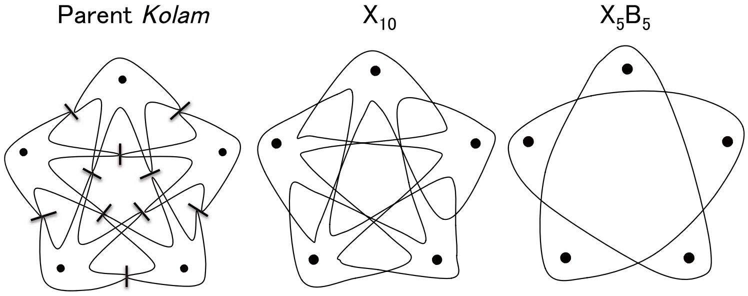

Fig. 11. An example of a parent *kolam* and two children *kolam*s for $N = 5$ and $J = 1$.

special case of the 3 dots arranged in an equilateral triangle, are shown in Fig. 6. Did we find all possible *kolam*s with $N = 3$? If so, how about the *kolam* on the left in Fig. 7? It turns out that this *kolam* is captured by the proposed method for $N = 4$, where an additional dot is placed in the middle of Fig. 7. This is discussed in the next section. The example is again illustrative of the fact that a *kolam*, once created, is distinctive in its own right, irrespective of the presence or absence of dots. *The characteristic N for a given kolam may be defined as the minimum number of dots required for generating the kolam with the above 5-step method*. However, note that when dots are removed or added to a *kolam*, the resultant *kolam*s may no longer be topologically equivalent to the original *kolam*.

5. Exploring *Kolam*s with 4 Dots ($N = 4$)

Three different configurations of parent *kolam*s are shown in Fig. 8 for $N = 4$.

It is possible to show that parents 1 and 3 are homotopic. Such equivalence is shown in Fig. 9a, and hence they form a single parent type. However, parent 2 forms a distinct parent type as shown in Fig. 9b since parent *kolam*s 1 and 2 cannot be distorted into each other without the lines crossing over the dots in two dimensions. The number of possible *kolam*s for any parent *kolam* with $N = 4$ following rules **M**1–**M**3 and **O**1, **O**2, and **O**5 can be computed from Eq. (1) as $K = 3^{1\times4\times(4-1)/2} = 3^6 = 729$.

The 729 possible *kolam*s from each parent is a large number, and so we choose here to impose additional restrictions in order to explore only a subset. For example, optional rule **O**6 suggests that symmetry equivalent junctions will have the same type of bond.

This allows for the symmetry of the parent phase to be preserved while bonds are formed. The various *kolam*s derived from three different parent *kolam*s (1, 2, and 3) in Fig. 8 under the rules of **M**1–**M**3 and **O**1, **O**2, **O**5, and **O**6 are shown in Fig. 10. For parent *Kolam* 1 in Fig. 10, there are 3 groups ($g = 3$) of symmetry equivalent junctions related by a vertical mirror symmetry. Thus the number of *Kolam*s with $J = 1$ is $K = 3^3 = 27$. For both the special cases of parent *Kolam* 2 (dots forming an equilateral triangle) and *Kolam* 3 (dots forming a square), $g = 2$ arising from a 3-fold and 4-fold rotational axes respectively, and hence $K = 3^2 = 9$ as shown. We note that B_3X_3 with $N = 4$ captures the *kolam* that was missed in Fig. 7 by $N = 3$.

6. Conclusions

We have demonstrated a method of generating countless *kolam*s from user-defined dot arrangement on a surface. This method can be mastered by anyone without the need to understand the detailed mathematics behind *kolam*s. For a give number, N, of dots in any spatial arrangement on a surface, the number of possible *kolam*s that follow only the mandatory rules **M**1–**M**3 is infinite, even for a 1-dot *kolam* ($N = 1$). However, by following additional optional rules **O**1 and **O**2, this number is finite as given by Eq. 1. Addition of rule **O**6 modifies this equation.

We show by example that for a given number of dots N, a set of parent *kolam* types exist, from which all possible *kolam*s can be generated. All parent *kolam*s within a single type are homotopic. Hence the resultant *kolam*s from these homotopic parents will also have corresponding homotopic cousins. Though a rigorous proof for such homotopy in general has not been presented, it can be argued based on the method of construction similar to that shown in Fig. 9.

*Kolam*s with higher N get richer and more complicated quickly. For example, Fig. 11 shows an example parent *kolam* for $N = 5$ and $J = 1$, and two possible children *kolam* arising from it. The readers are encouraged to try generating other parent and children *kolam*s for this case.

There are several advantages to this simple method:

(1) It is applicable for any number of dots, N.

(2) The dots can be arranged in any configuration in 2-dimensions.

(3) While the proposed method may not always guarantee aesthetics, it is simple enough for a user to impose additional aesthetically appropriate optional rules.

(4) A computer program can vary b, J, and N for generating numerous *kolam*s following the three mandatory rules, plus any number of user-defined optional rules.

This leads to the possibility of creating an interactive website or a mobile app that can help a user to generate *kolam*s at will. Such an app will get the user involved in the creative process, including young children who may be introduced to art, symmetry and topology through *kolam*s. The method is also applicable to generating other similar patterns such as some of the Chinese and Celtic knots.

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