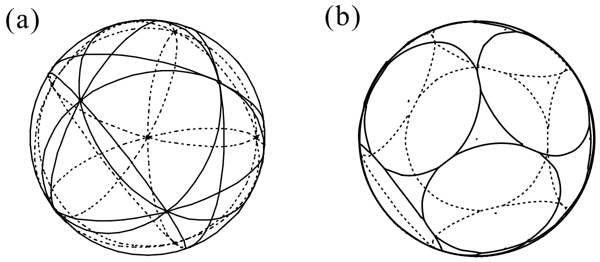

Fig. 11. (a) Our sequential covering for $N = 9$. (b) Our solution of Tammes problem for $N = 9$. Both viewpoints are (0, 0, 10). In this example, the coordinates of the centers are respectively (0, 0, –1), (0.23570, –0.91287, –0.33333), (0.94281, 0, –0.33333), (0.23570, 0.91287, –0.33333), (–0.82496, 0.45644, –0.33333), (–0.78567, –0.60858, 0.11111), (0.15713, –0.60858, 0.77778), (0.58926, 0.45644, 0.66667), and (–0.54997, 0.30429, 0.77778).

is graphically presented by the curve of $r \simeq 1.23096$ corresponding to $r = \cos^{-1}(1/3)$ in Fig. 10. Therefore, for $r = \cos^{-1}(1/3)$, we check numerically that $\cup_{\nu=1}^{7} C_\nu$ are in an extreme state if and only if M_7 is put at K_9 or K_{10}.

As mentioned above, in this paper, we put M_7 at K_9. Then, the uncovered region $(W_7)^c$ is the triangle $K_{10}K_6K_7$ on S that satisfies the relations $\pi/2 \geq d_s(K_{10}, H) = d_s(H, K_6) = d_s(K_{10}, K_6)/2 \geq d_s(H, K_7) > 0$. We note that H is the middle point of the geodesic arc $K_{10}K_6$. From Theorem 2 in Subsec. 2.4, the largest spherical distance in $(W_7)^c$ is $d_s(K_{10}, K_6)$. Namely, we find $\bar{r}_8 = \cos^{-1}(1/3)$. Hence, if M_8 is put at K_{10} or K_6, the set $\cup_{\nu=1}^{8} C_\nu$ which contains W_7 covers the spherical surface S except for a point. Then, due to the facts $K_6 \in \partial C_7$ and $K_{10} \in \partial C_7$, we find that $\cup_{\nu=1}^{8} C_\nu$ is in an extreme state. In this paper, we choose M_8 on K_6.

Then, $\cos^{-1}(1/3)$ satisfies the initial assumption $\tan^{-1}2 \leq r < r_8$. However, one can suspect this result is owing to the initial assumption. When r is in the range $(0, \tan^{-1}2]$, we check whether W_8 is able to cover S except for finite points. From the results of $r \simeq 1.10715$ in Figs. 3, 4, 7, and 10, we find the fact that our first to eighth spherical caps must leave an uncovered region on S when r is equal to $\tan^{-1}2 \approx 1.10715$. Hence, for $0 < r < \tan^{-1}2$, the uncovered region would become still bigger. Therefore, our upper bound r_9 for $N = 9$ does not exist in the range $(0, \tan^{-1}2]$ like the case of $N = 8$. Thus, we note that $\tan^{-1}2 < r < r_8$ is confirmed ($r = \tan^{-1}2$ is just excluded from the above consideration).

Finally, M_9 is put on the unique uncovered point K_{10}, and then $\cup_{\nu=1}^{9} C_\nu$ which contains W_8 covers the whole of S (see Fig. 11(a)). Thus, our consideration that the angular radius r_9 ($\bar{r}_8$) is equal to a side-length of spherical rhombus $K_8K_9K_6K_{10}$ which satisfies (17) is confirmed and $\cos^{-1}(1/3)$ is certainly a solution for $N = 9$.

4. Conclusion

In Sec. 3, we calculated the upper bound of r for $N = 2, ..., 9$, such that the set $\cup_{\nu=1}^{N} C_\nu$ which contains W_{N-1} covers the whole spherical surface S (see Table 2).