and (c) remain rigid if any one rod is broken. The structure in (c), moreover, remains rigid even if some two rods, e and e' for example, are broken. Thus, among the three structures, the structure in (b) only admits the property stated in Proposition 2. Indeed this structure satisfies (2A) and (2B).

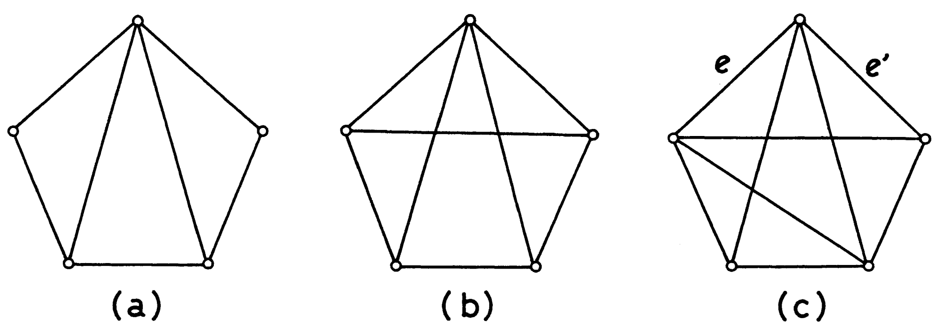

Fig. 2. Plane skeletal structures: the structure in (a), (b), and (c) are all rigid, but the structure in (a) becomes flexible if any one rod is broken, and the structure in (c) remains rigid if two rods (e and e' for instance) are broken; the structure in (b) only satisfies the conditions (2A) and (2B).

CORRESPONDENCE BETWEEN THE TWO SYSTEMS

Let P be a polyhedron, that is, a solid body bounded by a finite number of planar faces. Let V, E, and F be the set of vertices, that of edges, and that of faces, respectively, of P, and R be the set of vertex-face pairs (v,f) such that the vertex v is on the face f. Then, $G_P = (V,E)$ is a graph, which we call the graph induced by P, and $I_P = (V,F,R)$ is an incidence structure, which we call the incidence structure induced by P.

The next proposition is the main result of this paper.

Proposition 3. Let P be a polyhedron that is topologically equivalent to a sphere. Then the incidence structure I_P induced by P satisfies (1A) and (1B) if and only if the graph G_P induced by P satisfies (2A) and (2B).

Proof. First, (1A) is equivalent to (2A), because $\mu_I(F) = -\mu_G(V)-1$ follows directly from $|R| = 2|E|$ and Euler's formula $|V|+|F|-|E| = 2$.

Suppose that I_P satisfies (1A) and (1B). Let X be any proper subset of V such that $|X| \geqq 2$.

Case 1: Suppose that the graph $(X,E(X))$ is connected. Let F_1 $(\subsetneqq F)$ be the set of faces whose vertices are all in X, and let $F_2 = F-F_1$. Let V_0 be the set of vertices in X that belong to the boundaries of the faces in F_2, and let $V_1 = X-V_0$ and $V_2 = V-X$. Let, furthermore, E_0 be the set of edges in $E(X)$ that belong to the boundaries of the faces in F_2, and let $E_1 = E(X)-E_0$ and $E_2 = E-E(X)$. Then, $\{F_1,F_2\}$, $\{V_0,V_1,V_2\}$, and $\{E_0,E_1,E_2\}$ are partitions of F, V, and E, respectively.

Suppose that the polyhedral surface P is topologically deformed to a sphere, say K, and that the graph G_P is drawn on K as the vestiges of vertices, edges, and faces of P. Regions on K bounded by the edges correspond to the faces in F. If we delete the vertices in V_2 and the edges incident to them, the faces in F_2 are merged into connected regions, say A_i $(i = 1,\dots,k)$, on K. Note that $k \geqq 1$ because $X \neq V$.

Let F_2^i $(\subsetneqq F_2)$ be the set of the faces constituting A_i, and let us define $E_2^i = E(F_2^i)\cap E_2$, $E_0^i = E(F_2^i)\cap E_0$ $(i = 1,\dots,k)$, where, for any face set Y, $E(Y)$ denotes the set of edges belonging to the boundaries of the faces in Y. Similarly, let us define $V_2^i = V(F_2^i)\cap V_2$, $V_0^i = V(F_2^i)\cap V_0$ $(i = 1,\dots,k)$. Note that $|F_2^i| \geqq 2$ for any i $(1 \leqq i \leqq k)$, and $F_2^i\cap F_2^j = \phi$, $E_2^i\cap E_2^j = \phi$, $V_2^i\cap V_2^j = \phi$ for any i and j $(1 \leqq i < j \leqq k)$.

A connected region A_i $(1 \leqq i \leqq k)$ is bounded by edges in E_0^i; the edge in E_0^i form a closed path surrounding A_i. If one travels along the closed path around A_i, one passes each vertex in V_0^i at least once. Hence we get $|V_0^i| \leqq |E_0^i|$, where the equality holds when every vertex in V_0^i appears exactly once in the closed path.

Let R_i be the set of incidence pairs concerned with faces in F_2^i. Each face $f_j \in F_2^i$ had $|V(\{f_j\})|$ vertices, and $|R_i|$ equals the sum of such numbers of vertices over all faces in F_2^i: $|R_i| = \sum|V(\{f_j\})|$ where the summation is taken over all faces f_j in F_2^i. It can be proved that no vertex appears twice or more on the boundary of any face in F_2^i. Consequently, $|V(\{f_j\})|$ equals also the number of edges on the boundary of f_j. Thus, we get $|R_i| = 2|E_2^i|+|E_0^i|$, because in the summation elements of E_2^i are counted twice and those in E_0^i are counted once.

Then, we get

$$\mu_G(X) = 2(|V_0|+|V_1|)-(|E_0|+|E_1|)-3$$

$$= 3(|V_0\cup V_1|-|E_0\cup E_1|+|F_1|+k)-|V_0\cup V_1|+2|E_0\cup E_1|-3|F_1|-3k-3$$

$$= -(|V|+3|F|-|R|-4)+|V_2|+3|F_2|-3k-2|E_2|-1$$

$$= \sum_{i=1}^{k} (|V_2^i| + 3|F_2^i| - 2|E_2^i| - 3) - 1$$

$$\geqq \sum_{i=1}^{k} (|V_2^i \cup V_0^i| + 3|F_2^i| - 2|E_2^i| - |E_0^i| - 3) - 1$$

$$\geqq \sum_{i=1}^{k} (|V_2^i \cup V_0^i| + 3|F_2^i| - 2|E_2^i| - |E_0^i| - 4)$$

$$= \sum_{i=1}^{k} \mu_I (F_2^i) \geqq 0,$$

where the first equality is the definition of μ_G, the second one comes from simple counting, the third one follows from Euler's formula $|V_0 \cup V_1| - |E_0 \cup E_1| + |F_1| + k = 2$ for the subgraph $(V_0 \cup V_1, E_0 \cup E_1)$ and $|R| = 2|E| = 2(|E_0| + |E_1| + |E_2|)$, the fourth one comes from (1A), the next inequality follows from $|V_0^i| \leqq |E_0^i|$ for $1 \leqq i \leqq k$, and the last equality comes from $|R_i| = 2|E_2^i| + |E_0^i|$.

Case 2: Suppose that the graph $(X, E(X))$ is not connected. Then, (2B) can be derived easily from the fact that every connected component of $(X, E(X))$ satisfies $\mu_G(X_j) \geqq 0$, where X_j represents the set of vertices belonging to the jth connected component of $(X, E(X))$.

Therefore, (2B) is satisfied in both the cases.

Conversely, suppose that G_P satisfies (2A) and (2B). Let X be any subset of F such that $|X| \geqq 2$. Let us define $F_1 = X$, $F_2 = F - F_1$, $V_0 = V(F_1) \cap V(F_2)$, $V_1 = V(F_1) - V_0$, $V_2 = V(F_2) - V_0$, $E_0 = E(F_1) \cap E(F_2)$, $E_1 = E(F_1) - E_0$, $E_2 = E(F_2) - E_0$. Let $G_X = (V_0 \cup V_1, E_0 \cup E_1)$ denote the subgraph of G_P having the vertex set $V_0 \cup V_1$ and the edge set $E_0 \cup E_1$.

Case 1: Suppose that G_X is connected. Suppose that the graph G_P is drawn on the sphere K. If we delete the edges in E_2 from the graph, the faces in F_2 are merged into connected regions, say $A_1, \ldots, A_k$, on K. Let F_2^i ($\subseteq F_2$) be the set of the faces belonging to A_i, and let us define $V_0^i = V(F_2^i) \cap V_0$, $V_2^i = V(F_2^i) \cap V_2$, $E_0^i = E(F_2^i) \cap E_0$, $E_2^i = E(F_2^i) \cap E_2$ $(i = 1, \ldots, k)$. Note that $V_2^i \cap V_2^j = \phi$ and $E_2^i \cap E_2^j = \phi$ for $1 \leqq i < j \leqq k$. Moreover, note that for every edge in E_0, one side face belongs to F_1 and the other belongs to F_0, so that firstly we have $E_0^i \cap E_0^j = \phi$ for $1 \leqq i < j \leqq k$, and secondly we have $|V_0^i| = |E_0^i|$ for $1 \leqq i \leqq k$. If $k = 0$, then $X = F$ and hence $\mu_I(X) = 0$. If $k \geqq 1$, then $|V(F_2^i)| \geqq 3$ for any i $(1 \leqq i \leqq k)$ and hence we get

$$\mu_I(X) = |V_0| + |V_1| + 3|F_1| - (2|E_1| + |E_0|) - 4$$

$$= 3(|V_0 \cup V_1| + |F_1| + k - |E_0 \cup E_1|) + |E_1| + 2|E_0| - 2|V_0 \cup V_1| - 3k - 4$$

$$= |E| - 2|V| + 2 - |E_2| + |E_0| + 2|V_2| - 3k$$

$$= \sum_{i=1}^{k} (2|V_2^i \cup V_0^i| - |E_2^i \cup E_0^i| - 3)$$

$$= \sum_{i=1}^{k} \mu_G(V(F_2^i)) \geqq 0.$$

In the above equations, the third equality follows from Euler's formula for the graph $(V_0 \cup V_1, E_0 \cup E_1)$, and the fourth equality from (2A) and $|V_0^i| = |E_0^i|$.

Case 2: Suppose that the graph G_X is not connected. Then (1B) can be derived from the fact that each connected component of G_X satisfies $\mu_I(X_j) \geqq 0$, where X_j denotes the subset of X that are bounded by edges belonging to the jth connected component of G_X.

Therefore, (1B) is fulfilled in both the cases. Q.E.D.

Example 3. In Fig. 3, the line drawing in (a) represents a polyhedron (a cone with a quadrilateral base) and there are exactly 4 degrees of freedom in the choice of the polyhedron. Moreover, the property is preserved if one changes the positions of vertices slightly on the picture plane. Thus, from Proposition 1 the conditions (1A) and (1B) are fulfilled. The corresponding skeletal structure shown in (a') is rigid, remains rigid if any one rod is removed, but becomes flexible if two or more rods are removed. The property is preserved if the joint positions are perturbed. Thus from Proposition 2 the conditions (2A) and (2B) are fulfilled.

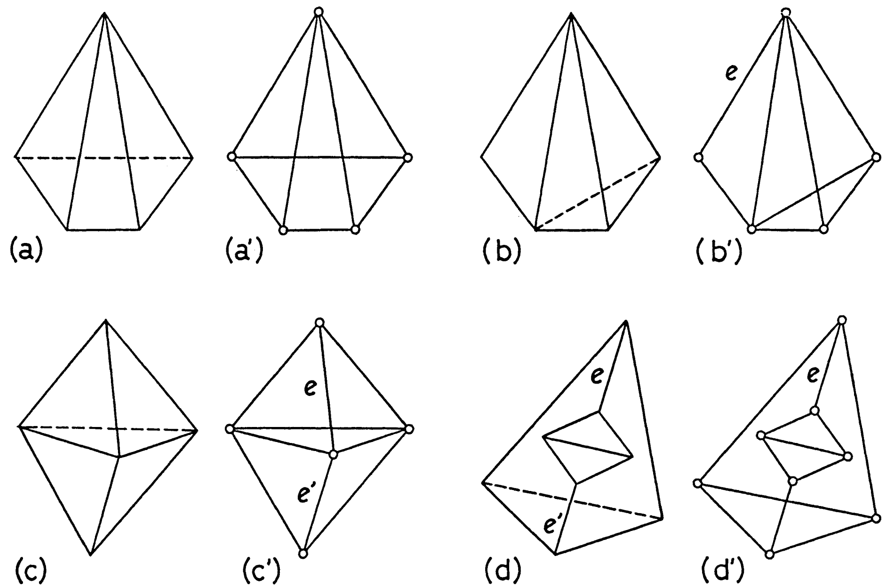

Fig. 3. Two ways of interpretations of diagrams: the pairs of (a) and (a') satisfies the conditions (1A), (1B), (2A), and (2B), but the other pairs do not.

In contrast, the line drawing in (b) does not represent any polyhedron, and the corresponding skeletal structure shown in (b') becomes flexible when only one rod, the rod e for instance, is deleted. Next, the line drawing in (c) represents a polyhedron, but there are 5 degrees of freedom in the choice of the object; indeed all the faces are triangular so that we have to specify the z coordinates of all the vertices in order to fix the object in the space. The corresponding skeletal structure shown in (c') remains rigid even if we delete two edges, the edges e and e' for example. Finally, the line drawing in (d) represents a polyhedron and there are exactly 4 degrees of freedom in the choice of the object, but the property is not preserved when some vertices are displaced on the picture plane; it represents a polyhedron correctly only when the two edges e and e' are collinear. The corresponding skeletal structure shown in (d') becomes flexible if only one rod, the rod e for instance, is removed.

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