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Black-and-White Symmetry, Magnetic Symmetry, Self-Duality and Antiprismatic Symmetry: The Common Mathematical Background

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Abstract. Seemingly distant areas of symmetry, such as mentioned in the title, are closely related to each other on an abstract level. The unified treatment is made possible by some common features of the symmetry groups used in the mathematical description of these types of symmetries. The key property is the way as a two-element group generated by an involutory transformation can extend a geometric symmetry group.

The close relationship between the first two kinds of symmetry is well known. The actual physical objects of the one kind whose symmetry is to be described mathematically are magnetic crystals, or a bit more precisely, "crystals with magnetic structure" (LANDAU and LIFSHITZ, 1986). The symmetry in stricter sense of any crystal is in essence a geometric symmetry, namely invariance with respect to a congruent transformation (that is, the symmetry transformation may be rotation, reflection, translation or combinations of these). However, in a magnetic crystal one has to take into account a non-geometric symmetry transformation—time reversal. Pure time reversal does not affect the geometric structure of the crystal; i.e. the spatial position of the particles remains the same. If the constituent particles, however, carry a magnetic moment, this momentum is reversed. To carry over the crystal to a state indistinguishable from the original, time reversal must be followed by a pure geometric transformation. To be precise, it must be followed *or* preceded by that—the actual order is unessential, whatever is the pure geometric symmetry. This commutativity is one of the key properties of such a special kind of transformation and in what follows it will be referred to as property (★).

It is quite a natural idea to conceive the two opposite directions of the magnetic moment as if the particle in question were painted black or white. In this approach the crystal is considered as a symmetrical material pattern which is coloured symmetrically by two colours. Then time reversal is interpreted as colour reversal, i.e. "black particles" are painted over to "white" and vice versa. This idea was already raised in 1929 by *Heesch* as well as by *Hermann* (JANSSEN, 1973; LANDAU and LIFSHITZ, 1986; GRÜNBAUM and

SHEPHARD, 1987). Thus we have come to objects of another kind—two-coloured symmetrical patterns, the most popular examples of which are perhaps works of *Escher* (see e.g. COXETER *et al.*, 1987). The close connection between these two areas of symmetry studies is well exemplified in (KOPTSIK, 1966), where in a catalogued description of the 1191 nontrivial magnetic space groups Koptsik uses red and black colouring. (The altogether 1651 magnetic space groups were derived in the fifties: ZAMORZAIEV, 1953, BELOV *et al.*, 1957.)

What is the adequate symmetry transformation of such a pattern? It turns out that it may be a pure colour reversal, a pure congruent transformation, or an appropriate combination of these. A pattern is considered symmetrical with respect to such a transformation, if two points which have the same colour before the transformation, will have the same colour afterward (this is essentially the principle formulated by *Loeb* as a "Consistency Postulate") (SENECHAL, 1975).

Yet, when the so-called dichromatic groups are used in the mathematical classification of these patterns, it is expedient to consider only congruent transformations which are or are not combined with colour reversal, and, as an invariance condition, colour preservation is required. This latter means that two points which are related to each other by such a transformation always have to be of the same colour (COXETER, 1985a, 1987). Following Coxeter, a "coloured symmetry group" of a black and white pattern consists of a group $\mathbf{G}$ in which the distinction of colour is disregarded, and a subgroup $\mathbf{G_1}$ of index 2 that preserves the colours. Accordingly, the factor group $\mathbf{G}/\mathbf{G_1}$ is isomorphic to the two-element group containing the colour reversal (the same symbol may serve for notation of the coloured symmetry group). (We note that this notation was introduced by ZAMORZAIEV, 1953, and also appears in KOPTSIK, 1967.)

As a simple example, let us see the ancient symbol *T'ai-chi T'u* (Fig. 1). Its coloured symmetry group can be given as $\mathbf{C_i}/\mathbf{C_1}$, where $\mathbf{C_i}$ is the Schoenflies symbol of the two-element group containing central inversion, and $\mathbf{C_1}$ stands for the trivial group.

It is clear that both pure time and colour reversal is an involutory transformation, that is, applying it twice one after the other results in identity transformation (for brevity, we denote this property with (★★)). Now if γ is a symmetry transformation with properties (★) and (★★), then *there are essentially two ways of extending a symmetry group by γ*, as Hermann WEYL (1952) pointed out . The role of γ is played in Weyl's book by central inversion, and it serves to build the whole set of the finite groups of congruences of the Euclidean space of dimension 3 starting from the groups of pure rotations. In the study of magnetic symmetry, Zamorzaiev used the two ways of extension already in 1953

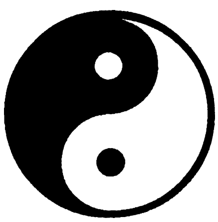

Fig. 1. *T'ai-chi T'u.*

(ZAMORZAIEV, 1953, KOPTSIK, 1966; LANDAU and LIFSHITZ, 1986), and γ is regarded here as either time or colour reversal.

Let us see these two cases.

Case I. This is a simple adjunction of γ to $\mathbf{G_1}$ for obtaining the extended group $\mathbf{G}$. Then $\mathbf{G}$ is generated by $\mathbf{G_1}$ and γ, thus $\mathbf{G}$ is isomorphic to the direct product $\mathbf{G_1} \times \langle\gamma\rangle$, where the second term denotes the two-element group containing γ. For the magnetic crystals, this is the case when pure time reversal is taken into consideration as a symmetry transformation, and the groups containing it are called the trivial magnetic groups. For a 2-coloured pattern this case has no significance, since in accordance with our above convention, pure colour reversal is not considered as a proper symmetry operation by itself.

Case II. If $\mathbf{G_1}$ is a subgroup of index 2 in a geometric symmetry group $\mathbf{G'}$, then we have the decomposition $\mathbf{G'} = \mathbf{G_1} + \varphi\mathbf{G_1}$, where φ is a non-identity element in $\mathbf{G'} \backslash \mathbf{G_1}$. Then $\mathbf{G} = \mathbf{G_1} + (\varphi\gamma)\, \mathbf{G_1}$, i.e. the extended symmetry group $\mathbf{G}$ contains γ only in product form. For magnetic crystals, this is the case of non-trivial magnetic groups, and also, for 2-coloured patterns this case was already mentioned above.

This simple group theoretical principle proves to be useful in other symmetry studies as well. For example, the duality relation between polyhedra can be formulated in the context of transformations. In fact, between two polyhedra P and $P^\star$ dual to each other there is a bijective mapping δ which sends the faces, edges and vertices of P to the vertices, edges and faces of $P^\star$, respectively, while preserving incidence. (This latter invariance property means that if two such constituting elements of the one polyhedron are incident, e.g. a vertex v takes place on an edge e, then their image $\delta(v)$ and $\delta(e)$ are incident as well.) Here δ is called a duality map. A polyhedron may be dual to itself as well, i.e. it may be self-dual (tetrahedron is the most simple example), and in this case δ is a self-duality map. Hence the symmetry group of a self-dual polyhedron can be extended so that it include self-dualities. Such extended groups are called self-duality groups (ASHLEY *et al.*, 1991).

Among these groups, both cases considered above may and do occur. For example, the self-duality group of the regular tetrahedron belongs to the Case I: it is isomorphic to the direct product $\mathbf{T_d} \times \langle\gamma\rangle$, where the first symbol denotes the symmetry group of the regular tetrahedron and in the second term γ can be taken as central inversion. (On the other hand, this product is isomorphic to the symmetry group $\mathbf{O_h}$ of the regular octahedron.) To take examples of dimension 2, the self-duality group of every k-sided polygon belongs to Case I if k is odd, and to Case II, if k is even. Within Case II, however, one has to take into account two further cases (for polyhedra, or more generally, for d-polytopes, $d \geq 3$). For, in general, δ^2 need not be the identity, i.e. property (★★) does not always hold, despite the well-known fact that "the dual of the dual is the original". Moreover, it is possible that a self-dual polyhedron does not possess involutory self-duality at all—this is a possibility to which GRÜNBAUM and SHEPHARD (1988) called the attention recently.

Thus, Case IIa is when a self-duality group contains involutory self-dualities but neither of them commute with all the other transformations, and Case IIb is when neither of the self-dualities is involutory. In the context of abstract groups, these two cases can be appropriately treated in terms of the theory of group extensions, due to *Schreier* from the twenties (SCOTT, 1964). Without going into the details, we only mention that Case IIa corresponds to an extension $\mathbf{G}$ of $\mathbf{G_1}$ by $\langle\gamma\rangle$, such that $\mathbf{G}$ is isomorphic to the semi-direct product of $\mathbf{G_1}$ and $\langle\gamma\rangle$ (then we say that the extension is splitting). On the other hand, Case

IIb corresponds to an extension where such splitting does not exist.

We have seen three areas of symmetry studies together with distinct interpretations of an abstract mapping γ which has the properties (★) and (★★). It is not hard to find other areas in which our simple group extension scheme plays some role. Construction of highly symmetrical antiprisms may be such an area, especially in the case of higher dimensional analogues of the ordinary 3-antiprisms. The idea of such higher antiprisms raised Grünbaum. Here we omit the details and only refer to his book (GRÜNBAUM, 1967). In the special case when the bases of a d-antiprism are two copies of a self-dual (d–1)-polytope the symmetry group of the whole polytope can be obtained as an extension of the symmetry group of the basis. (This condition is automatically fulfilled for an ordinary antiprism in 3-space, i.e. when $d = 3$.) There are two ways of construction, depending on how one can extend the group in question.

For an example, we mention the following construction. Take two congruent copies of the four-dimensional regular 24-cell (COXETER, 1948) such that they are polar of each other (polarity is a relation stronger than duality (GRÜNBAUM, 1967). After suitably translating one of them into a parallel position in the 5-dimensional space, they serve as bases of a 5-antiprism. Then the symmetry group of this antiprism is isomorphic to a 4D symmetry group denoted in (COXETER, 1985b) by [[3, 4, 3]].

We note, in conclusion, that these four distinct areas of application do not exhaust at all the possibilities. Further examples (such as e.g. the higher-dimensional subperiodic symmetry groups, etc.), however, are out of scope of this paper.

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