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Katachi and Symmetry
Part 2: The Points in Common and the Points of Difference

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Abstract. The expected roles or significance of our philomorphical interdisciplinary activity, *Katachi* and *Symmetry*, is discussed. The following subjects are contained; art and science in organically balanced culture in future, and intellectual delights which are consistent with a noble heart.

1. Introduction

It is the purpose of the paper, Part II, to describe the underlying philosophy of organizing the symposium. In Part I in the proceeding of KUS (OGAWA, 1996), the following subjects are emphasized: basic principle in interdisciplinary cooperation and in intercultural cooperation, international standards, diversity of culture, the extension of science, peace, etc. This paper is mainly devoted to the explanation of the viewpoint of Katachi.

Kepler's model of solar system is used as the warp of the paper. The model is described in Sec. 2. He introduced the model to explain the radii of six planets from the natures of five Platonic solids. Extension of the model is proposed and solved in Sec. 3. Symmetry is powerful as a guiding principle to solve the problem as in sciences. The problem of truth (science) and that of beauty (esthetics) are interrelated but different with each other. The solution of beauty is not unique but multiple. Personal view of the author on the necessary element of beauty, grasping eyes and worth long watching, is described in Sec. 4. The mathematical beauty, easier than beauty itself, is discussed in Sec. 5. Another important element, ethical aspects, is discussed in Sec. 6. Concluding remarks is in Sec. 7.

2. Kepler's Model of Solar System

In 1596, German astronomer Johannes Kepler (1571–1630) proposed a geometrical model (KEPLER, 1596) in order to explain systematically the size of orbits of all of six planets known in that period; Mercury, Venus, Earth, Mars, Jupiter, and Saturn. It is well known that the number of the regular polyhedra, so-called Platonic Solids is just five; tetrahedron, hexahedron (i.e. cube), octahedron, dodecahedron, and icosahedron.

The fact that the number of known planets was six was convenient for him to introduce the polyhedral model. He assigned five Platonic solids properly to five intervals between the six orbits so that the two radii of orbits, larger and smaller correspond to those of two spheres, circumscribing sphere and inscribing sphere of a properly chosen regular polyhedron. Figure 1 shows a part of the 2D-correspondence. If the idea explains the six radii well and then it also explained that the number of planets should be six at the same time.

The data of five Platonic solids are summarized in Table 1. The ratio of inscribed radius to circumscribed radius for five Platonic solids are 1/3 = 0.3333, $1/\sqrt{3}$ = 0.5774, $1/\sqrt{3}$ = 0.5774, $\sqrt{(4\tau+3)/15}$ = 0.7947, and $\sqrt{(4\tau+3)/15}$ = 0.7947 respectively for tetrahedron, hexahedron, dodecahedron, and icosahedron. It is noted that the values for mutually dual ones are the same as generally proved easily. Therefore the possibility of the assignment is reduced to only $5!/2^2 = 30$ among $5! = 120$ possibilities from our modern viewpoint. But Kepler based on some metaphysical idea including perfect circular orbits and the reasoning of Platonic solids. Kepler made a paper model by the above-mentioned idea and presented it as a gift to Friedrich in Wurzburg in 1596. G. Gruppenbach made a model basing on the paper model in 1597. What we look sometimes as the Kepler's model (Fig. 2) is the picture of the last one.

Watch the way of the arrangement of polyhedra in Fig. 2: For example, the direction of tetrahedron. It is interesting to note that the tetrahedron is placed in the most stable way as if the model is under the action of gravity though the model is that in the Copernican system.

Anyway, he withdrew the model by himself. The polyhedral model was necessary for him as a step and finally he arrived at his three lows including the elliptic planetary orbits. Kepler is the first scientist who took the orbit other than circle and gave up the undoubted

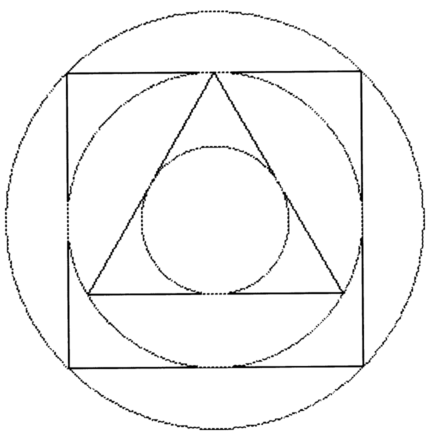

Fig. 1. 2D correspondence of the Kepler model of solar system.

Table 1. Kepler's Model of Solar System.

Planets	Orbit radius C	Orbit radius K	Orbit radius P	Ratio of adjacent ones (inner / outer) C	Ratio K	Ratio P	Kepler's assignment of polyhedra to five intervals
Mercury	0.36	0.43	0.39				
				0.50	0.56	0.54	Octahedron 0.5774
Venus	0.72	0.76	0.72				
				0.72	0.76	0.72	Icosahedron 0.7947
Earth	1.00	1.00	1.00				
				0.66	0.69	0.66	Dodecahedron 0.7947
Mars	1.52.	1.44	1.52				
				0.29	0.27	0.29	Tetrahedron 0.3333
Jupiter	5.24	5.26	5.20				
				0.57	0.57	0.54	Hexahedron 0.5774
Saturn	9.16	9.16	9.55				

a : the edge length
r : the radius of the inscribing sphere
ρ : the distance of the mid-edge from the center
R : the radius of the circumscribing sphere

$$\frac{1}{2}\sqrt{\frac{11\tau+7}{5}}=1.1135,\quad \frac{\tau+1}{2}=1.3090,\quad \frac{\sqrt{3}\tau}{2}=1.4013,$$

$$\frac{\sqrt{3}(\tau+1)}{6}=0.7586,\quad \frac{\tau}{2}=0.8090,\quad \frac{\sqrt{\tau+2}}{2}=0.9511,$$

where $\tau \equiv (1+\sqrt{5})/2=1.6190$ is so-called golden number.

C: the values by Copernicus
K: the values by Kepler
P: the values at present

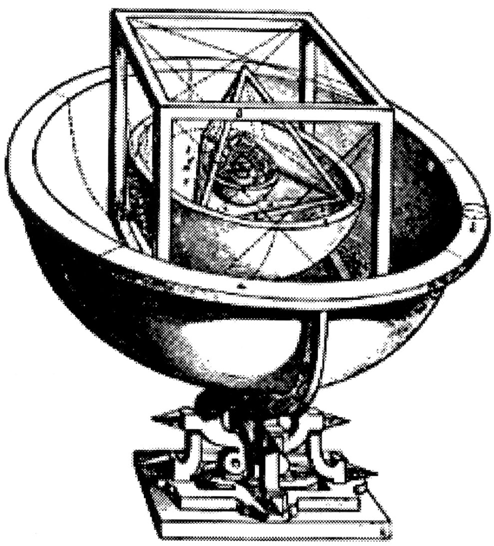

Fig. 2. Kepler's model of the solar system.

principle to introduced elliptic orbits (KEPLER, 1609). It is very interesting to follow the tracks of his mind and thought.

3. Extension of Kepler's Model

Therefore the spheres are not leading part any more. Nowadays, nobody believes the model in the original sense. The role of spheres is no more main one. What is the most proper modification of the model now? Let us amuse ourselves with the extended Kepler's geometrical model where spheres play secondary roles. What is the best order of five regular polyhedra among all the possible concentric Platonic solids? What are their relative directions? The following requirements are assumed as the rule of a game.

3.1. Rule of a game

1. All of the five platonic solids are used once and only once.
2. They are placed so that their centers coincide with each other.
3. The relative size of two adjacent polyhedra is decided so that a sphere circumscribes the outer polyhedron and inscribes the inner polyhedron as in the original Kepler's model.
4. The spheres are not drawn since their roles are secondary.

3.2. Solution for the game

The number of possible combinations of two adjacent Platonic solids is $5 \times 4 = 20$ if any constraint is not imposed. Now some guiding principles to the solution are introduced.

The relationship of two adjacent polyhedra is not visible unless the vertices of the inner one touch the surface of the outer one so long as the sphere is not drawn. Therefore it is assumed that *some* of the vertices of the inner one touch at the center of some surface of the outer one (the weak assumption). The number of the possible pairs is 10 shown in Table 2 and Fig. 3. There are 10 solutions, which consist of these pairs, for the present game as shown in Fig. 4. If the assumption is strengthen so that *all* of the vertices of the inner one touch at the center of some surface of the outer one, the number of the possible pairs is reduced to seven and the number of the solution is only one.

Table 2. Possible combinations of two Platonic solids.

	Inner (-hedron)	Outer (-hedron)	r_i/R_i	r_0/R_0	$r_i r_0/R_i R_0$
1	4	8	0.3333	0.5774	0.1925
2	4	20	0.3333	0.7947	0.2649
3	6	8	0.5774	0.5774	0.3333
4	6	20	0.5774	0.7947	0.4589
5	8	6	0.5774	0.5774	0.3333
6	12	20	0.7947	0.7947	0.6315
7	20	12	0.7947	0.7947	0.6315
(8)	6	4	0.5774	0.3333	0.1925
(9)	12	4	0.7947	0.3333	0.2649
(10)	12	8	0.7947	0.5774	0.4589

Now, the solution is uniquely obtained under the requirements. How mighty is the power of symmetry principle! It can be regarded as a sort of scientific way. Principle of this kind can be understood by anybody and very persuasive. It is successfully applied evenly anytime, anywhere, and by anybody. Success of the Western culture may be stated as the success in the formulation in such style. Only the most essential thing is taken in the first scientific models, by ignoring the minor things. The models are revised later by taking the neglected effects. Then the scientific truth is always restricted with expectation of asymptotically approach to the facts. To the author, the importance of symmetry seems as similar to the science in this sense. The powerfulness and the importance of symmetry of them seem inseparably connected.

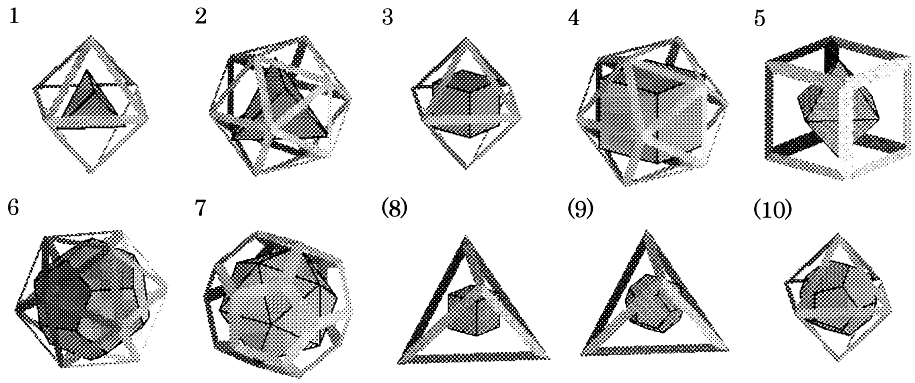

Fig. 3. Possible pairs of adjacent polyhedra. The last three among them are excluded by the argument in the second stage.

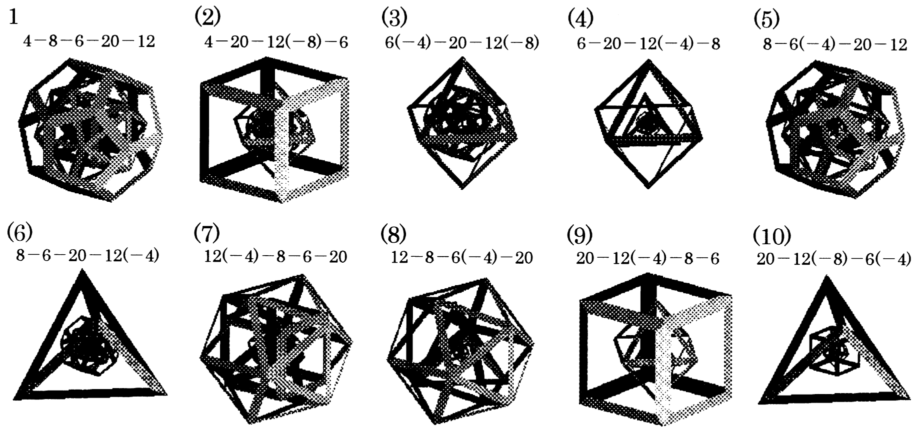

Fig. 4. Ten solutions of the game. Only the first among the ten fills the strong requirement.

4. Beauty and Clarity

The above-mentioned solution is at least one of the reasonable solutions. However, it is harder to say the most beautiful arrangement within the condition. There must be more beautiful arrangements though it is difficult to find and it is furthermore difficult to prove which is more beautiful. The author personally thinks that the symmetry in a narrow sense is not the most important element of beauty for esthetic sense in Japan. There are many Japanese designs that are symmetrical. In many of them, however, the most important is not the symmetrical frame structure itself but other elements worth long watching; balance, harmony, pattern-background relationship, elaboration, sophistication, ambiguity, uniformity in some extended sense, etc. See the patterns in Fig. 5, the collection of seventeen kinds of wallpaper patterns by a mathematician Urabe, participants of this symposium, are very attractive to the author from this point of view. They are in his web-site (URABE, 1999) and consist of Japanese patterns, most of that are rather popular.

The author regards something additional to symmetry in a narrow sense here as an important element of *Katachi*. He thinks that the similar aim lies in the scope of symmetry interdisciplinary movement too. He also thinks that symmetry itself is not the starting point in designing in Japanese thinking, but Japanese way might be able to contribute to the extension of symmetry concept.

5. Mathematical Beauty

Mathematical beauty may be easier to discuss than artistic beauty. Golden section is beautiful at least mathematically besides the beauty of artistic side. Golden ratio is not

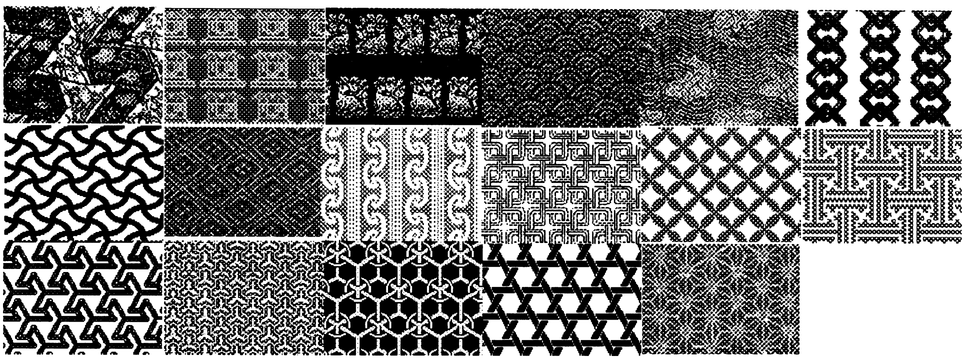

Fig. 5. Seventeen kinds of wallpaper patterns in Japanese design collected by Urabe (URABE, 1999) and rearranged by the author. The most of them are traditional Japanese designs. Only the exception, a modern Japanese design, is at the very left in the bottom row. Original figures are colorful. Upper row; p1 (*Bingata*), pm (*Yakata-Mon*), pg (*Hana-Usagi-Kinran*), cm (*Seikaiha*), p2 (*Ariso-Donsu*), pmm (*Yoshiwara-Tsunagi*), middle row; pgg (*Fundo-Tsunagi*), cmm (*Takeda-Bishi*), pmg (*Shikan-Jima*), p4 (*Rokuyata-Koshi*), p4m (*Sippo-Tsunagi*), p4g (*Sayagata*), bottom row; p3, p31m (*Bisyamon-Kikko*), p3m1 (*Kasane-Kikko-ni-Wa*), p6 (*Kagome*), p6m (*Asanoha*).

merely a ratio but it is reproduced or propagates. There are the cases in which a certain special features appear only at some special value of parameter

5.1. View axis for a cube

Are there any special directions to watch a cube? Easiness to understand and beauty are both desirable. Some examples of view of a cube are shown in Fig. 6. The view get flat and the information about the depth is reduced, if the view axis coincides with the symmetry axis as in case 1 and case 2. View axes in case 2 and case 3 are related with golden ratio and may be beautiful though the information about the depth is also lost in case 2. The case 4 is more three-dimensional; it is from the axis (β, α, 1) where α and β is the length of minor and major diagonal of regular heptagon respectively. The set of numbers 1, α, and β fill the following mathematically beautiful relations; any product of α and β can be expressed as the linear combination of 1, α, and β with integral coefficients. They can be regarded as a generalization of those for golden ratio (OGAWA, 1994)
$\alpha^2 = 1 + \beta$, $\alpha\beta = \alpha = \beta$, $\beta^2 = 1 + \alpha + \beta$. Unit vector ($u$, v, w) ≡ (.7370, .5910, .5280) is parallel to (β, α, 1). It makes a set of Cartesian coordinate together with ($-w$, u, $-v$) and ($-v$, w, u), both of which are expressed only by similar type of coordinates.

In all of the figures of the extended Kepler model in Sec. 2, the view axis of case 4 is adopted since case 3 just coincides with two-fold axis of dodecahedron and icosahedron.

5.2. Geometry of the beauty: Cooperation with an artist

In 1998, the author visited an attractive exhibition of a painter K. Enomoto in Tokyo (ENOMOTO, 1998). Her works were motivated by *Melencolia I* by Albrecht Dürer (1471–1528). She analyzed numerically the truncated rhombohedron in *Melencolia I* and reached the idea of a kind of hierarchical structure. She made a lot of pieces due to the idea with various materials. Her aim was not to ascertain the Dürer's original idea as a historical investigator, but to find the most attractive case for herself as an artist.

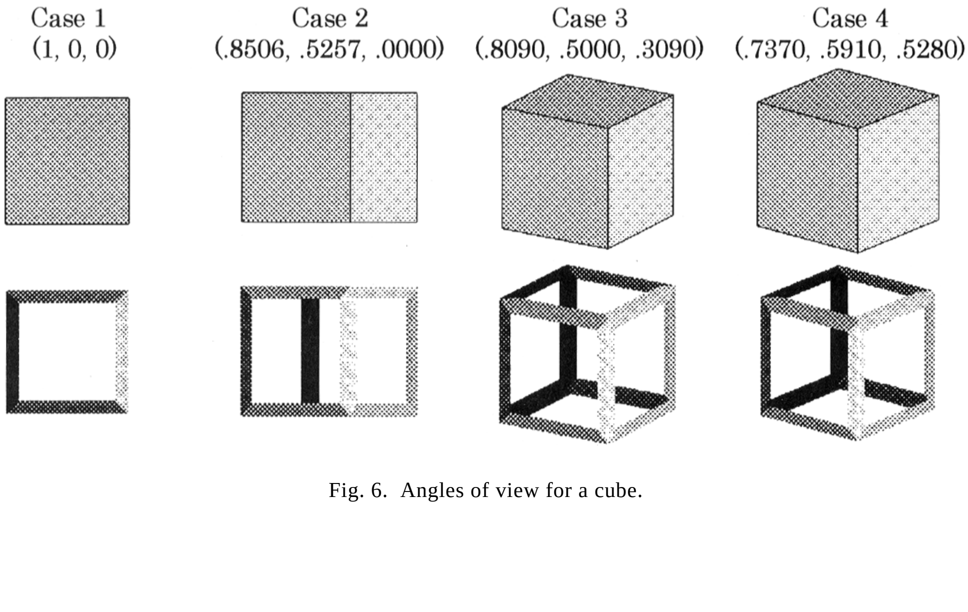

Fig. 6. Angles of view for a cube.

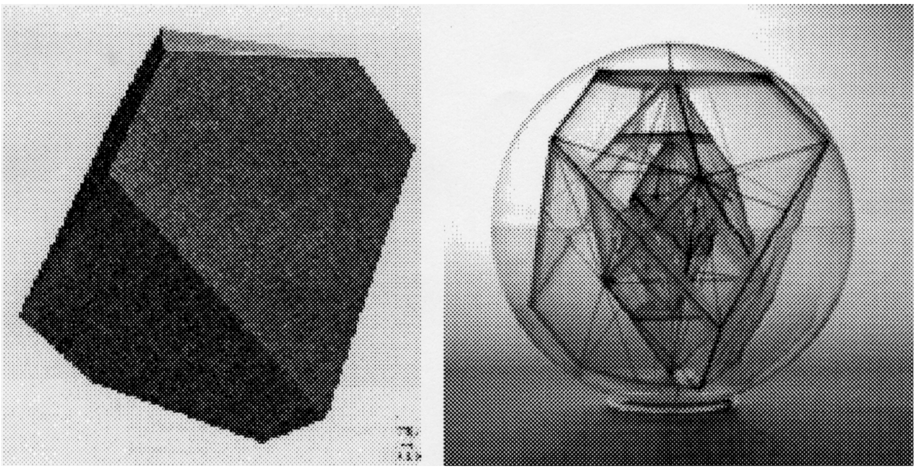

Fig. 7. From “Infinite Vision: The Octahedron” by K. ENOMOTO (1998).

Starting with rhombi decided by 72° and 144°, the rhombohedron is constructed so that the angles at the vertices of rhombi at the both ends of symmetry axis are 72°. The truncations are performed so that final polyhedron has circumscribing sphere. Then, an original rhomboheron of small scale can be inscribed to the truncated rhombohedron. Truncation and scale-down can be theoretically repeated infinite times to have hierarchical structure. Video of CG of her piece (NAKAMURA, 1999) was presented at the Coinciding Exhibition of KUS2.

Her attempts stimulated the author to study the problem of geometry of the beauty in mathematical sense. Systematic or analytical study of rhombohedra and truncation (OGAWA, 1998, 1999) lead to the conclusion that her result should be revised to as follows. The angle 72° should be read as $72.53° = \cos^{-1}(3/5)$. Her numerical estimation should be regarded as surprisingly well. The value of a quantity corresponding to $\sqrt{3-\tau}/2 = 0.587785$ is equated to the value of another quantity corresponding to $\sqrt{29-16\tau}/3 = .587977$ in her numerical evaluation. It is regarded as reasonable.

A mathematical fruit of the cooperation is described. Take any rhombohedon A. A rhombohedron has a three-fold symmetry axis. The 3/5-miniature of A is referred to as B. B is placed inside A so that the symmetry axes coincide and the six vertices of B except two on the symmetry axis contact on the diagonals of six rhombic surfaces of A. It is possible for any A. The contact point divides a diagonal into three and two.

The author wishes to cooperate successfully with artists so that $1 + 1 > 2$ and both sides are really satisfactory.

6. Ethical Aspect of Katachi and Symmetry

In forgoing sections, truth and beauty (in other words, Science and Esthetics) were discussed with the extension of the Kepler’s model and other geometrical topics as the subject. Goodness (in other words Ethics) is the remaining one.

Human genome project is going on. From the viewpoints of curiosity and an inquiring mind, it is interesting to the author. He, however, also feel some danger. Such an analysis may reveal the distance between such and such races. What kind of effect does such scientific knowledge bring? Some different races may feel familiar to each other and other different races may dislike each other basing on their cultural relationship. He regards it as rather natural and reasonable. But new information may unnaturally and unreasonably affect their relationship. Not only art and science relationship but also relationship with ethics should be also included in the scope of our cooperation. Ethics may be rephrased as constructing the logic connecting culture and human society.

It is heard that advanced nations collect genetic resources from developing countries. Whom the ownership of such resources should belong to? What is the best social system? This kind of problems are also one of the *Katachi* problems; *Katachi* of systems.

Nowadays, everybody knows that economy cannot grow endlessly and that resources are finite. Before, people may be able to wait till one's turn comes in some day. Then the problem of allocation has been avoided. Now importance of the allocation problems of resources grows year by year. What is the most fair or the most even (symmetrical to people or area) allocation? The allocation problem is not only quantitative but there is qualitative side too. Where and where should finite number of medical apparatus, for example, artificial dialysis equipment, be placed? It is one of the most important realistic problems of bio-ethics. Another important problem is to find the best mesh, as uniform and as locally isotropic as possible, for a spherical surface or for that with some thickness. Such mesh is necessary to simulate global problems with high accuracy. The maximum number of the exactly equivalent points on a sphere is 120 though the corresponding arrangement has anisotropy. The corresponding number of isotropic case is only 20.

Thus, the arrangement problems have contemporary importance. Note that geometry in this sense does not necessarily belong to mathematics in a narrow sense, where nothing is above proof, but to a branch of science, where geometrical phenomena are accepted as facts. Such an extended geometry is important in the Katachi-Symmetry cooperation. However, the Katachi-Symmetry cooperation is not confined to geometry but everything is contained. More abstract concepts are also included in Katachi.

7. Concluding Remarks

The first contact of two interdisciplinary movements, Katachi and Symmetry, was in Budapest in August of 1989. It was just before the break down of the cold war and the outset to that took place close to there. In 1994, five years after then, Katachi U Symmetry symposium was successfully held as the first achievement of the cooperation of two movements. Next five years was over and the cooperation is more steadily developing now. It is our task to break the wall between disciplines, the wall of prejudice and discrimination of any kind, the wall by inequality (asymmetry), and the wall by formalism and formality (negative side of Katachi, or Katachi only in superficies).

Rather many people think that scientists work for one's private achievements. Unfortunately, the author cannot deny it as a general tendency. There are, however, many romantic scientists: some of them research merely for the pursuit of truth, some of them from aesthetic motivation, and others for enjoying intellectual activity as a kind of performance, fun or amusement. Ideally, they should have a noble heart at the same time. Many participants of our interdisciplinary activities belong to such a category. We have some problems how to organize the younger generations to this category. Usually, one starts his research in some discipline and gets an interdisciplinary scope after then. Starting *interdisciplinary* subject may yield a *new discipline*.

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