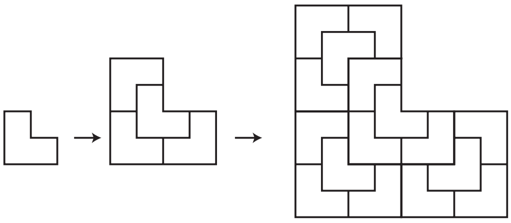

Fig. 1. Non-periodic L-tiling.

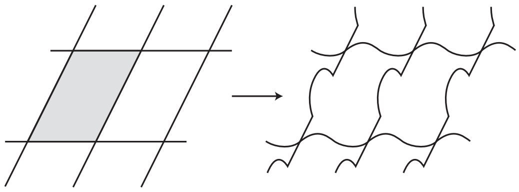

Fig. 2. Perturbation of edges of parallelogram.

Fig. 3. An edge.

Fig. 4. Examples of perturbed edges.

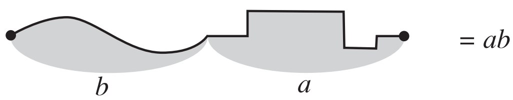

Fig. 5. An example of a product of perturbed edges.

turbed edge which is a little perturbed to avoid the restriction.

Next, we define a product of perturbed edges.

DEFINITION 2.4 (PRODUCT OF EDGES) *Let* $a_1, a_2, \cdots, a_k$ *be perturbed edges. If they are placed on a straight line from right to left and form a row, then we call it a product of* $a_1, a_2, \cdots, a_k$ *and we denote this product by* $a_1 a_2 \cdots a_k$. *See Fig. 5.*

For a perturbed edge a, we define two operations $\overline{a}$, and a^{-1}. For a perturbed edge a, $\overline{a}$ is a symmetry (right-side-left) image of a. In the same way, a^{-1} is an upside-down image of a. See Fig. 6.

It is easy to show the following lemma.

LEMMA 2.5 *(1)* $\overline{(\overline{a})} = a$, $(a^{-1})^{-1} = a$
(2) $\overline{ab} = \overline{a}\,\overline{b}$, $(ab)^{-1} = b^{-1}a^{-1}$
(3) $\overline{(a^{-1})} = (\overline{a})^{-1}$

In a tiling, if a tile with a perturbed edge a and another tile with a perturbed edge b are neighbors at a and b, we

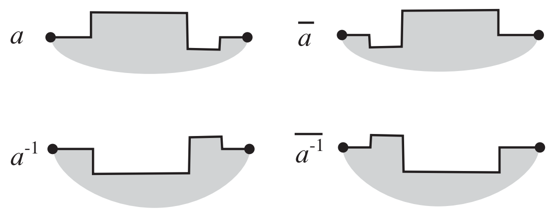

Fig. 6. Definition of $\overline{a}$, and a^{-1}.

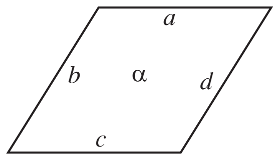

Fig. 7. Edges of the parallelogram α.

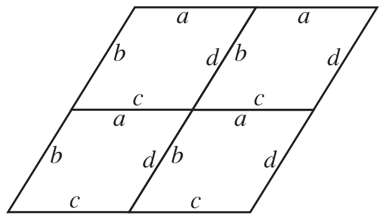

Fig. 8. Matching of the tiling (P1).

have $a = \overline{b^{-1}}$. We denote this relation by $\frac{a}{b}$. We often say that a matches b.

The following lemma is trivial.

LEMMA 2.6 *(1)* $\frac{a}{b}$ *if and only if* $\frac{b}{a}$
(2) If $\frac{a}{b}$ *and* $\frac{a}{c}$ *then* $b = c$
(3) $\frac{ab}{cd}$ *if and only if* $\frac{a}{d}$ *and* $\frac{b}{c}$

Let $\mathcal{T}$ be a tiling with respect to a protoset $\mathcal{S}$. Suppose that all prototiles are polygons. Here we assume that there is no vertex of a tile lying on an edge of another tile.

DEFINITION 2.7 (ESCHERIZATION, ESCHER DEGREE) *(1) Let* $\mathcal{T}$ *and* $\mathcal{S}$ *be as above. If we perturb edges of prototiles such that the perturbed prototiles give another tiling, we call this process escherization.*

(2) If the set of escherization of $\mathcal{T}$ *is parametrized by some perturbed edges, the escher degree is the number of the parameters.*

Example. Let α be a parallelogram and (P1) a tiling of $\mathbf{E}^2$ as in Fig. 2. Let a, b, c, d be edges of α as in Fig. 7.

From the matching of the tiling, we have $\frac{a}{c}$ and $\frac{b}{d}$. That is, if we perturb a, then the edge c changes such that $c = \overline{a^{-1}}$, and we can perturb b independently of a. Then the edge d changes such that $d = \overline{b^{-1}}$. See Fig. 8.

We call relations obtained from the tiling property *edge-matchings*.