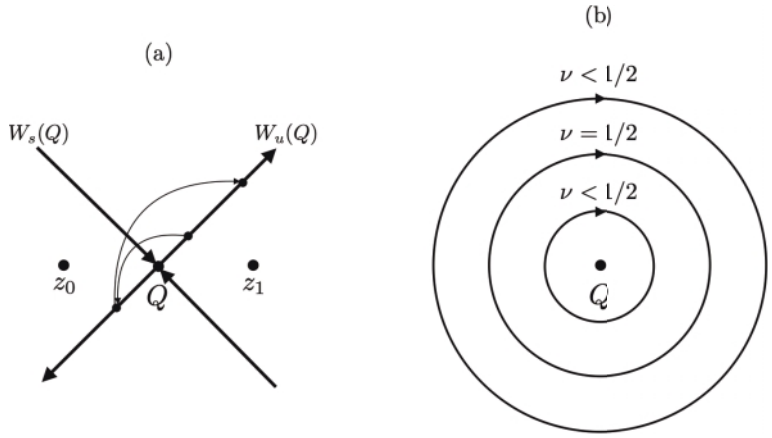


Fig. 1. (a) The situation after the regular period-doubling bifurcation ($z_1 = Tz_0,\ z_0 = Tz_1$). (b) The configuration around $Q$ when the anomalous period-doubling bifurcation happens. Here, $\nu$ represents the rotation number.

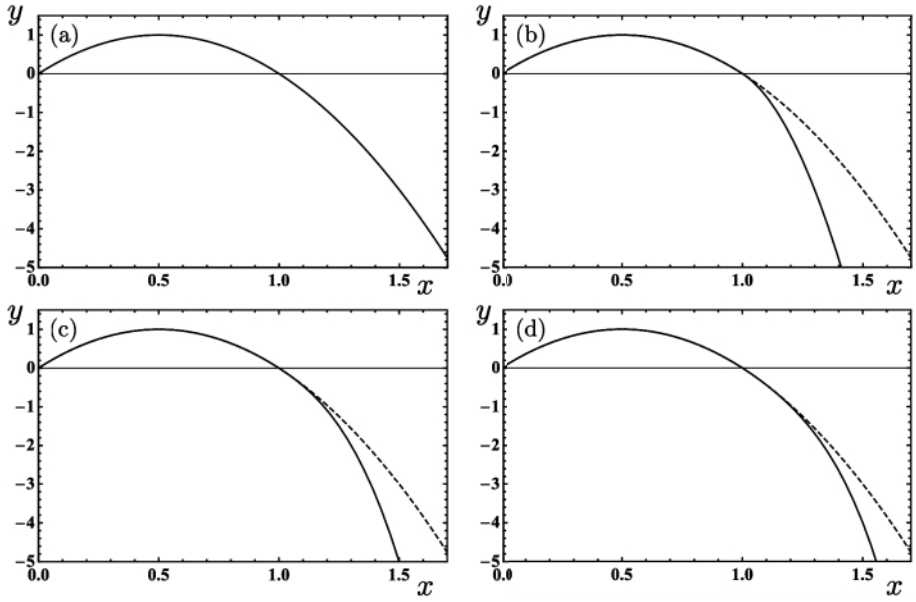


Fig. 2. (a) Analytic function ($b = 0$). (b) $C^1$-class function ($m = 2, b = 16$). (c) $C^2$-class function ($m = 3, b = 16$). (d) $C^3$-class function ($m = 4, b = 16$). The dotted line in the region $x > 1$ represents $y = a(x - x^2)$. $a = a_c^{\mathrm{pd}} = 4$.

## 2. Preliminaries

### 2.1 Properties of the mapping function

The properties of the mapping function $f(x)$ are summarized. The function $f(x)$ with $b = 0$ is analytic and $f(x)$ with $b > 0$ is of $C^{m-1}$-class.

For example, consider $f(x)$ with $m = 2$ and $b > 0$. We have $f_l'(1) = -a = f_r'(1)$ and $f''_l(1) = -2a \neq f''_r(1) = -2a - 2b$. Thus, $f(x)$ with $m = 2$ is of $C^1$-class.

Several mapping functions are depicted in Fig. 2. Fig. 2(a) represents the analytic function. The mapping functions with $m = 2, 3$, and 4 are displayed in Figs. 2(b)–(d). The dotted line in the region $x \geq 1$ is $y = a(x - x^2)$. Here, we increase the value of $m$ at the fixed value of $b > 0$ and observe that the mapping functions accumulate at the analytic one.

### 2.2 Critical value of the period-doubling bifurcation

The first derivative of $f(x)$ is continuous at $x = 1$. Thus, we can use it for the linear stability analysis. The linearized matrix $M_Q$ at $Q = (1, 0)$ is obtained as

$$M_Q = \begin{pmatrix} 1 & -a \\ 1 & 1-a \end{pmatrix}. \tag{3}$$

The determinant of $M_Q$ is 1. This means that the map is area and orientation preserving. The eigenvalues are determined by the following characteristic equation.

$$\lambda^2 - (2 - a)\lambda + 1 = 0. \tag{4}$$

We have the discriminant $D = a^2 - 4a$. The fixed point $Q$ is a stable elliptic point at $0 < a < 4$ and is a saddle point with reflection at $a > 4$. Thus, $a = 4$ is the critical value $a_c^{\mathrm{pd}}$ at which the period-doubling bifurcation of $Q$ happens.