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On a General Method to Calculate Vertices of *N*-Dimensional Product-Regular Polytopes

Motonaga ISHII

Graduate School of Human and Environmental Studies, Kyoto University, Yoshida-nihonmatu-cho, Sakyo-ku, Kyoto 606-8501, Japan

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Abstract. A set of *N*-dimensional product-regular polytopes (*N*-dimensional PRP-set) can be produced from a regular polytope by taking the product space of *N* solid models which are derived from the boundary figures (vertices, edges, faces, and so on) of the regular polytope.

I found a method to calculate all vertices of the $N(\geq 2)$-dimensional PRP-set derived from an *N*-dimensional regular polytope.

The PRP-set derived from a regular polytope is similar to the set of semi-regular polytopes derived from the regular polytope in point of that both include simple semi-regular polytopes which have the same rotational symmetry with the regular polytope. But the PRP-set differs from the set of semi-regular polytopes because that the PRP-set includes the regular polytope itself, and it doesn't include the snub-type semi-regular polytopes, regular prisms, nor regular anti-prisms, all of which have no same rational symmetry with regular polytopes.

In this paper, I calculated and displayed only 2 to 6-dimensional PRP-sets by using computer.

1. Definitions and Main Result

This paper shows how to calculate vertices of a set of *N*-dimensional product-regular polytopes (*N*-dimensional PRP-set) derived from an *N*-dimensional regular polytope.

Definition 1 *An N-dimensional polytope is defined recursively as follows*;

$$\begin{cases} \text{a point} & \text{if } N = 0, \\ \text{a segment} & \text{if } N = 1, \\ \text{a polygon} & \text{if } N = 2, \\ \text{a polyhedron} & \text{if } N = 3, \\ \text{an } N\text{-dimensional polyhedron which is constructed of} & \\ (N-1)\text{-dimensional boundaries} & \text{if } N \geq 4. \end{cases}$$

Definition 2 *An N-dimensional product-regular polytope* (*say* PRP) *derived from an N-dimensional regular polytope* $\mathscr{A}$ *is defined as follows*:

(a) *The normal of each* $(N-1)$*-space is parallel to some of the vectors which span from the center of* $\mathscr{A}$ *to vertices or centers of the boundaries of* $\mathscr{A}$.

(b) *The length of every edge is constant.*

(c) *The condition around every vertex is regular.*

The set consisting of all PRPs *derived from* $\mathscr{A}$ *is called N-dimensional* PRP-*set.*

Definition 3 *An N-dimensional simple semi-regular polytope is defined as follows*:

(a) *It has the same rotational symmetry with any of N-dimensional regular polytopes.*

(b) *It is not an N-dimensional regular polytope.*

Theorem *Each vertex* v *of every* PRP *can be calculated by*

$$\boxed{v = TT'v_0}\ , \tag{1.1}$$

where $T \in \mathscr{T}_{\mathscr{A}}$ (*see Subsection* 2.1), T' *is the matrix derived from* T (*see Subsection* 2.2) *and* $v_0 \in \mathscr{V}_0$ (*see Subsection* 2.3).
Furthermore for each v_0

$$\{v = TT'v_0 \mid T \in \mathscr{T}_{\mathscr{A}}\} \tag{1.2}$$

is the set of all vertices of one PRP.

2. Complements

2.1. The set of matrices $\mathscr{T}_{\mathscr{A}}$

The set of $N \times N$ matrices $\mathscr{T}_{\mathscr{A}}$ is defined as:

$$\begin{aligned}\mathscr{T}_{\mathscr{A}} = \{T \mid T = (&\boldsymbol{a}_0\ \ \boldsymbol{a}_1\ \cdots\ \boldsymbol{a}_n\ \ \boldsymbol{a}_{n+1}\ \cdots\ \boldsymbol{a}_{N-1}),\\ &(\boldsymbol{a}_0) \subset (\boldsymbol{a}_1) \subset \cdots \subset (\boldsymbol{a}_n) \subset (\boldsymbol{a}_{n+1}) \subset \cdots \subset (\boldsymbol{a}_{N-1})\},\end{aligned} \tag{2.1}$$

where $\boldsymbol{a}_n$ is a vector from the center of $\mathscr{A}$ to the center of each n-dimensional component polytope (say $(\boldsymbol{a}_n)$), provided that each n-dimensional polytope $(\boldsymbol{a}_n)$ belongs to an $(n+1)$-dimensional polytope $(\boldsymbol{a}_{n+1})$.

Each T is equivalent to a symmetrical region of $\mathscr{A}$ on an N-dimensional sphere, and the number of the elements in $\mathscr{T}_{\mathscr{A}}$(say $L_{\mathscr{A}}$) equals to the number of symmetrical regions of $\mathscr{A}$. The number of vertices is not more than $L_{\mathscr{A}}$, because some vertices coincide. The number of vertices is exactly $L_{\mathscr{A}}$ when

$$v_0 = (1, 1, \ldots, 1). \tag{2.2}$$

2.2. Diagonal matrix T'

A diagonal matrix T' is calculated from a given T as:

$$T' = \begin{pmatrix} p_{1,1} & & & 0 \\ & p_{2,2} & & \\ & & \ddots & \\ 0 & & & p_{N,N} \end{pmatrix}, \tag{2.3}$$

$$p_{i,i} = \sqrt{\sum_{n=1}^{N} \left|\overline{T}_{i,n}\right|^2}, \tag{2.4}$$

where $\left|\overline{T}_{i,j}\right|$ is the cofactor of (i,j)-th entry of T.

2.3. The set of vectors $\mathscr{V}_0$

Let

$$\mathscr{V}_0 = \{v_0 = (v_{0,1}, v_{0,2}, \ldots, v_{0,N}) \mid v_0 \neq (0, 0, \ldots, 0), \quad v_{0,i} = 0 \text{ or } 1 \text{ for } i = 1, 2, \ldots, N\}. \tag{2.5}$$

Then, the number of all PRPs derived from $\mathscr{A}$ is

$$\#\mathscr{V}_0 = 2^N - 1. \tag{2.6}$$

If an $\mathscr{A}$ is self-dual, then two PRPs which are derived from a v_0 and from the reverse ordered vector are congruent each other.

3. An Example in 3-Space

3.1. Symmetrical regions

Let a regular octahedron be $\mathscr{A}_1$. The edge-contact polytope is a rhombic dodecahedron $\mathscr{A}_2$, and the point-contact, i.e. the dual, is a cube $\mathscr{A}_3$. By projecting the edges of these three polytopes from the center onto the circumsphere of $\mathscr{A}_1$, the lines of symmetry is derived as in Fig. 1 (COXETER, 1973). All of the intersected points of the lines become all vertices, mid-points of edges, and centers of faces of $\mathscr{A}_1$. These lines, i.e. great circles, divide a sphere into 48 spherical triangles. In other words, the discs each of whose boundaries are the great circles divide the planes which include the origin divede the inner space of the sphere the 3-space into 48 regions.

Generally, an N-dimensional regular polytope $\mathscr{A}$ can be written as:

$$\mathscr{A} = \{\{k_1, k_2, \ldots, k_N\}\}, \tag{3.1}$$

where k_N is the number of $(N-1)$-dimensional polytopes $\{\{k_1, k_2, \ldots, k_{N-1}\}\}$ which compose $\mathscr{A}$ on its boundaries. Then, the $(N-1)$-spaces of symmetry divide the N-space into $L_{\mathscr{A}}$ symmetrical regions;

$$L_{\mathscr{A}} = k_1 \times k_2 \times \cdots \times k_N$$
$$= \prod_{n=1}^{N} k_n. \tag{3.2}$$

There is a one-to-one correspondence between these regions and the matrices in $\mathcal{T}_{\mathscr{A}}$ (refer to Subsection 2.1).

3.2. Setting of a point in a spherical triangle

Put a point P inside a spherical triangle ABC. The centers of $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}_3$ are fixed at the origin O. They can not be rotated nor moved, but can be similarly extended or contracted. If the surfaces of these three polytopes include P which is not on an edge, then the product space of them makes a 26-hedron as in Fig. 2. This 26-hedron is produced by symmetrical reflections, and it constructs a semi-regular octagon (each angle is constant, but there are two kinds of edges) around A according to the property of symmetry. Similarly, it constructs a rectangle around B, and a semi-regular hexagon around C. The shapes around A, B and C are;

$$\text{around A} \begin{cases} \text{a point} & \text{if P} = \text{A}, \\ \text{a square} & \text{else if P} \in \widehat{\text{AC}} \cup \widehat{\text{AB}}, \\ \text{a semi-regular octagon} & \text{otherwise}, \end{cases} \tag{3.3}$$

$$\text{around B} \begin{cases} \text{a point} & \text{if P} = \text{B}, \\ \text{a segment} & \text{else if P} \in \widehat{\text{BA}} \cup \widehat{\text{BC}}, \\ \text{a rectangle} & \text{otherwise}, \end{cases} \tag{3.4}$$

$$\text{around C} \begin{cases} \text{a point} & \text{if P} = \text{C}, \\ \text{a regular triangle} & \text{else if P} \in \widehat{\text{CA}} \cup \widehat{\text{CB}}, \\ \text{a semi-regular hexagon} & \text{otherwise}, \end{cases} \tag{3.5}$$

where, $\widehat{\text{AB}}$ means an arc AB, for example.

3.3. The conditions on the position of P to construct a PRP

In Fig. 2, if the length of every edge is constant, the shape is a PRP which is equivalent to a semi-regular polytope or a regular polytope. The conditions for having the shapes of PRPs are;

$$\overline{\text{PQ}} \neq 0, \quad \overline{\text{PR}} = \overline{\text{PS}} = 0, \tag{3.6}$$

$$\overline{\text{PR}} \neq 0, \quad \overline{\text{PQ}} = \overline{\text{PS}} = 0, \tag{3.7}$$

$$\overline{\text{PS}} \neq 0, \quad \overline{\text{PQ}} = \overline{\text{PR}} = 0, \tag{3.8}$$

$$\overline{\text{PQ}} = \overline{\text{PR}}(\neq 0), \quad \overline{\text{PS}} = 0, \tag{3.9}$$

$$\overline{\mathrm{PQ}} = \overline{\mathrm{PS}}(\neq 0), \quad \overline{\mathrm{PR}} = 0, \tag{3.10}$$

$$\overline{\mathrm{PR}} = \overline{\mathrm{PS}}(\neq 0), \quad \overline{\mathrm{PQ}} = 0, \tag{3.11}$$

$$\overline{\mathrm{PQ}} = \overline{\mathrm{PR}} = \overline{\mathrm{PS}}(\neq 0), \tag{3.12}$$

where, $\overline{\mathrm{PQ}}$, for example, is the distance from P to Q.

3.4. Vertex v

In Fig. 2, let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ be;

$$\boldsymbol{a} = \overrightarrow{\mathrm{OA}}, \tag{3.13}$$

$$\boldsymbol{b} = \overrightarrow{\mathrm{OB}}, \tag{3.14}$$

$$\boldsymbol{c} = \overrightarrow{\mathrm{OC}}, \tag{3.15}$$

and T and v_0 be;

$$T = (\boldsymbol{a} \ \ \boldsymbol{b} \ \ \boldsymbol{c}), \tag{3.16}$$

$$v_0 = (\alpha, \beta, \gamma), \tag{3.17}$$

then,

$$T' = \begin{pmatrix} |\boldsymbol{b}\times\boldsymbol{c}| & 0 & 0 \\ 0 & |\boldsymbol{c}\times\boldsymbol{a}| & 0 \\ 0 & 0 & |\boldsymbol{a}\times\boldsymbol{b}| \end{pmatrix}, \tag{3.18}$$

$$v = TT'v_0 \tag{3.19}$$

$$= (\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}) \begin{pmatrix} |\boldsymbol{b}\times\boldsymbol{c}| & 0 & 0 \\ 0 & |\boldsymbol{c}\times\boldsymbol{a}| & 0 \\ 0 & 0 & |\boldsymbol{a}\times\boldsymbol{b}| \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{3.20}$$

$$= \alpha|\boldsymbol{b}\times\boldsymbol{c}|\boldsymbol{a} + \beta|\boldsymbol{c}\times\boldsymbol{a}|\boldsymbol{b} + \gamma|\boldsymbol{a}\times\boldsymbol{b}|\boldsymbol{c}. \tag{3.21}$$

3.5. Distance from a point to a plane

The next equation is equivalent to Eq. (3.6) as is shown in Fig. 3;

$$\text{P-OAB} = \text{P-OBC} = \text{P-OCA}, \tag{3.22}$$

where, P-OAB, for example, is the distance from P to a plane OAB.

In N-space, let an $(N-1)$-space x be;

$$\boldsymbol{n} \cdot \boldsymbol{x} = K, \tag{3.23}$$

K: a constant.

The distance d from a point ν to an $(N-1)$-space x is;

$$d = \frac{\boldsymbol{n} \cdot \nu + K}{|\boldsymbol{n}|}. \tag{3.24}$$

If an $(N-1)$-space $\boldsymbol{x}$ includes the origin O, then K is 0.

In this case, planes OAB, OBC, and OCA are;

$$\text{plane OAB: } (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{x} = 0, \tag{3.25}$$

$$\text{plane OBC: } (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{x} = 0, \tag{3.26}$$

$$\text{plane OCA: } (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{x} = 0. \tag{3.27}$$

Substitute Eqs. (3.21) and (3.25) into Eq. (3.24);

$$\text{P - OAB} = \frac{(\boldsymbol{a} \times \boldsymbol{b}) \cdot \left(\alpha|\boldsymbol{b} \times \boldsymbol{c}|\boldsymbol{a} + \beta|\boldsymbol{c} \times \boldsymbol{a}|\boldsymbol{b} + \gamma|\boldsymbol{a} \times \boldsymbol{b}|\boldsymbol{c}\right)}{|\boldsymbol{a} \times \boldsymbol{b}|} \tag{3.28}$$

$$= \frac{(\boldsymbol{a} \times \boldsymbol{b}) \cdot \left(\gamma|\boldsymbol{a} \times \boldsymbol{b}|\boldsymbol{c}\right)}{|\boldsymbol{a} \times \boldsymbol{b}|} \tag{3.29}$$

$$= \gamma \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}), \tag{3.30}$$

because $(\boldsymbol{a} \times \boldsymbol{b}) \perp \boldsymbol{a}$ and $(\boldsymbol{a} \times \boldsymbol{b}) \perp \boldsymbol{b}$.

Similarly,

$$\text{P-OBC} = \alpha \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}), \tag{3.31}$$

$$\text{P-OCA} = \beta \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}), \tag{3.32}$$

and,

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}), \tag{3.33}$$

therefore,

$$\text{P-OBC} = \text{P-OCA} \Leftrightarrow \alpha = \beta, \tag{3.34}$$

$$\text{P-OCA} = \text{P-OAB} \Leftrightarrow \beta = \gamma, \tag{3.35}$$

$$\text{P-OAB} = \text{P-OBC} \Leftrightarrow \gamma = \alpha. \tag{3.36}$$

When $(\alpha, \beta, \gamma) = (1, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$, each of them satisfies each of Eqs. (3.6)–(3.12), and it is equivalent to each vector v_0 in Subsection 2.3.

3.6. Vector product in N-space

The extension of 3-dimensional vector product to N-space is shown in IWAHORI (1982) as follows;

$$v_1 \times v_2 \times \cdots \times v_{N-1} = \left(v_{1,1}, v_{1,2}, \ldots, v_{1,N}\right) \times \left(v_{2,1}, v_{2,2}, \ldots, v_{2,N}\right) \times \cdots \times \left(v_{N-1,1}, v_{N-1,2}, \ldots, v_{N-1,N}\right)$$

$$= \det \begin{vmatrix} \boldsymbol{e}_1 & v_{1,1} & v_{2,1} & \cdots & v_{N-1,1} \\ \boldsymbol{e}_2 & v_{1,2} & v_{2,2} & \cdots & v_{N-1,2} \\ \boldsymbol{e}_3 & v_{1,3} & v_{2,3} & \cdots & v_{N-1,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{e}_N & v_{1,N} & v_{2,N} & \cdots & v_{N-1,N} \end{vmatrix}, \tag{3.37}$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ are the fundamental orthonormal vectors in N-space. It needs $(N - 1)$ vectors to produce an N-dimensional vector product. In N-space, we can use Eq. (3.37) as the normal of an $(N - 1)$-space which spans $v_1, v_2, \ldots$, and v_{N-1}.

4. Conclusion

In case of 2-space, all PRPs are regular 2-polytopes. In case of 3-space, all PRPs are regular and simple semi-regular 3-polytopes. In case of 4-space, PRPs are regular and simple semi-regular 4-polytopes.

The vertices of all $N(\geq 5)$-dimensional regular polytopes are already shown (MIYAZAKI and ISHIHARA, 1989), and we can determine vertices of $N(\geq 5)$-dimensional PRP-set according to Eq. (1.1). The 5-dimensional PRP-set derived from the regular $(5 + 1)$-tope is shown in Fig. 4, the 5-dimensional PRP-set from the regular 2^5-tope and (2×5)-tope in Figs. 5 and 6, the 6-dimensional PRP-set from the regular $(6 + 1)$-tope in Figs. 7 and 8, and the 6-dimensional PRP-set from the regular 2^6-tope and (2×6)-tope in Figs. 9–12.

In 2-space, all PRPs are regular polytopes. They include no semi-regular polytopes. On the other hand, in $N(\geq 3)$-space, the PRP-set derived from a regular polytope includes all of regular polytopes and simple semi-regular polytopes derived from the regular polytope. The coordinates of all vertices of a PRP are easily derived from a regular polytope according to Eq. (1.1). Therefore, all vertices of $N(\geq 3)$-dimensional simple semi-regular polytopes can be calculated.

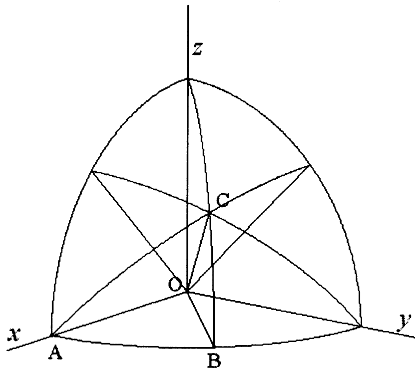

Fig. 1. Symmetrical regions.

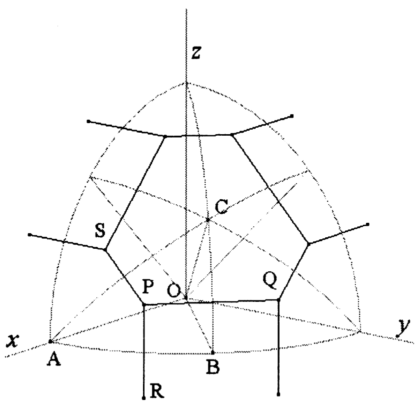

Fig. 2. Vertices P, Q, R, and S of a 26-hedron.

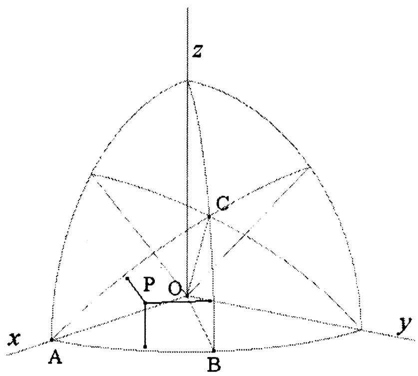

Fig. 3. Distance from P to 3 planes OAB, OBC, and OCA.

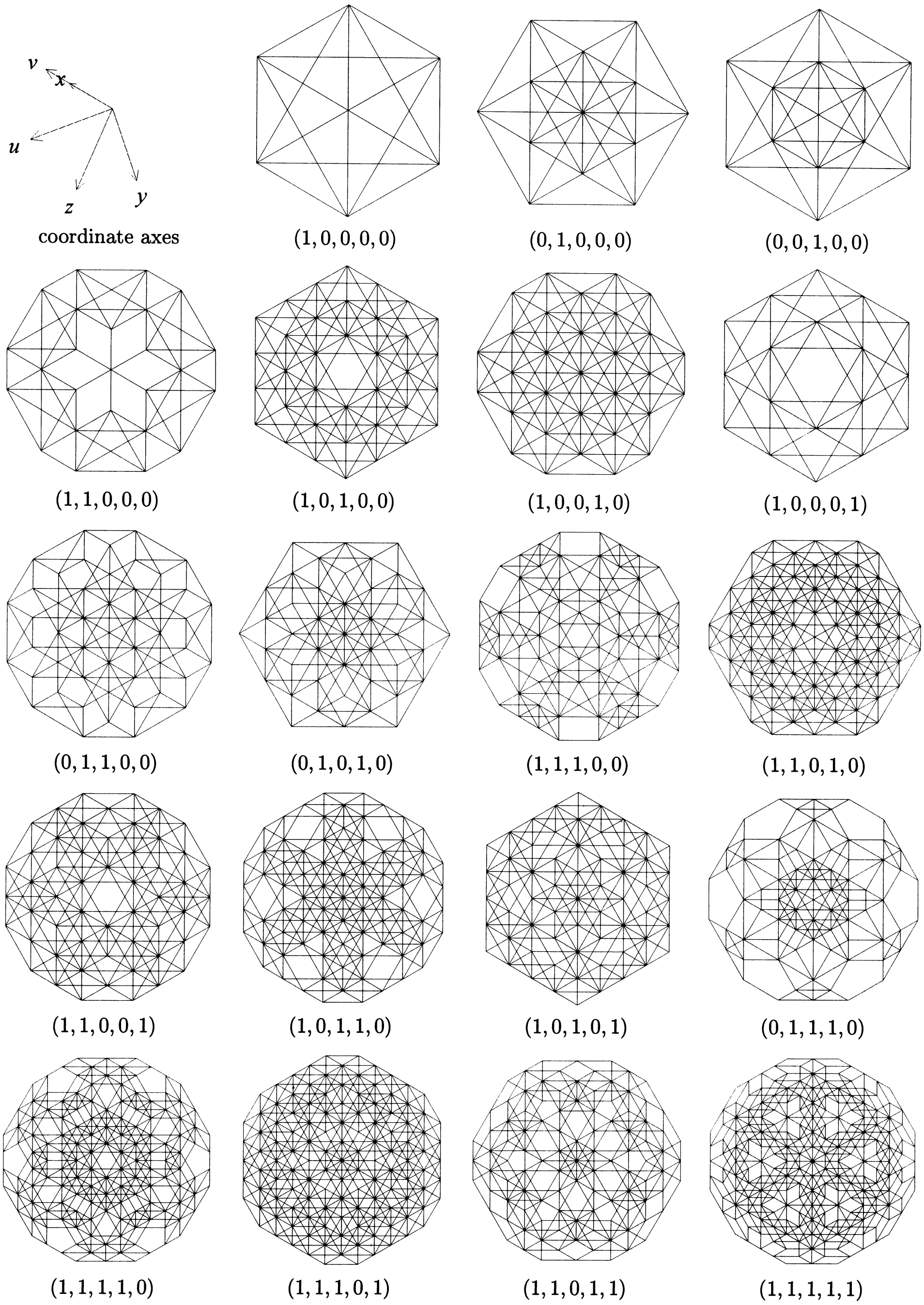

Fig. 4. The 5-dimensional PRP-set derived from a regular (5 + 1)-tope.

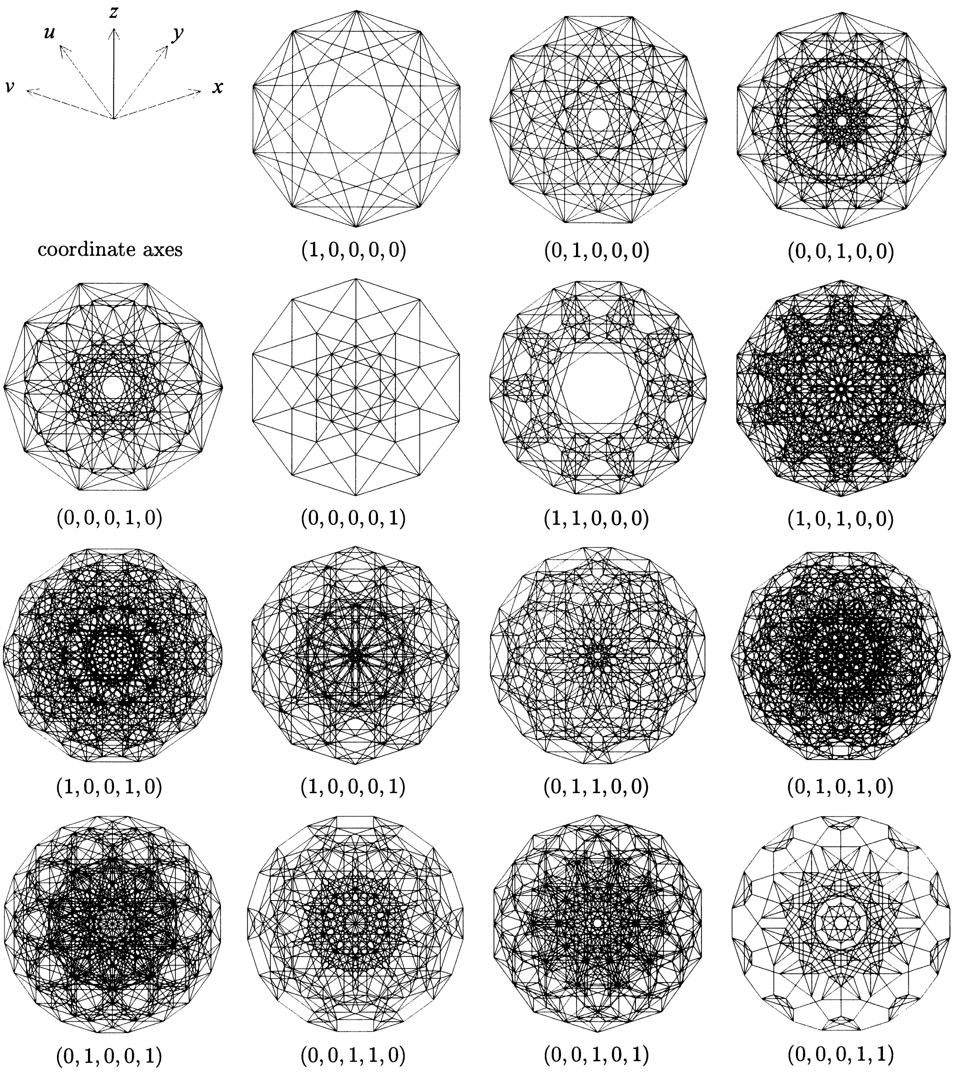

Fig. 5. The 5-dimensional PRP-set derived from a regular 2^5-tope (1/2).

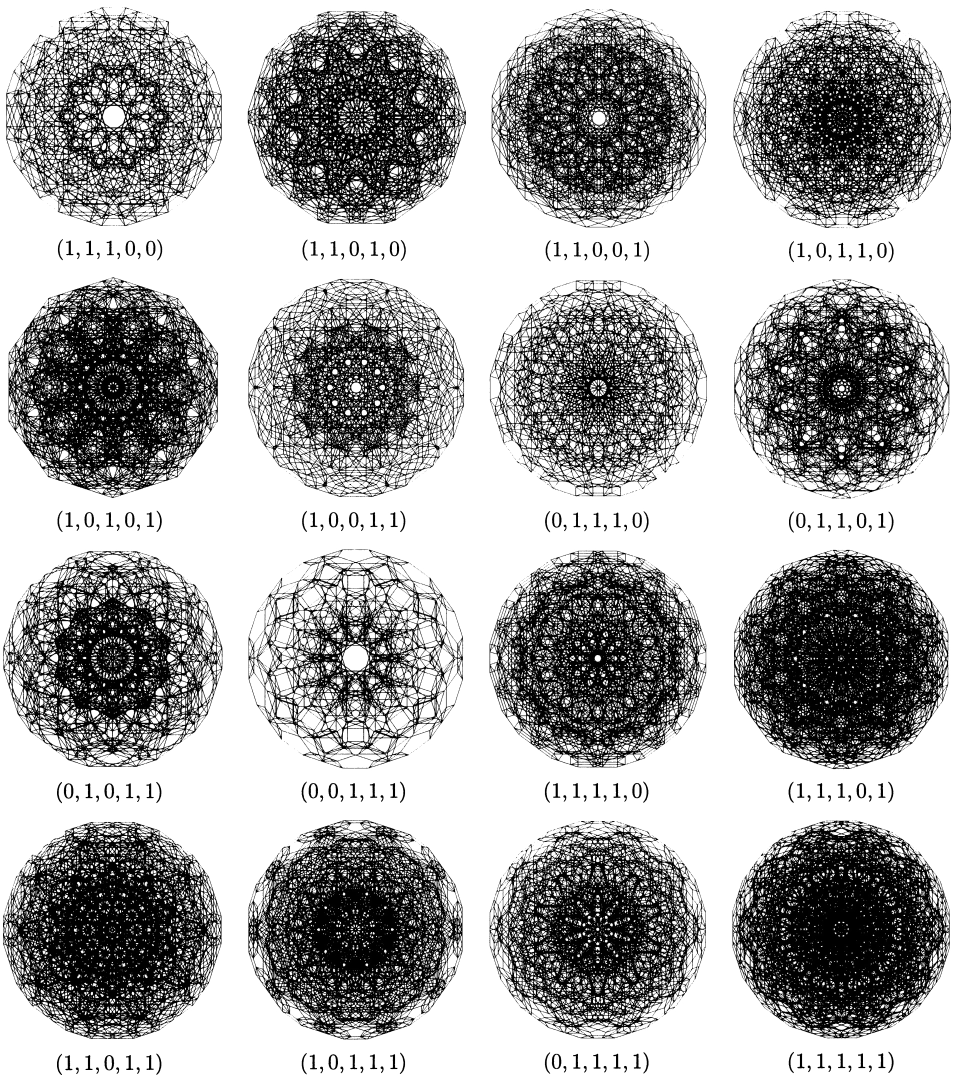

Fig. 6. The 5-dimensional PRP-set derived from a regular 2^5-tope (2/2).

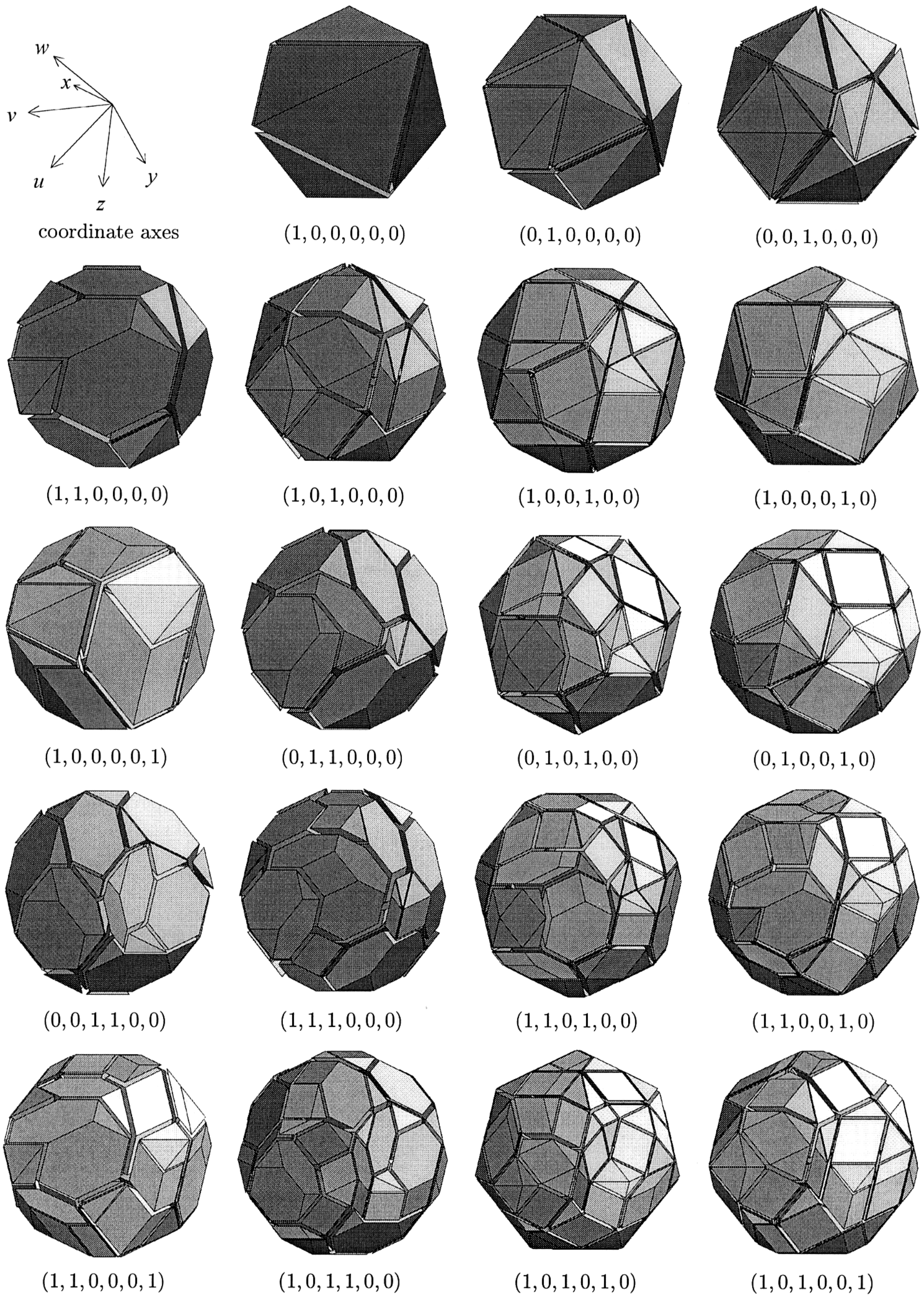

Fig. 7. The 6-dimensional PRP-set derived from a regular (6 + 1)-tope (1/2).

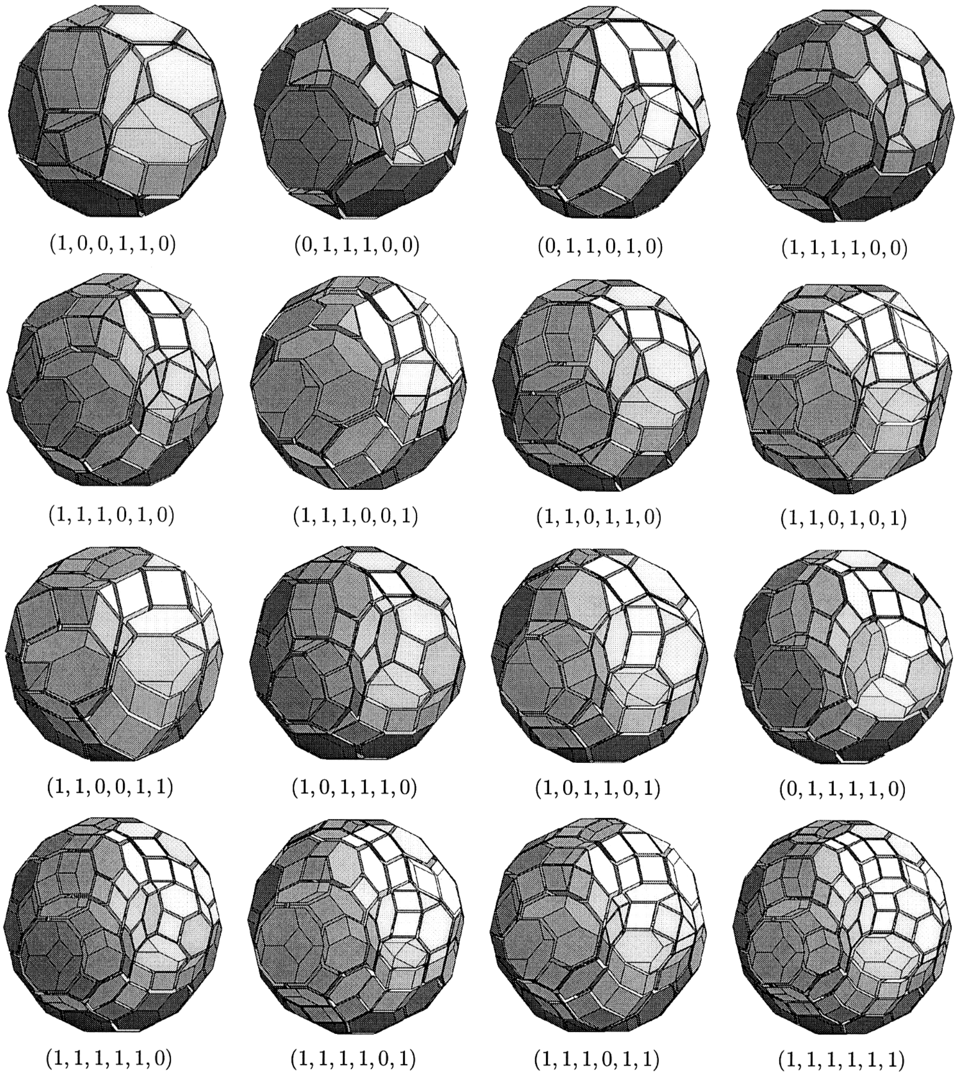

Fig. 8. The 6-dimensional PRP-set derived from a regular (6 + 1)-tope (2/2).

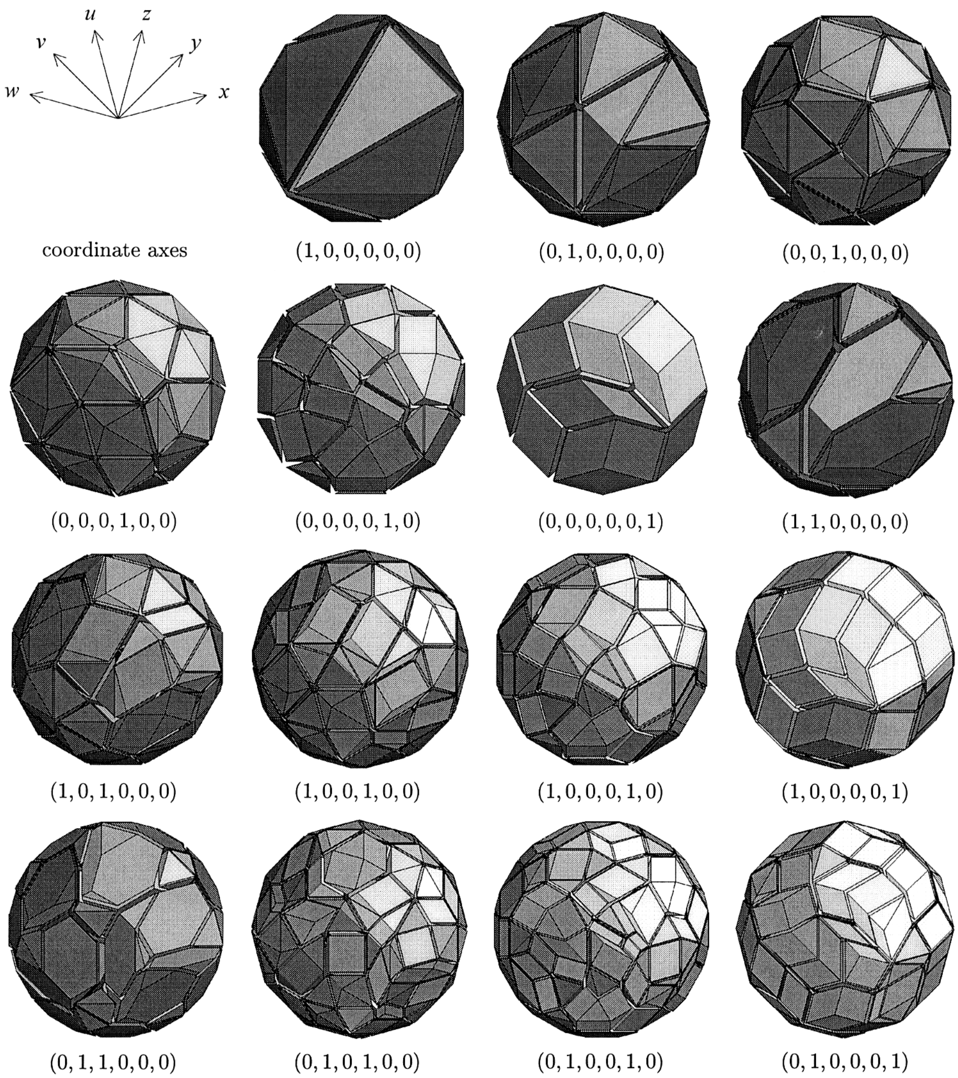

Fig. 9. The 6-dimensional PRP-set derived from a regular 2^6-tope (1/4).

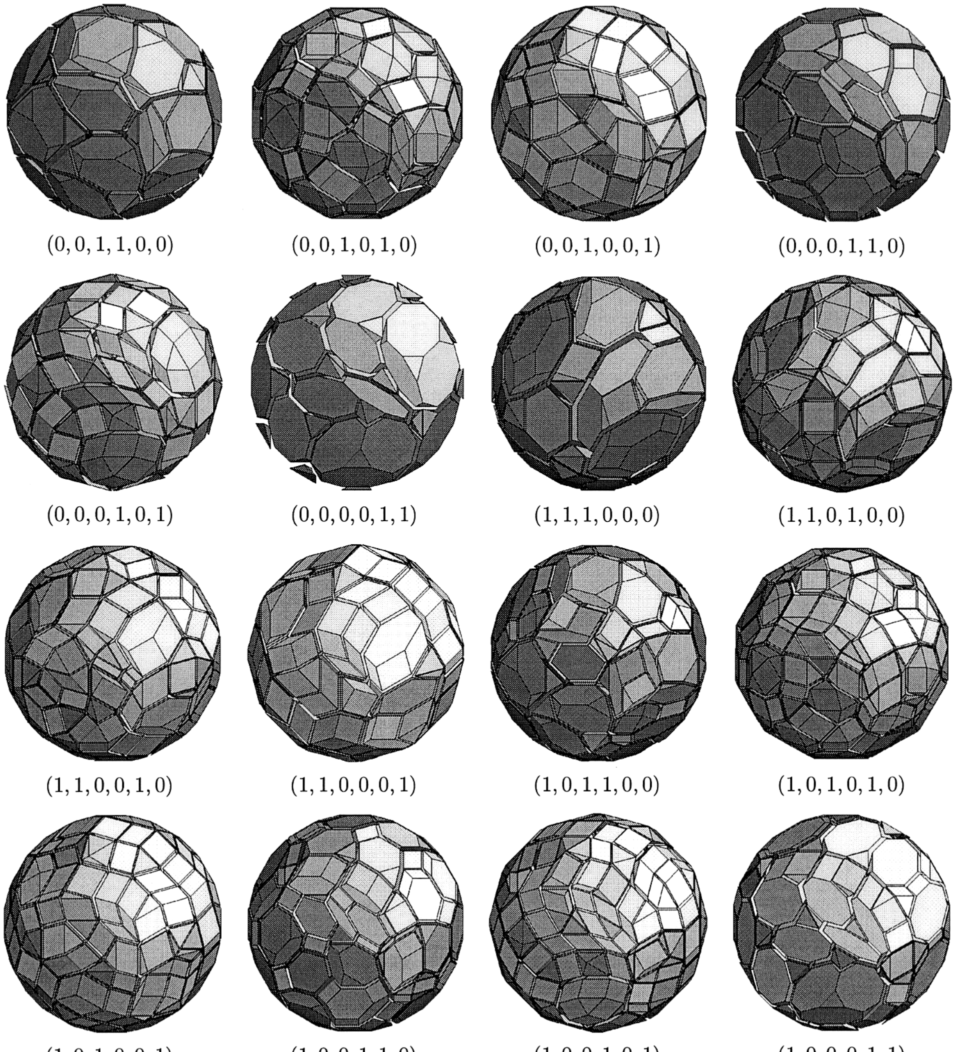

Fig. 10. The 6-dimensional PRP-set derived from a regular 2^6-tope (2/4).

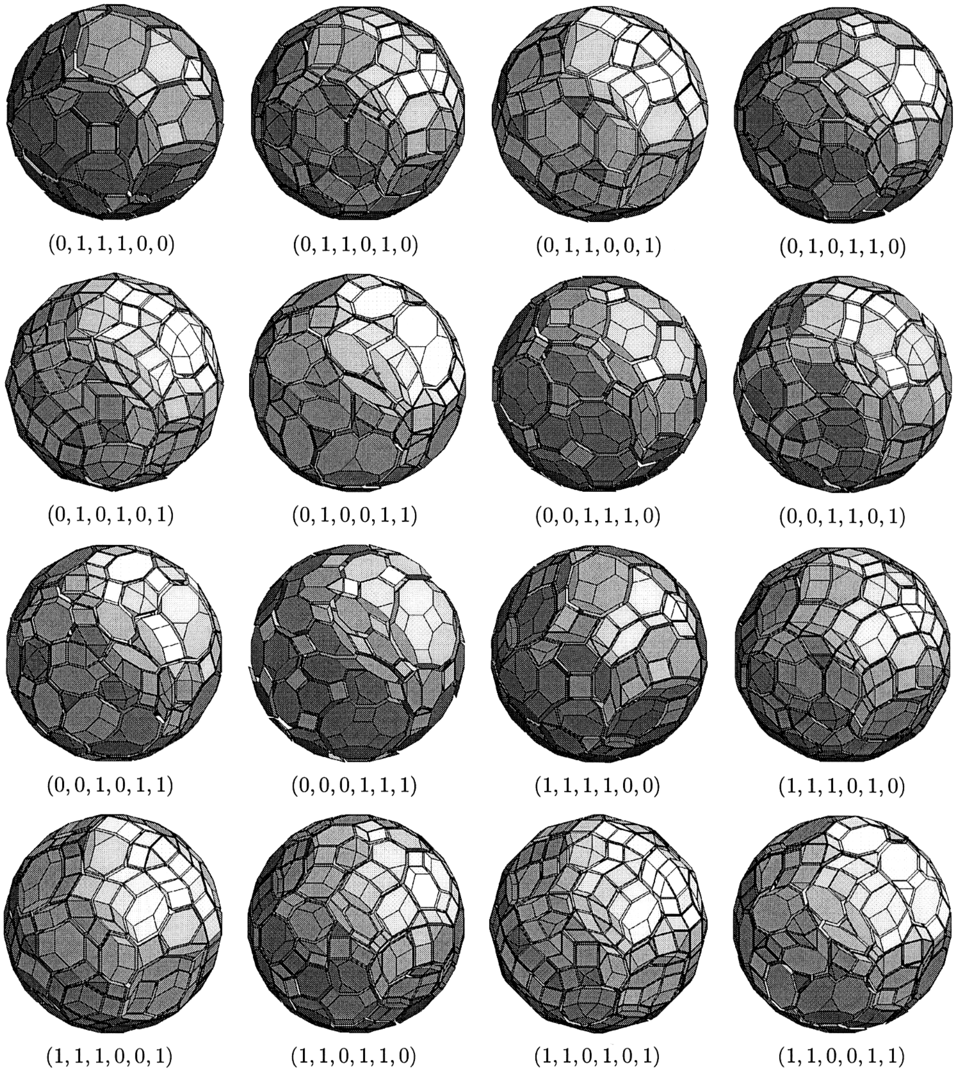

Fig. 11. The 6-dimensional PRP-set derived from a regular 2^6-tope (3/4).

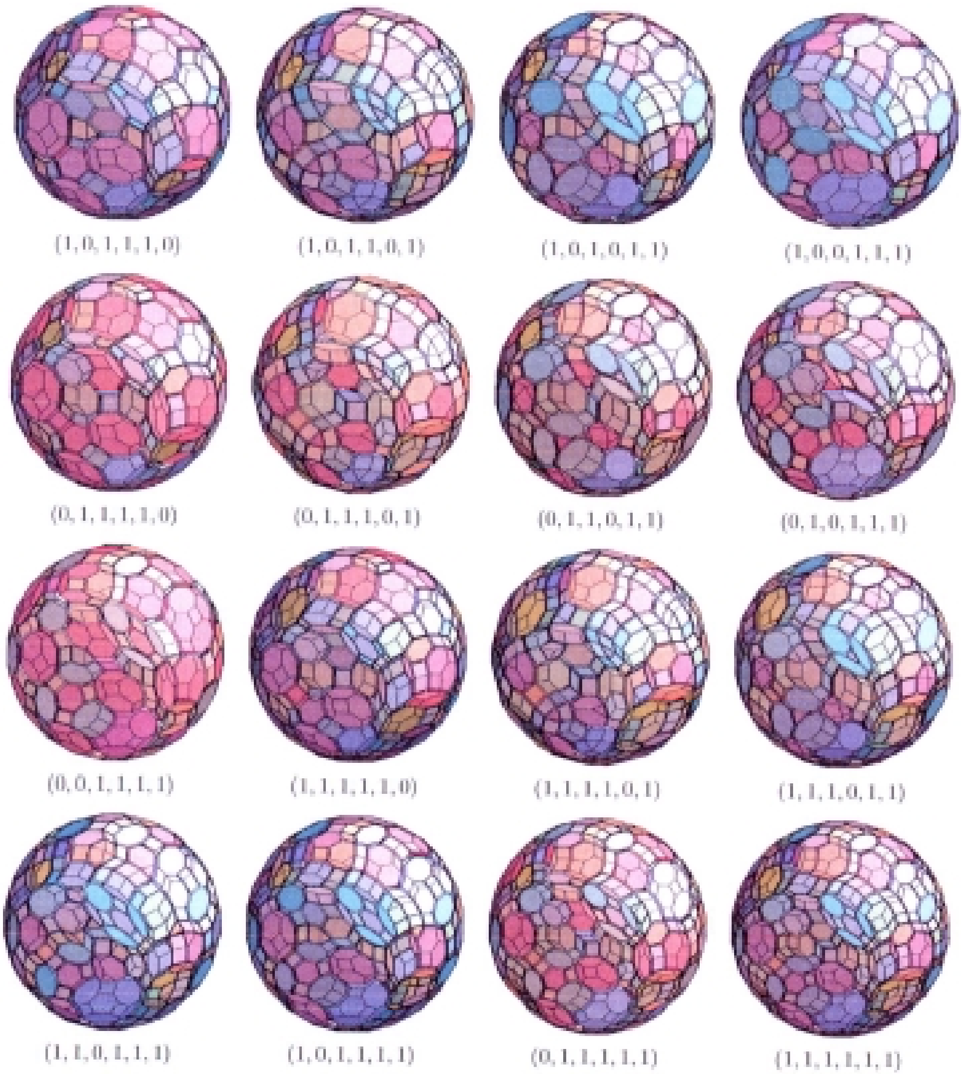

Fig. 12. The 6-dimensional PRP-set derived from a regular 2^6-tope (4/4).

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