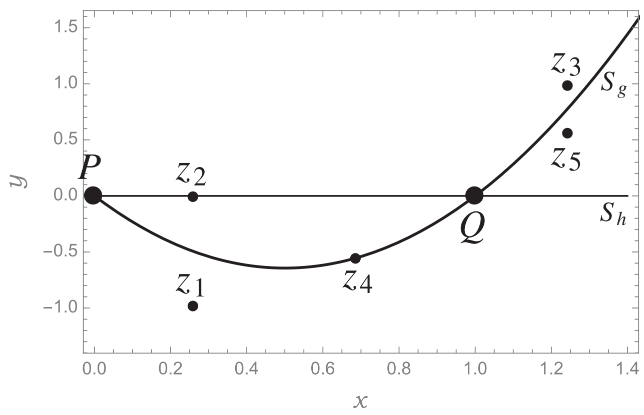

Fig. 1. Saddle periodic orbit with rotation number 2/5 where $z_1 = Tz_5$, $z_2 = Tz_1$, $z_3 = Tz_2$, $z_4 = Tz_3$, and $z_5 = Tz_4$. This orbit rotates around Q twice during one period. The orbital point z_2 locates on S_h, and z_4 locates on S_g. $a = 5.15$.

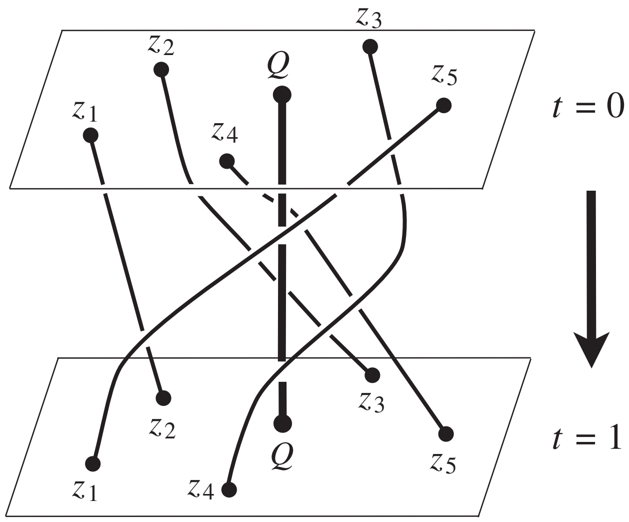

Fig. 2. How to make the braid β_5. The upper plane represents the configuration at $t = 0$ and the lower one represents the configuration at $t = 1$. Thick arrow represents the direction of the time evolution. For example, the strand starts at z_1 in the upper plane goes down and arrives at z_2 in the lower plane. The thick strand starts at Q in the upper plane goes down and arrives at Q in the lower plane. Each strand rotates around thick strand and goes down.

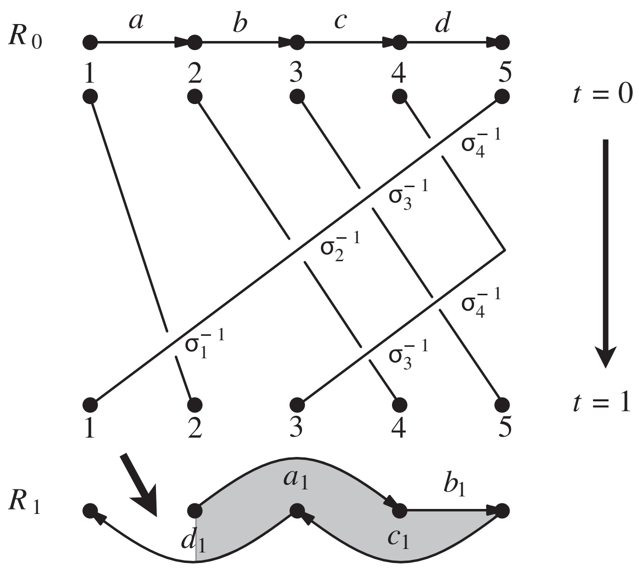

Fig. 3. Braid β_5. The upper region of braid represents the configuration at $t = 0$ and the lower one the configuration at $t = 1$. Time flows from the upper region to the lower one (see thick arrow). Let R_0 be the set of Strings $\{a, b, c, d\}$ located above the braid, and R_1 be the set of Strings $\{a_1 = \beta_5 a, b_1 = \beta_5 b, c_1 = \beta_5 c, d_1 = \beta_5 d\}$ located below the braid. Thick arrow in R_1 represents the entrance of tunnel (gray region). Here, σ_k^{-1} $(k = 1, 2, 3, 4)$ are the generators to represent the braid.

coordinate representation (θ_k, r_k) is used. Here, we measure an angle clockwise from the segment QP. We observe the angular velocities of the orbital points from $z_1 = (\theta_1, r_1)$ to $z_4 = (\theta_4, r_4)$. During three iterations, the orbit rotates around Q approximately once $(\theta_4 - \theta_1 \approx 2\pi)$. The orbit from z_1 to z_4 rotates slowly. On the other hand, during two iterations, the orbit from z_4 to z_1 rotates around Q approximately once. Thus, the orbit from z_4 to z_1 rotates rapidly.

The slow rotation is characterized by the rotation number $1/3$ and the rapid one by the rotation number $1/2$. Thus, the total rotation is characterized by the rotation number $2/5$, which is divided as follows.

$$\frac{2}{5} = \frac{1+1}{3+2} \equiv \frac{1}{3} \circ \frac{1}{2}. \tag{4}$$

In the situation that the periodic orbit satisfying Eq. (4) exists, there exists the periodic orbit with the rotation number $1/3$ and that with the rotation number $1/2$ (Yamaguchi and Tanikawa, 2009, 2011). In fact, the former orbit appears at $a = 3$ through the rotation bifurcation of Q. At $a = 3$, two periodic orbits appear. One orbit is an elliptic orbit and the other one is a saddle orbit. At $a = 4$, the periodic orbit with the rotation number $1/2$ appears through the period-doubling bifurcation of Q. The periodic orbit rotates about 180 degree per one iteration around Q. Using the coexistence of the saddle periodic orbit with rotation number $1/3$ and the periodic orbit with rotation number $1/2$, we discuss the properties of BSP in Subsec. 3.1.

2.2 Braid and braid stirring pattern

Using the braid $\beta_5 = \sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_4^{-1}\sigma_3^{-1}$, we explain how to make BSP. All strands go down from the upper plane to the lower one (Fig. 3). Time progresses towards the lower plane from the upper one and the inverse progress never occur. The braid β_5 means the action to describe the time evolution.

Let the positions at which the strands start be Position P_k $(k = 1, 2, \cdots, 5)$. In Fig. 3, the abbreviated notations 1, 2, 3, 4 and 5 are used. In fact, the first strand starts at Point 1 and reaches at Point 2, the second one starts at Point 2 and reaches at Point 4, and so on.

In order to reproduce the pattern of boundary curve on the surface of fluid, we prepare the four strings. At $t = 0$, we set String a connecting from Point 1 to Point 2. Similarly, we also set String b, String c, and String d. These lengths are assumed to be one. These are deformed by the action of β_5. Let R_0 be the set of Strings $\{a, b, c, d\}$. In the following, we study the structure of $R_k = \beta_5^k R_0$ $(k \geq 0)$, which inherits the properties of β_5.