Science on Form: Proceedings of the First International Symposium for Science on Form,
General Editor: S. Ishizaka, Editors: Y. Kato, R. Takaki, and J. Toriwaki, 379–385.

Tooth Pattern: Towards a Mathematical Principle to Determine the Patterns

Zen-Ichiro Horii

LA JOLLA INSTITUTE, 478-25 Omachi, Okayama city 700, Japan

Keywords: Point-interval, Self-organization

We described on the process of a regular pattern formation from random array of points in an uniform enviroment and in a field with gradient. The modes of regular patterns are different between the cases with and without influence on interaction between points. In so-called 'self-organization', it is certain that physical value presented by the point-interval is regulated on the process. Hence, the scheme is available to many examples of biological pattern formation by conversion of the physical value.

INTRODUCTION

Tooth pattern is determined by the shape of the dental cusp of the tooth. Human dental arch is much variable and in short, the individuality of the pattern is very high. The dental arch consists of the succession of dental cusps, so that irregularities in the placement of the dental crowns produce the high individuality. Since the dental arch is decided by the mode of the teeth arrangement, the problem of the dental arch formation is replaced by the question how the teeth are arranged. The question is further converted to the study on the eruptive mechanism of tooth because the mode of the teeth arrangement is determined by the location of the teeth and the directions of the tooth axes.
An approach towards finding of a mathematical principle how to determine the position and the direction on the eruptive process was attempted (Horii and Yamauchi). If only a couple of factors determine the two physical values, the

eruptive process will be determinative. On the contrary, if a lot of agents in enviroments participate in the determination, the eruption will belong to a stochastic process. In the latter case, the individuality of the pattern can be very high. Further, the occlusal relation between the both jaws must be decided on a stochastic process. However, regardless of any occlusal relation, the intercuspation between both teeth in maxille and in mandible will be effective. Here, we shall find a keyrole to resolve the problem on the mechanism to determine tooth patterns in a mathematical viewpoint.

THEORY

Various cellular patterns in living and non-living worlds can be represented quite well by Dirichlet's domain with appropirate assigned centers. Honda (1978,1980) first reported on epidermal cells in mammlian skin. Then, Saito (1982) formulated the mathematical relation for the formation of regular pattern of points as the Voronoi centers from the random array in the cases of one to three dimensions. For certain, Dirichlet's domain presents a key to resolve the problems on various events due to self-organization in biological field.

(A) Points on the basal layer are randomly distributed. Each line on the upper layer presents the center of two adjacent points on the basal layer. Then, the mid-position between the neibouring lines presents the balanced position to the point between the both lines. Hence, the points move to the balanced positions as shown on the second layer. Point distribution becomes regular towards the upper layer by repeating the process. Thus, the regular pattern of point array is formed in Fig. 1-A.

Tooth pattern

An endless system with numerious numbers of points has been formulated by Saito (1982) but we need an information of a limited system with some numbers of points in a definite region. Then, we pursue the process to form a regular pattern from random array of 16 points in a limited region. The intervals between each two of 16 points and between the boundary and the point at the end are represented by one Vector $\vec{X}^{(o)} = (x_1^{(o)}, x_2^{(o)}, x_3^{(o)}, \ldots, x_{17}^{(o)})$ on the original array. The Vector $\vec{X}^{(o)}$ is called 'point-intervals'. When we set a line at the center of two adjacent points, the 'line-intervals' are defined by another kind of Vector $\vec{A}^{(o)} = (a_1^{(o)}, a_2^{(o)}, a_3^{(o)}, \ldots, a_{18}^{(o)})$. The relation between the both Vectors $\vec{X}^{(o)}$ and $\vec{A}^{(o)}$ is represented by

$$a_i^{(o)} = \frac{1}{2}(x_i^{(o)} + x_{i+1}^{(o)});\quad a_1^{(o)} = \frac{1}{2}x_1^{(o)} \text{ and } a_{18}^{(o)} = \frac{1}{2}x_{17}^{(o)}.$$

On the first movement of each point to the balanced positions, the point-intervals are changed to another Vector $\vec{X}^{(1)} = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_{17}^{(o)})$. The relation between the both Vectors $\vec{X}^{(o)}$ and $\vec{X}^{(1)}$ is represented by

$$x_i^{(1)} = \frac{1}{4}(x_{i-1}^{(o)} + 2x_i^{(o)} + x_{i+1}^{(o)});\quad x_1^{(1)} = \frac{1}{4}(3x_1^{(o)} + x_2^{(o)}) \text{ and } x_{17}^{(1)} = \frac{1}{4}(x_{16}^{(o)} + 3x_{17}^{(o)}).$$

Further, by repeating the process, we obtain

$$x_i^{(2)} = \frac{1}{4^2}(x_{i-2}^{(o)} + 2x_{i-1}^{(o)} + 4x_i^{(o)} + 2x_{i+1}^{(o)} + x_{i+2}^{(o)}).$$

Now, in the matrix notation, it is described as follows;

$$\vec{X}^{(1)} = \frac{1}{4}\begin{vmatrix} 3 & 1 & 0 & 0 & \cdots & & & 0 \\ 1 & 2 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 2 & 1 & \cdots & & & 0 \\ 0 & 0 & 1 & 2 & \cdots & & & 0 \\ \cdots & & & & & & & \\ \cdots & & & & & & & \\ 0 & 0 & \cdots & & 0 & 1 & 2 & 1 \\ 0 & 0 & \cdots & & 0 & 0 & 1 & 3 \end{vmatrix} \vec{X}^{(o)} = N^{(1)}\vec{X}^{(o)}.$$

Then, $\vec{X}^{(2)} = N^{(1)}\vec{X}^{(1)} = N^{(1)} \cdot N^{(1)}\vec{X}^{(o)} = N^{(2)}\vec{X}^{(o)}$.

Further, $\vec{X}^{(3)} = N^{(1)} . \vec{X}^{(2)} = N^{(1)} . N^{(2)}\vec{X}^{(o)} = N^{(3)}\vec{X}^{(o)}$.

In a general formula, it is $\vec{X}^{(k)} = N^{(1)}\vec{X}^{(k-1)} = N^{(1)}N^{(k-1)}\vec{X}^{(o)}$

$$= N^{(k)}\vec{X}^{(o)}.$$

We carried out the computation of the above formula through the progrum Fort N, Run LIBT/PAMMLT,R by N6950 (NEC, Japan) at Okayama University Computor Center.

$$\vec{N}^{(k)} = \left| \begin{array}{c} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots \beta_1 \; \beta_0 \; \beta_1 \; \beta_2 \cdots\cdots \\ \cdots\cdots \beta_2 \; \beta_1 \; \beta_0 \; \beta_1 \cdots\cdots \\ \cdots\cdots \beta_3 \; \beta_2 \; \beta_1 \; \beta_0 \cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array} \right|$$

Here, the value of β_0, β_1, β_2..... are on a Gaussian distribution at the k (β_0 is defined as the maximum value).

$\vec{X}^{(k)} = N^{(k)}\vec{X}^{(o)} = \beta_0 . N'^{(k)}\vec{X}^{(o)}$.

Since it is $\beta_1 / \beta_0 \rightarrow 1$, $\beta_2 / \beta_0 \rightarrow 1$ .....for $k \rightarrow \infty$, each component of the matrix $N^{(k)} \rightarrow 1$ and $\beta_0 \rightarrow 1/n$.

Then, we obtain $x_i^{(k)} = \frac{1}{n} \sum_{i=1}^{n} x_i^{(o)}$. Hence, the position of $x_i^{(k)}$ ($k \rightarrow \infty$) is determined by an average of all points on the original array. Here, an equal interval is proven.

(B) A gradient to have an influence on the intervals between points is taken into consideration on the process of regular pattern formation. Then, a different type of regular pattern is formed on the new process. In the mathematical viewpoint, for instance, it is a linear conversion to the point array on parabolic curve from that on line. In the previous case, the point array was investigated in an uniform enviroment, If we take into account on interactions between points, points will be equivalently interacted in the enviroment.

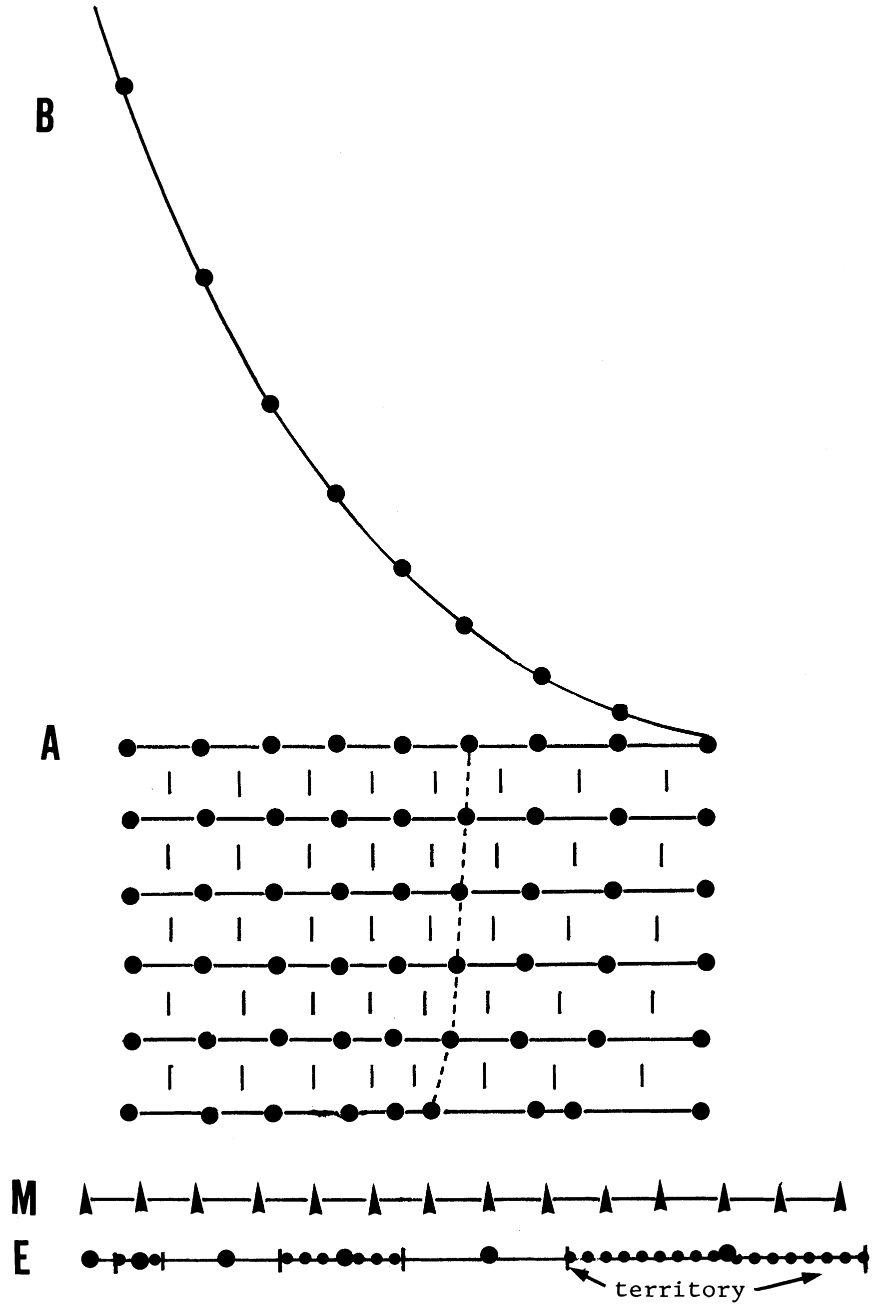
Consequently, an equal interval is obtained. On the other hand, if we take into account on the interaction in field with a gradient, the equivalent interactions are interfered. In the results, intervals between points on the array are determined by the gradient. An example is shown in Fig. 1-B.

APPLICATIONS OF THE THEORY

In the case of (A), we investigated a point array in a limited region. The interval at the end is much influenced by the boundary rather than determined by the interaction with the other points. Only some points located at the center of the array are equivalently interacted with the others. Hence, the intervals far from the end can be as equal as in an endless system. However, we consider that the boundary or the termination of point array may be flexible in growing organs and be movable with the development. Upon the assumption, the intervals would be equal with no effect of the boundary as if it were an endless system. Thus, the theory in an endless system is available even to a limited system in growing organs.

Now, in an example, a point presents a tooth germ in Fig. 1-A. The germs are settled at the regular positions regardless of the development. Then, the tooth development commerces at the definite positions. Alternatively, in another example, a point presents a dental crown. The crowns are settled at the regular positions with the development. In the latter case, the regulation is performed on the eruptive process. In general description, if intervals between points present the distance, locational control is illustrated in the figure. On the other hand, if intervals present degree of direction, directional control is illustrated. It is significant what is presented by the point-intervals in Fig.1-A. The results on tooth development will be published elsewhere.

If the intervals present space to be gaverned by the point, the figure showes territory of each point (Tanemura & Hasegawa, 1980). The equal territory produces a homogeneous cell-



▲ :M-cell, ● :E-cell, •: the descendant of E-cell.

Fig.1, A regular pattern formation from random array.
(A) in an uniform enviroment, (B) in field with gradient.

linage in an organ when a point presents a cell. On the contrary, if the territory of point is determined by a gradient, the number of descendants is different in cell-linage. Now, descendants of cell-linage is termed by polyclone (Crick & Lowrence,1975). An organ with gradient consists of different sizes of polyclones. For instance, M-cells are distributed homogeneously and the territory of M-cell is always constant. If original E-cells are distributed in a gradient, the size of polyclone of E-cell is due to a space of territory. If the relation between E- and M-cells is concerned with the pattern formation (Slavkin et al., 1977), distinctive patterns will be formed in the organ. In this article, we described a general scheme of pattern fromation in biological system on the basis of Honda's model. Then, it should be remarked that the process of self-organization is described by diffusion-equation with variable of physical value presented by the interval in Fig.1-A. In actual, we have succeeded in the description by the reciprocal process of diffusion about the direction-control on tooth eruption (Horii & Yamauchi). Further, in feature, we will have an attempt of approach to tooth pattern formation.

Acknowledgement : I should like to express my appreciation to my wife for her contribution to the work. Also, I appreciate to Dr.Honda (Kanebo Institute) and Dr.Tanemura (The Institute of Statistical Mathematics) for their helpful discussion.

REFERENCES

Crick, F.H.C. and Lawrence, P.A. : Science, vol. 189,340-345(1975)

Honda, H. : J. of theor. Biol., vol.72, 523-543 (1978).

Honda, H. : International Reviews of Cytology, vol. 81, 191-248 (1983).

Horii, Z-I and Yamauchi, K. : submitted to J. of dent. Research.

Slavkin, H.C. et al. : Cell and Tissue Interactions,(J.W.Lash & M.M.Burger Ed.) 29-45 (1977).

Tanemura, M. and Hasegawa, M. : J of theor. Biol., vol.82, 477-496 (1980).

Saito, N. : J. of theor. Biol., vol.95, 591-599 (1982).