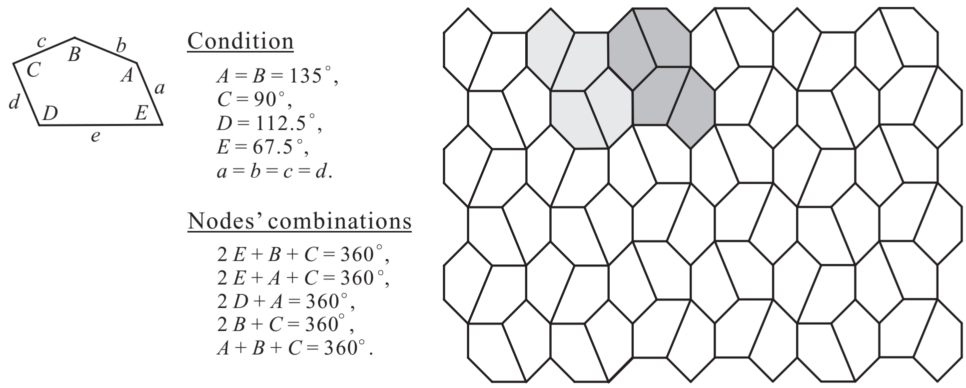

Fig. 2. Tiling by convex pentagons that belong to both type 1 and type 7.

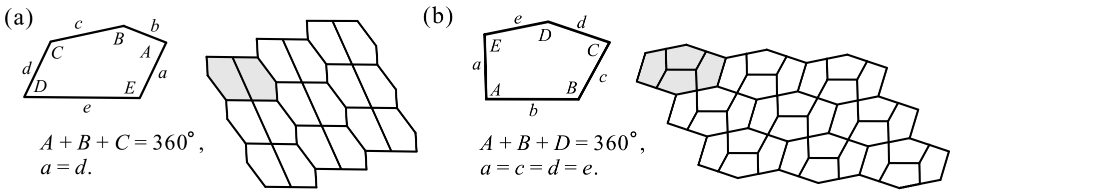

Fig. 3. (a) Edge-to-edge tiling by convex pentagonal tiles that belong to type 1. (b) Edge-to-edge tiling by convex pentagonal tiles that belong to type 2.

common (Grünbaum and Shephard, 1987). In an edge-to-edge tiling by polygons, the number of adjacents of a polygon is equal to its number of edges. Therefore, from Statement 3.3.5 ("A normal tiling in which every tile has the same number of adjacents is balanced") in Grünbaum and Shephard (1987), the tiling $\mathcal{J}$ is balanced. Thus, we derive the limit

$$\lim_{\rho\to\infty} \frac{N(F)}{P(F)} = \frac{3}{2} \tag{2}$$

by using Euler's Theorem for Tilings (see Statement 3.3.3 in Grünbaum and Shephard (1987)). If a node belongs to t pentagons from F, then the share of each of these pentagons for this node is $1/t$. Since every pentagon in F has m nodes of valence 3 and $5 - m$ nodes of valence k, we obtain

$$N(F) = (P(F_1) + P(F_2))\left(\frac{m}{3} + \frac{5-m}{k}\right) + \varepsilon, \tag{3}$$

where ε is the sum of difference between the contributions of the vertices of all boundary pentagons for F and $\varepsilon/P(F) \to 0$ as $\rho \to \infty$. From (1), (2), and (3), as $\rho \to \infty$,

$$\frac{m}{3} + \frac{5-m}{k} = \frac{3}{2}. \tag{4}$$

Let $K(F)$ be the sum of valences of $N(F)$ nodes in F. Then, the limit $\lim_{\rho\to\infty} K(F)/N(F)$ is called the average valence of nodes in $\mathcal{J}$. From Proposition in Sugimoto and Ogawa (2006b), the average valence of nodes in $\mathcal{J}$ is $10/3 \approx 3.33\cdots$. That is, since the average valence is not an integer, there does not exist the balanced tilings by pentagons with all nodes of the same valence. Hence $0 < m < 5$. In addition, from Proposition 2.1 ("In each edge-to-edge tiling of the plane by uniformly bounded pentagons, there exists a tile with at least three nodes of valence three") in Bagina (2004), a pentagon in $\mathcal{J}$ has at least three nodes of valence three. Therefore, $3 \le m < 5$.

For $m = 3$, from (4), $\frac{3}{3} + \frac{2}{k} = \frac{3}{2}$. Therefore, $k = 4$. Thus, $(m, k) = (3, 4)$.

For $m = 4$, from (4), $\frac{4}{3} + \frac{1}{k} = \frac{3}{2}$. Therefore, $k = 6$. Thus, $(m, k) = (4, 6)$. □

3. Other Property

In this section, other property which can be obtained by the same proof as Proposition 1 is shown.

PROPOSITION 2 *Let P_a be a pentagon that has m_1 nodes of valence* 3 *and* $5 - m_1$ *nodes of valence k, and P_b be a pentagon that has m_2 nodes of valence* 3 *and* $5 - m_2$ *nodes of valence k* ($0 \le m_2 < m_1 \le 5, k \ge 4$), *respectively. If an edge-to-edge tiling $\mathcal{J}_1$ that is formed of pairs of P_a and P_b is normal, then* $(m_1, m_2, k) = (4, 2, 4)$, $(m_1, m_2, k) = (5, 1, 4)$, *or* $(m_1, m_2, k) = (5, 3, 6)$.

Proof of Proposition 2. The number of nodes of the tiling in F on $\mathcal{J}_1$ is

$$N(F) = (P(F_1) + P(F_2)) \cdot \left(\frac{1}{2}\left(\frac{m_1}{3} + \frac{5-m_1}{k}\right) + \frac{1}{2}\left(\frac{m_2}{3} + \frac{5-m_2}{k}\right)\right) + \varepsilon \tag{5}$$

where $\varepsilon/P(F) \to 0$ as $\rho \to \infty$. From (1), (2), and (5), as $\rho \to \infty$,

$$\left(\frac{m_1}{3} + \frac{5-m_1}{k}\right) + \left(\frac{m_2}{3} + \frac{5-m_2}{k}\right) = 3. \tag{6}$$

From Proposition 2.1 in Bagina (2004), one of m_1 and m_2 is certainly equal to three or more. Therefore, in this report, we assume $3 \le m_1 \le 5$, $0 \le m_2 < 5$ and $m_2 < m_1$.