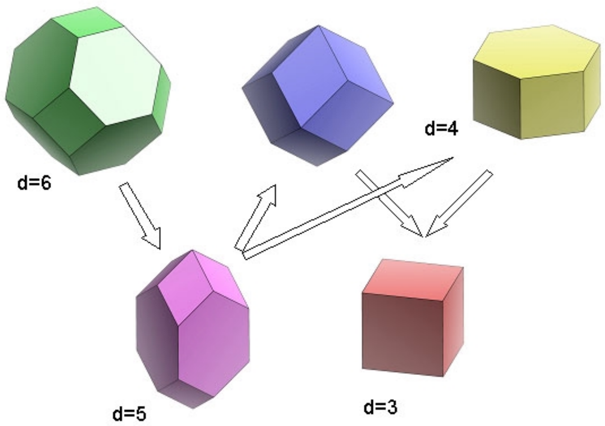

Fig. 3. Processes to reproduce four members of Fedorov's parallel polyhedra from the truncated octahedron. These figures are provided by Ishii (2008, private communication).

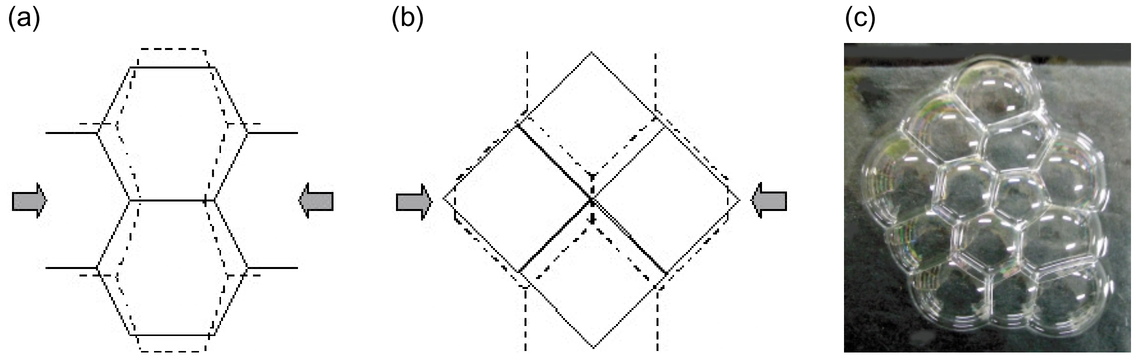

Fig. 4. Examples of stability of 2-dimensional arrangements. (a) Hexagon packing (stable), (b) square packing (unstable), where solid and dashed lines show arrangements before and after pressure, respectively. (c) Soap bubbles arranged on a horizontal plate.

3. Stability of Space Division

The usual meaning of stability is a property of construction of objects, where a small deformation of construction causes responses either restoring to the original one (stable case) or leading to larger deformation (unstable case). The concept of stability discussed in this section is similar to this definition, while some notes are necessary for better understanding. The deformation here means change of positions of elements (for example, due to an external pressure), while its response is the change of types of element shapes, i.e. whether the square remains square, or the hexagon remains hexagon.

We begin with 2-dimensional cases. It is well known that the regular polygons filling the plane with equal members are limited to the triangle, the square and the hexagon. Among these arrangements the triangle and square cases have different property from the hexagonal case. In the latter case each hexagon touches neighbors through edges, while in the former cases each triangle or square has neighbors which it touches only through a vertex. In this case a small deviation of positions of polygons leads to change of types of shapes. A simple example of this situation is shown in Fig. 4, where the hexagons pressed horizontally remain hexagons (Fig. 4(a)) while the squares change to hexagons (Fig. 4(b)). It should be noted here that in the stable case always three edges meet at a vertex. Ensemble of soap bubbles arranged on a horizontal plate has this kind of stability (see Fig. 4(c)).

In the 3-dimensional case the stable division must satisfy the following conditions, that 4 vertices of 4 polyhedra touching each other gather at one point, and that 3 edges of 3 polyhedra touching each other gather at one line segment. These conditions are satisfied by most space divisions, both in biological and non-biological systems.

The idea of stable division can be understood in terms of Voronoi division. In the most point arrangements in the plane or in the space their Voronoi divisions are stable. Rare cases such as the point arrangement in the square lattice in the plane or the cubic lattice in the space produce unstable division.

Stability of space division is closely connected to the mechanical stability in cell arrangements in biological tissues. If a force is applied to a cell in a 2-dimensional stable arrangement, this force balances with restoring forces appearing in the two cells touching the first cell (see Fig. 5(a)). In the same way the force applied on a cell in 3-dimensional stable arrangement balances with the three forces appearing in the three cells touching the first cell (see Fig. 5(b)). When an external force is applied, to a cell in unstable division, the cell and neighboring cells change its type of shapes. If the cells are made of rigid material, the balance of forces cannot be determined uniquely. This situation is similar to that