[Proof]

We shall show the truth by use of the Binet's formula.

$$
\begin{aligned}
&\frac{\phi^n + F_{n-2}}{\phi^2} \\
&= \phi^{n-2} + \phi^{-2}\frac{1}{\sqrt{5}}\{\phi^{n-2} - (-1)^{n-2}\phi^{2-n}\} \\
&= \frac{1}{\sqrt{5}}\{\sqrt{5}\phi^{n-2} + \phi^{n-4} - (-1)^{n-2}\phi^{-n}\} \\
&= \frac{1}{\sqrt{5}}\{(\phi^2 - \phi^{-2})\phi^{n-2} + \phi^{n-4} - (-1)^{n}\phi^{-n}\} \\
&\qquad\qquad (\because \phi^2 - \phi^{-2} = \sqrt{5}) \\
&= \frac{1}{\sqrt{5}}\{\phi^{n} - \phi^{n-4} + \phi^{n-4} - (-1)^{n}\phi^{-n}\} \\
&= \frac{1}{\sqrt{5}}\{\phi^{n} - (-1)^{n}\phi^{-n}\} \\
&= F_n.
\end{aligned}
$$

[Q.E.D.]

Theorem I: the generalised golden right triangle

We generalise the golden right triangle of (the short leg, the long leg, the hypotenuse) by

$$(\sqrt{F_{n-2}}, \phi^{n/2}, \sqrt{F_n}\phi) \quad \text{for} \quad \forall n \geq 1,$$

as shown in Fig. 2. Then the known ϕ-related right triangles, i.e., the Kepler triangle and its kin, are all covered by the formula above.

In case $n = 1$, the Kepler triangle, the golden right triangle, is defined by the set of $(1, \phi^{1/2}, \phi)$.

In case $n = 3$, the silver right triangle is defined by the set of $(1, \phi^{3/2}, 2^{1/2}\phi)$.

In case $n = 4$, Olsen's square-root-three ϕ right triangle (Olsen, 2002) is defined by the set of $(1, \phi^2, 3^{1/2}\phi)$.

We should note that, in case $n = 2$, the triangle is degenerated to the segment of the set $(0, \phi, \phi)$.

We may call the ultimate golden right triangle, as n tends to infinity. Although the set itself is divergent, we can determine the ratios amongst the sides. By use of the Binet's formula, we retain the leading terms of the sides such that

$$
\begin{aligned}
&(\sqrt{F_{n-2}}, \phi^{n/2}, \sqrt{F_n}\phi) \to (5^{-\frac{1}{4}}\phi^{\frac{n-2}{2}}, \phi^{\frac{n}{2}}, 5^{-\frac{1}{4}}\phi^{\frac{n}{2}}\phi) \\
&= (\phi^{-1}, 5^{1/4}, \phi)5^{-1/4}\phi^{n/2},
\end{aligned}
$$

as $n \to \infty$. The set of values in the parentheses above last satisfies the Pythagorean theorem, because

$$\phi^2 = \sqrt{5} + \phi^{-2}.$$

[Proof]

Due to Eq. (1) of Lemma I, the generalised golden right triangle satisfies the Pythagorean theorem:

$$F_n\phi^2 = \phi^n + F_{n-2}. \tag{2}$$

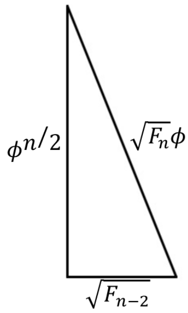

Fig. 2. The generalised golden right triangle (a generic image).

We confirm the uniqueness of the generalisation about the combination between integers and the golden ratio. Suppose the integer sequences a_n and b_n, completely different from the Fibonacci sequence, exist for the generalisation, then these must satisfy the relation such that

$$a_n\phi^2 = \phi^n + b_n. \tag{3}$$

Taking difference between Eqs. (2) and (3), we obtain the result below.

$$(F_n - a_n)\phi^2 = F_{n-2} - b_n.$$

The left-hand side of the equation above is irrational, whilst the right-hand side is integral. Therefore, the equation above holds true, if and only if $F_n - a_n = 0$ and $F_{n-2} - b_n = 0$. That is, $a_n = F_n$ and $b_n = F_{n-2}$. Hence the generalisation is unique.

[Q.E.D.]

3. Conclusion

It is our major success to generalise the definition of the Kepler triangle and its kin by use of the recursion formula of ϕ and the Fibonacci numbers:

$$
\begin{aligned}
&\text{(the short leg, the long leg, the hypotenuse)} \\
&= (\sqrt{F_{n-2}}, \phi^{n/2}, \sqrt{F_n}\phi) \quad \text{for} \quad \forall n \geq 1.
\end{aligned}
$$

This formalism covers all the known ϕ-related right triangles, i.e., the Kepler triangle and its kin. In case $n = 2$, the triangle is degenerated to the segment of the length ϕ. As n tends to infinity, the ultimate golden right triangle is found to have the ratios of the sides:

$$\text{(the short leg, the long leg, the hypotenuse)} \to (\phi^{-1}, 5^{1/4}, \phi).$$

Our definition becomes a nice classroom model to explore the relation amongst the golden ratio, the Fibonacci numbers and the Pythagorean theorem.

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