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On Example of Inverse Problem of Variation Problem in Case of Columnar Joint

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Abstract. Taking the columnar joint as example, we illustrate how we solve the inverse problem to the variation problem posed by Nature giving the hexagonal shape to the joint. We notice that the joint is formed by straight line segments, three of which crosses at one vertices. From this geometric information, we characterize a class of evaluation function for the joint. We prove that the subset consists essentially of the functions only in the length. Since the hexagonal shape is known to minimize the length, we see that the hexagonal shape of the joint is a necessary consequence of the geometric information under the assumption that the joint is the solution to the variation problem satisfying several mathematical coditions.

1. Introduction

Variation theory is consisted of a pair (X, *f*) of a set X and an evaluation function *f* on it. The variation problem is to find a point of X minimizing the evaluation function *f* and describe the property of the solution. The inverse problem for the variation problem is in the simplest sense a problem to find out the evaluation function from the property of the solutions of the variation problem. Since the information contained in the solution is not big enough to determine one single evaluation function in general, we need to classify evaluation functions by an equivalence relation to point out one single class of evaluation functions. Thus the inverse problem is understood to introduce an equivalence relation to meet the practical requirement and to find out a method to determine one single class in the equivalence class effectively.

We may consider that a real phenomenon is the solution to a hidden variation problem with a hidden evaluation function. Therefore the inverse problem in practical application is to determine a class of evaluation functions from the properties of given phenomena and to predict further properties of the solution on the knowledge of the evaluation function.

We shall illustrate the philosophy and the practical method the inverse problem as a problem minimizing curves, i.e. the columnar joint formed by the lava when it cools down.

2. Observation of Columnar Joint

Hot and melted lava reduces its volume gradually, when it cools down. Then the lava cracks and produces columns, whose horizontal section shows a pattern of hexagons in typical cases. Together with the hexagonal pattern the columns are called the columnar joint. Photo 1 shows an example in Genbudo area, Tottori Japan.

A question naturally arises why and how the joint is generated. We consider this question as a typical inverse problem of variation problem and try to give a mathematical answer to it.

We may restrict ourselves to the horizontal section of the columns and assume that Nature posed a variation problem with a hidden evaluation function for the plane curves along which the lava cracks. The crack pattern is considered as the solution to the problem from which we seek for the hidden evaluation function.

We notice the following characteristics in the pattern of the plane curves on the section:

1) Mostly (straight) line segments form the pattern.
2) At the vertices, they cross to form sharp wedges.
3) Mostly three lines meet at the crossing at 120 degree.

In the following, we solve the inverse problem using the local property 1), 2) above as the key information and deduce the global property 3) as a necessary consequence from the evaluation function obtained as the solution to the problem under several mathematical condition. We then explain the reason for the hexagonal shape of the pattern mathematically.

3. Characterization of the Curves by Invariant Function

A plane curve $c(t)$ is said to be parameterized by its arc length, if the arc length of $c(t)$ between two points with parameters 0 and t is equal to t, or equivalent to say, $|d(c(t))/dt|=1$.

The curvature $\kappa(t,c)$ of the curve c at the parameter t is invariant under the Euclidean motion group. Conversely if the curves $c(t)$ and $c'(t)$ parameterized by its arc length have the same curvature κ at any parameter t, then they are congruent under the motion group. This is the theorem of Frenet- Serret for the plane curves.

Thus we see that only the curvature and the arc length are the essential parameters for the plane curve from the viewpoint of congruence by motion group, or in other words, as a geometric figure.

For this columnar joint, we may assume that every part of the figure contributes equally to form the crack, hence the evaluation function should be decomposed into the sum of functions defined on each small pieces of the figure. Thus, the evaluation function is represented by an integration of a function of the figure.

The function to be integrated should be an invariant function on the curve and therefore a function in the length and the curvature. We can formulate these facts as in the following theorem:

Theorem 1 *The evaluation function to generate the columnar joint on a curve c is represented as the integration on c of a function $F(t,\kappa(t,c))$ of the length and the curvature κ of c*:

$$E(c)= \int F(t,\kappa(t,c))dt.$$

4. Decision Process of the Evaluation Function

In order to solve the inverse problem and to determine the type of the function $F(t,\kappa(t,c))$, we assume that $F(t,\kappa(t,c))$ is a polynomial in t and $\kappa(t)$.

Since the evaluation function may be assumed additive on each segment of the curve, $F(t,\kappa(t,c))$ should be independent of the length. In fact, consider a line segment L so that the curvature is zero on it and let $G(t)$ be an indefinite integral of F, then, for any division of L into line segments L', L'' in L with length T', T'', we have

$$E(L' \cup L'') = G(T' + T''),$$
$$E(L) + E(L') = G(T') + G(T'').$$

Thus, additiveness implies

$$G(T' + T'') = G(T') + G(T'').$$

This indicates that the integral G is linear and that F is independent of the length. Hence we see the following proposition.

Proposition 1 *The additive evaluation function E(c) on c is represented by an integration of F, which is a function only of the curvature on c*.

As a good approximation, we may assume that the function F is a polynomial of the curvature. We know that the evaluation function $E(c)$ is independent of the orientation of the curves. Since the curvature changes its sign when the orientation is reversed, the polynomial function of the curvature F should be independent of the sign of its argument. Therefore, the function F is a function in the square of curvature, hence we have the following proposition.

Proposition 2 *The evaluation function E(c) on c is represented by the integration of a polynomial function F, which is a function in the square of curvature of c*.

Suppose the highest term of the polynomial F has a negative coefficient. Then the value of F decreases infinitely if the absolute value of the curvature increases infinitely, and the pattern generated by the minimization of the integration of F should consist of curves with high curvature. But, the actual columnar joint does not show the curves of high curvature, thus we may conclude the following:

Proposition 3 *In order to generate the columnar joint as the minimization solution, the highest term of F has to have positive coefficient*.

The argument above is based on the behavior of regular point of curves on the columnar joint. The singular point of the curves also gives important information, which indicates that the order of the highest term of F should be less than or equal to one.

In fact, observation of the columnar joint around a point P where several curves intersect tells us that a curve XY of higher curvature is favorable to minimize the evaluation function F than the curve X′Y′ of lower curvature, (see Fig. 1). Where the notion of the curvature is generalized by using the distribution theory, to the details we return in near future.

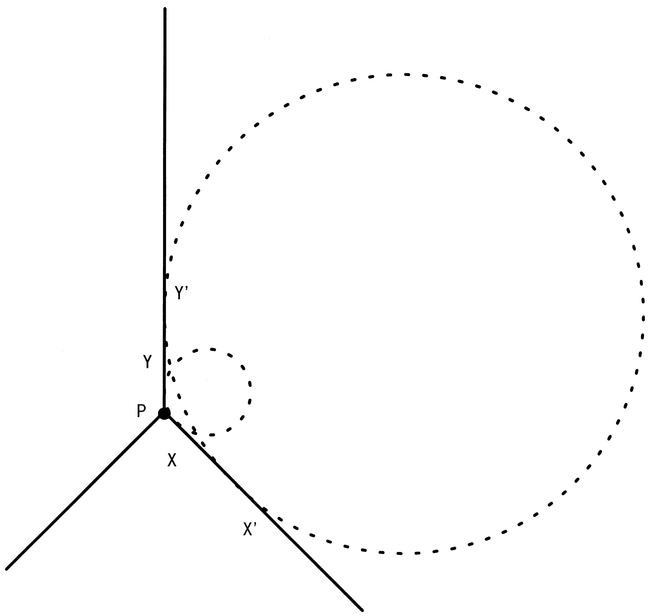

Fig. 1. Small and large circles touching to the angle P at X, Y and X′, Y′, respectively.

Suppose the order of the highest term of F is two, then Schwartz's inequality yields that

$$(\int \kappa(t,c)dt)(\int \kappa(t,c)dt) \le (\int dt)(\int \kappa(t,c)\kappa(t,c)dt).$$

Since curvature is expressed as the differential of the tangent vector, the integral of left-hand side is equal to the difference of the tangent vectors at the starting point X and the end point Y of the curve XY, which has the angle∠XPY. Therefore, F tends to infinity, if the length of the curve XY is made small. But in the case of the columnar joint, the curve XY of smaller length is much more frequent, which means lower F. Thus we deduce a contradiction, indicating that the highest term of F can not be two.

A successive application of Schwartz's inequality yields that the order of the highest term of F can neither be two to power n for any positive integer n, for F to be an evaluation function of columnar joint. Then a standard technique to separate the domain of the integration into two domains satisfying $|\kappa(t,c)| \le 1$, $|\kappa(t,c)| \ge 1$ extends the conclusion over natural numbers.

Hence we see that any integer except zero is impossible as the highest order of F to be the evaluation function of columnar joint.

Theorem 2 *The function F for the columnar joint should be constant.*

We notice here that we deduced the above theorem only from local observation of the joint pattern, that is, the curvature at regular point and the attaching shape at the singular point.

5. The Hexagonal Pattern of Columnar Joint

Thus determined evaluation function is nothing but the length function for curves (as a geometric measure, because the sign should be always positive). Hence we can say that the columnar joint is generated so as to make the shortest length.

On this ground, we can deduce the reason for the hexagonal shape in the columnar joint as follows:

Consider three points ABC and take a point P in the triangle ΔABC (Fig. 2). The problem to minimize the sum of the length of curves jointing A to P, B to P and C to P is reduced to the problem to minimize the sum of the distance $|AP| + |BP| + |CP|$, which is given by a point P making all three angles equal:

$$\angle APB = \angle BPC = \angle CPA = 2\pi/3.$$

Hence we can conclude that hexagon in the columnar joint is a necessary consequence of the assumption that it is a solution to the minimization problem of an evaluation function and the local observation of the columnar joint, as is stated at the end of the preceding section.

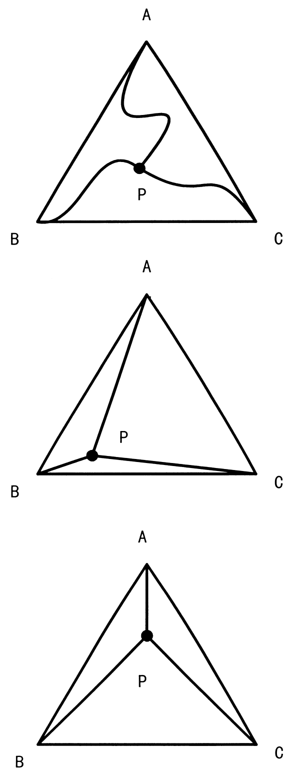

Fig. 2. The total length of P to the vertices is minimized by the line segments (top and middle), then it is minimized by replacing the point P so as to make angle 120 degree to any edge of the triangle (bottom).

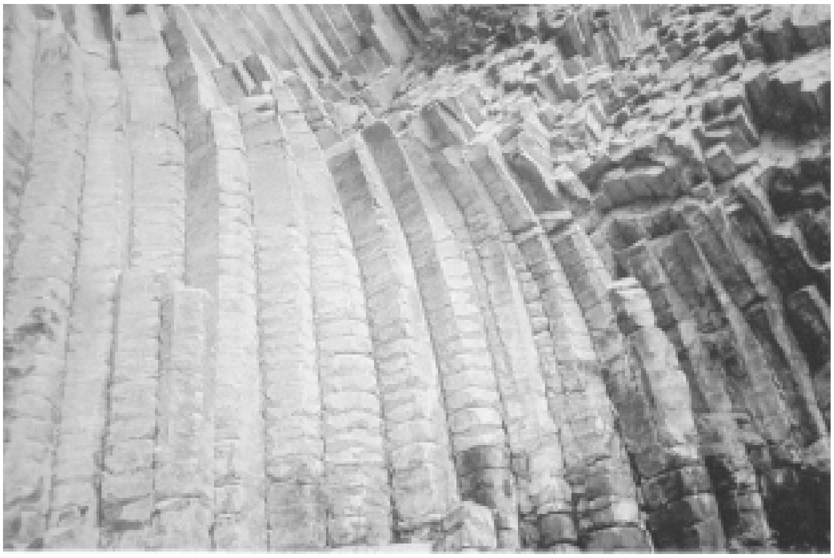

Photo 1. Global view of columnar joint at Genbudo area.

Photo 2. Cross section of a columnar joint.

Photo 3. Close up of a cross section.

The photos we used here are reproduced from the joint work with Suwa *et al*., we are grateful to all the authors, especially Professor Suwa for his favor and encouragement.

REFERENCE

SHIKATA, Y., SUWA, K., YASUNO, M. and HIRAIWA, I. (1991) Columnar joint of Basalt, *1991 Bull. Nagoya Univ. Furukawa Museum*, No. 7, 19–40 (in Japanese).