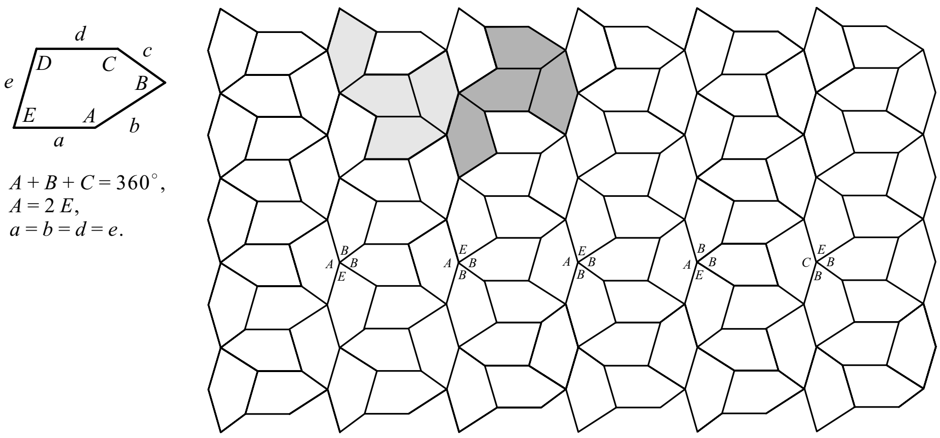

Fig. 4. Tiling that satisfies the property of $(m_1, m_2, k) = (4, 2, 4)$ in Proposition 2.

For $m_1 = 3$, from (6),

$$k = \frac{3(7 - m_2)}{6 - m_2}. \tag{7}$$

In the cases of $m_2 = 0$, 1, and 2, each value of (7) is not an integer.

For $m_1 = 4$, from (6),

$$k = \frac{3(6 - m_2)}{5 - m_2}. \tag{8}$$

In the cases of $m_2 = 0$, 1, and 3, each value of (8) is not an integer. On the other hand, the value of (8) for $m_2 = 2$ is four.

For $m_1 = 5$, from (6),

$$k = \frac{3(5 - m_2)}{4 - m_2}. \tag{9}$$

First, $m_2 < 4$ since k is finite ($\mathcal{J}_1$ is normal). In the cases of $m_2 = 0$ and 2, the value of (9) is not an integer. Then, the value of (9) for $m_2 = 1$ and $m_2 = 3$ is four and six, respectively.

The value of k must be an integer since k is the valence of nodes in edge-to-edge tiling. Thus, we obtain $(m_1, m_2, k) = (4, 2, 4)$, $(m_1, m_2, k) = (5, 1, 4)$, and $(m_1, m_2, k) = (5, 3, 6)$. □

4. Discussion

Here, let us investigate the properties of tilings by congruent convex pentagons using the propositions presented in the previous sections. First, the edge-to-edge tilings of type 1 and type 2 in Fig. 3 and the tilings of type 4, type 6, type 7, type 8, and type 9 in Fig. 1 satisfy the property of $(m, k) = (3, 4)$ in Proposition 1. When the tilings are edge-to-edge, as far as we know, the convex pentagonal tiles that belong to type 1 or type 2 can form tilings that satisfy the property of $(m, k) = (3, 4)$ in Proposition 1. Then, the edge-to-edge tiling of type 5 in Fig. 1 satisfies the property of $(m, k) = (4, 6)$ in Proposition 1. On the other hand, the tiling in Fig. 4 is an example of the tiling which satisfies the property of $(m_1, m_2, k) = (4, 2, 4)$ in Proposition 2 (Schattschneider, 1978, 1981; Sugimoto and Ogawa, 2004, 2009). Note that, since the pentagonal tiles in Fig. 4 belong to type 1, the tiling that satisfies the property of $(m, k) = (3, 4)$ in Proposition 1 can be formed of these tiles (Sugimoto and Ogawa, 2006a, 2009). We do not know the tiling by congruent convex pentagons that satisfy the properties of $(m_1, m_2, k) = (5, 1, 4)$ or $(m_1, m_2, k) = (5, 3, 6)$ in Proposition 2.

A tiling is called isohedral if the symmetry group operates on the set of tiles of the tiling transitively. In Schattschneider and Dolbilin (1997) is given a local criterion for 2-dimensional isohedral tilings. "The perfect list" of all planigons (incl. pentagons), prototiles of isohedral tiling, has been given independently and simultaneously in papers by Grüenbaum, Shephard and Delone, Dolbilinb, Shtogrin. See Section 9.1 in Grünbaum and Shephard (1987) about the results of convex polygons which admit isohedral tilings. The convex pentagonal tiles that belong to type 1, type 2, type 3, type 4, or type 5 can form the isohedral tiling. But the tilings of type 6, type 7, type 8, and type 9 are not isohedral. In particular, we should emphasize that the properties in this paper are applicable also to the non-isohedral edge-to-edge tiling by congruent convex pentagons.

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