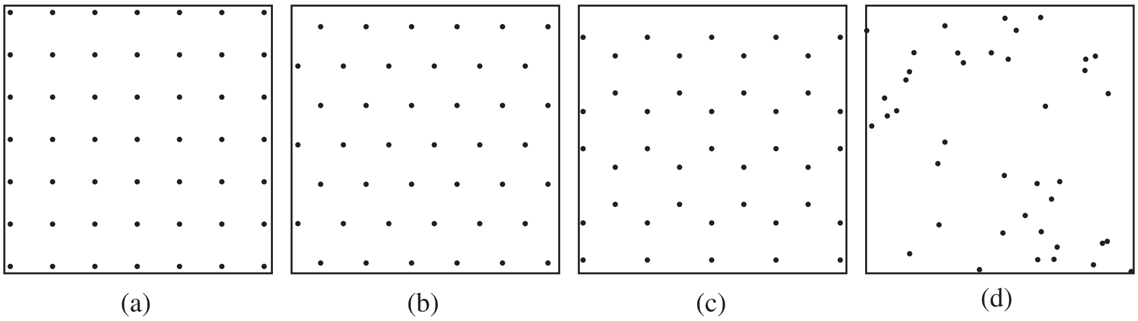

Fig. 1. Regular and random patterns of facilities: (a) Square lattice; (b) Triangular lattice; (c) Hexagonal lattice; (d) Random.

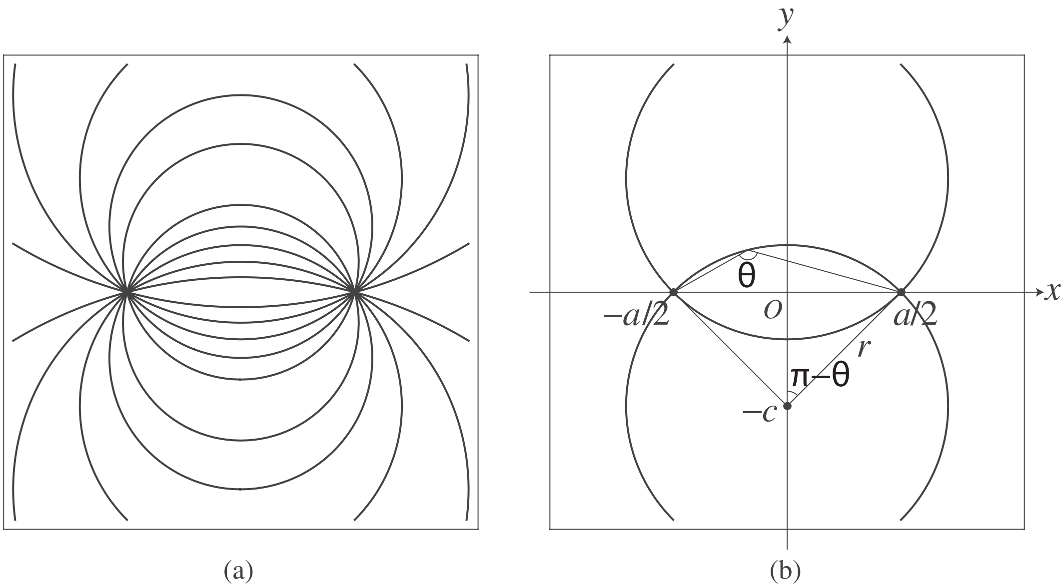

Fig. 2. (a) Contour of the angle; (b) Locus such that $\Theta = \theta$.

extremes serve as a basis for empirical analysis of actual patterns. In fact, the regular and random patterns have frequently been used in location analysis (O'Kelly and Murray, 2004; Sadahiro, 2005; Miyagawa, 2009). If customers are uniformly distributed, the optimal location that minimizes the average distance to the nearest facility is the triangular lattice (Fig. 1b) (Leamer, 1968; Iri *et al.*, 1984; Du *et al.*, 1999).

Let Θ be the angle between the directions from a randomly selected location in a study region to the first and second nearest facilities. The contour of the angle Θ is given by a circle passing through facilities, as depicted in Fig. 2a. Recall that angles subtended at the circumference by the same arc of a circle are equal. The locus such that $\Theta = \theta$ is obtained as follows. Set the coordinate system as shown in Fig. 2b, where facilities are at $(-a/2, 0)$, $(a/2, 0)$. Note that θ is the angle subtended at the circumference by the facilities. Let $(0, -c)$ and r be the center and radius of the circle, respectively. Since

$$\begin{aligned} c &= \frac{a}{2\tan(\pi-\theta)} = -\frac{a}{2\tan\theta}, \\ r &= \frac{a}{2\sin(\pi-\theta)} = \frac{a}{2\sin\theta}, \end{aligned} \tag{1}$$

the locus such that $\Theta = \theta$ is the circles expressed as

$$x^2 + \left(y \pm \frac{a}{2\tan\theta}\right)^2 = \frac{a^2}{4\sin^2\theta}. \tag{2}$$

2.1 Square lattice

Suppose that facilities are regularly distributed on a square lattice with spacing a. Let $F(\theta)$ be the cumulative distribution function of Θ, that is, the probability that $\Theta \le \theta$. $F(\theta)$ is given by

$$F(\theta) = \frac{S(\theta)}{S}, \tag{3}$$

where S and $S(\theta)$ are the area of the study region and the area of the region such that $\Theta \le \theta$ in the study region, respectively. The study region can be confined to the region where two facilities are the first and second nearest, which is the square centered at the midpoint of the facilities with side length $a/\sqrt{2}$, as shown in Fig. 3. The area of the study region is then $S = a^2/2$. The region such that $\Theta \le \theta$ is given by the dark gray region in Fig. 3. Thus, $S(\theta)$ is obtained by subtracting the area of the intersection of the