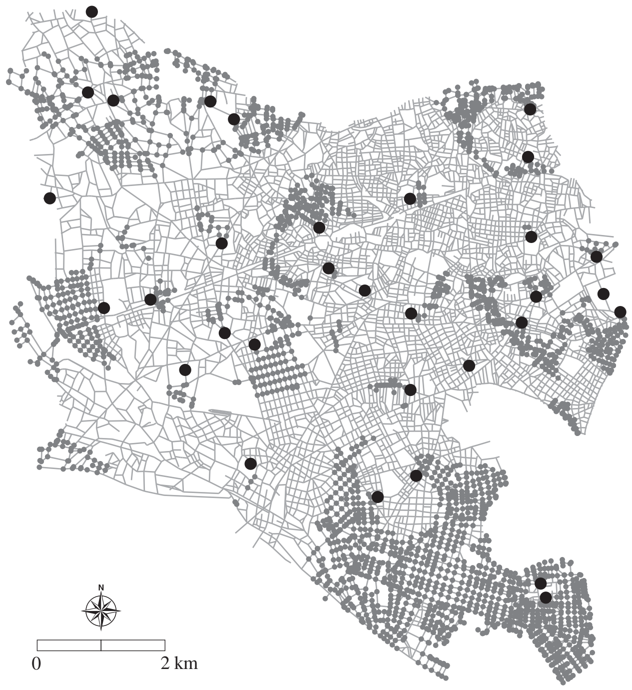

Fig. 6. Hospitals and nodes such that $\Theta < \pi/4$ in Setagaya, Japan.

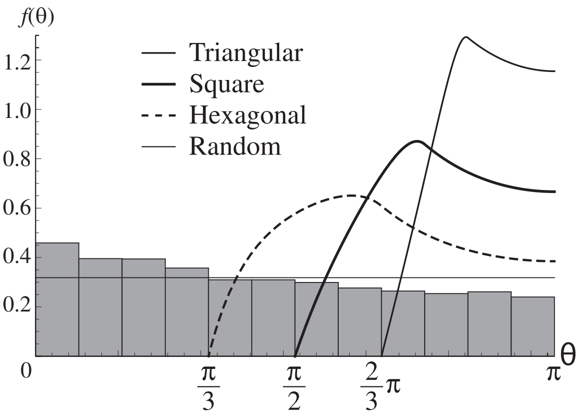

Fig. 7. Distribution of the angle.

Table 1. Average and standard deviation of the angle.

	Average	Standard deviation
Hospital	1.39 ($\approx 80°$)	0.91 ($\approx 52°$)
Square	2.45 ($\approx 140°$)	0.39 ($\approx 22°$)
Triangular	2.71 ($\approx 155°$)	0.25 ($\approx 14°$)
Hexagonal	2.09 ($\approx 120°$)	0.54 ($\approx 31°$)
Random	1.57 ($\pi/2 = 90°$)	0.91 $\left(\pi/\left(2\sqrt{3}\right) \approx 52°\right)$

where

$$\alpha = \frac{1}{4\cdot 3^{3/4}}\left(3 - \frac{\sqrt{3}}{\tan\theta} - \frac{1}{\sin\theta}\sqrt{2+\sqrt{3}\sin 2\theta + \cos 2\theta}\right). \tag{12}$$

The distribution of the angle $f(\theta)$ is obtained from Eq. (8) and shown in Fig. 7.

2.4 Random

Suppose that facilities are uniformly and randomly distributed. The probability that a region of area S contains exactly x facilities, denoted by $P(x, S)$, is given by the Poisson distribution as

$$P(x, S) = \frac{(\rho S)^x}{x!}\exp(-\rho S), \tag{13}$$

where ρ is the density of facilities (Clark and Evans, 1954). The probability $P(x, S)$ is independent of the location and shape of the region. The angle Θ is then uniformly distributed over the interval $[0, \pi]$ as

$$f(\theta) = \frac{1}{\pi}, \quad 0 \le \theta \le \pi. \tag{14}$$

The distribution of the angle $f(\theta)$ is shown in Fig. 7.

3. Actual Facility Location

In this section, we examine the distribution of the angle for actual facility location to discuss whether the model of the regular and random patterns can be applied to actual patterns. As an example, we consider 32 hospitals in Setagaya, Japan, as shown in Fig. 6, where black circles represent hospitals.

Let Θ be the angle between the directions from a node to the first and second nearest hospitals. The angle between the directions from a node $\mathbf{q}$ to hospitals $\mathbf{p_1}$, $\mathbf{p_2}$ is given by

$$\Theta = \arccos\frac{(\mathbf{p_1}-\mathbf{q})\cdot(\mathbf{p_2}-\mathbf{q})}{|\mathbf{p_1}-\mathbf{q}||\mathbf{p_2}-\mathbf{q}|}. \tag{15}$$

Dark gray circles in Fig. 6 represent nodes such that $\Theta < \pi/4$. It can be seen that the boundary of the set of the nodes forms a circle, as shown in Fig. 2. The normalized histogram of the angle for all nodes is shown in Fig. 7. The distribution for the actual pattern is similar to that for the random pattern. The average and standard deviation of the angle are summarized in Table 1. The average angle for the actual pattern is smaller than that for the regular and random patterns, and the standard deviation is as large as that for the random pattern. Note that the average angle for the triangular lattice is the largest among three regular patterns. It follows that the triangular lattice is suitable for the location of refuges. Note also that the standard deviation for the triangular lattice is the smallest, which leads to a small disparity in service level among customers.

4. Conclusions

This paper has derived the distribution of the angle between the directions of the first and second nearest facilities. The analytical expressions for the distribution for regular and random patterns are useful for location models using the direction of facilities as follows. First, they give an estimate for the service level of actual facility location. By comparing distributions, we can evaluate the efficiency of actual patterns. For example, if the angle for the location of refuges is much smaller than that for the regular patterns, relocating some refuges should be considered. Second, they have all the information about the angle. The minimum, average, and standard deviation of the angle, which can be