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Some Expressions of Ovaloid and Form Defined by Supporting Function

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Abstract. A relation between Supporting function *P* derived from the normal line of hyper tangent plane and the coordinates of its contact point is clarified. Supporting functions as the definition of objects (Oval, Ovaloid, Hyper-Ovaloid etc.) are given, and coordinates of the contact point are obtained as parametric equations. Some examples and their CG are shown.

1. Introduction

The ovaloid such as an egg shape is an interesting form. Many researchers made studies of its differential geometry (KUBOTA, 1967). In these researches, the ovaloid is defined to have a property, i.e, all points on the segment connecting any two points inside the curved surface exist within this surface. In addition, there is a theorem that at least one closed geodesic line and four or more umbilici exist on the ovaloid.

However, in spite of its usefulness of the ovaloid in designing, optics, biology and physics, no practical tool to express it is available, because the precise drawing is difficult. It is noted that the ovaloid is not a quadratic surface like spheroid, but algebraically a quaternary surface. Hence its projective drawing was almost impossible in time without computer. Whereas, the recent development of CG applications has made it possible to perform parallel processing with manipulation of formula and numerical calculations, allowing visualization of various curved surfaces (GRAY, 1996). Therefore, I planned to visualize the ovaloid as a closed convex surface and realized that the concept of Supporting function (KUBOTA, 1967) used in the studies of differential geometry is promissing. Then, I contrived the method of deriving parametric expression of ovaloid with Supporting function made of elementary function. Furthermore, in order to express the ovoid by extending an ovaloid into the four-dimensional space, I derived the parametric expression from a typical four-dimensional Supporting function. This allowed us to draw some egg shapes like oval or ovaloid and other forms using functional graphic applications. These results are reported in this paper.

2. Parametric Expressions of Supporting Function and Its Original Graphic and Their CGs

Here, we describe the procedure of deriving four-dimensional parametric expressions. According to this procedure, we derived a lower-dimensional formula. In addition, for the spheric surface and ellipse as typical ovaloids, we confirmed the relation between the Supporting function and the parametric expression of its original graphic. Then, by giving various Supporting functions, we obtained their CGs with Maple V software.

2.1. Derivation of parametric expressions of original graphics from general four-dimensional Supporting functions

Here, if there is an ovoid within the four-dimensional space and a distance from a given point within it to its super tangent plane is expressed as the function of angle in the direction of the normal of this hyper-tangent plane, the function is called the Supporting function of ovoid. On the other hand, this tangent plane is called the Supporting hyperplane of ovoid.

Now, for the direction cosine N in the direction of the normal of four-dimensional Supporting hyperplane (that is, in the direction of perpendicular from the origin to the Supporting hyperplane), if the direction angles from $y(Y_2)$, $z(Y_3)$, $w(Y_4)$ among four axes x, y, z, w are expressed by x_1, x_2, x_3 instead of s, t, u in order to clarify the dimensional expandability, it is given as ($\sin x_3 \sin x_2 \sin x_1$, $\sin x_3 \sin x_2 \cos x_1$, $\sin x_3 \cos x_2$, $\cos x_3$).

We also assume that the space coordinates (x, y, z, w) of a point of contact between this hyperplane and ovoid is $Y(Y_1, Y_2, Y_3, Y_4)$. At this time, the Supporting function P can be expressed as the inner product of vector N and vector Y, namely,

$$P = Y_1 \cdot \sin x_3 \sin x_2 \sin x_1 + Y_2 \cdot \sin x_3 \sin x_2 \cos x_1 + Y_3 \cdot \sin x_3 \cos x_2 + Y_4 \cdot \cos x_3. \quad (1)$$

In order to solve this on $Y(Y_1, Y_2, Y_3, Y_4)$ by giving P, Eqs. (2), (3) and (4) are respectively generated by performing a partial differential with x_3, x_2, x_1 on each member of the above formula in the manner of obtaining the envelope:

$$\partial P/\partial x_3 = Y_1 \cdot \cos x_3 \sin x_2 \sin x_1 + Y_2 \cdot \cos x_3 \sin x_2 \cos x_1 + Y_3 \cdot \cos x_3 \cos x_2 - Y_4 \cdot \sin x_3, \quad (2)$$

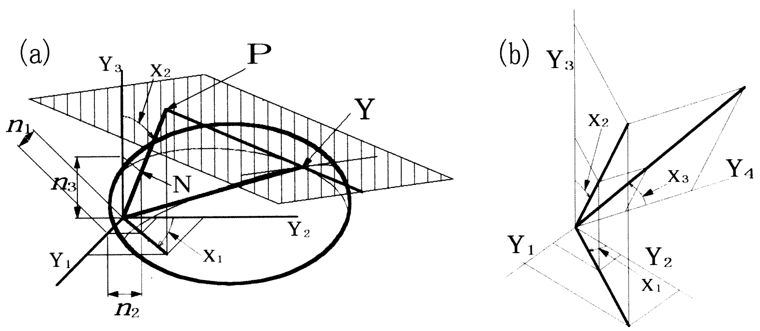

Fig. 1. (a) $N \cdot Y = P$, (b) four-dimensional polar coordinates.

$$\partial P/\partial x_2 = Y_1 \cdot \sin x_3 \cos x_2 \sin x_1 + Y_2 \cdot \sin x_3 \cos x_2 \cos x_1 - Y_3 \cdot \sin x_3 \sin x_2, \tag{3}$$

$$\partial P/\partial x_1 = Y_1 \cdot \sin x_3 \sin x_2 \cos x_1 - Y_2 \cdot \sin x_3 \sin x_2 \sin x_1. \tag{4}$$

Now, we express Px_1, Px_2, Px_3 with the result of partially differentiating the function $P(x_1, x_2, x_3)$ with variables x_1, x_2, x_3, respectively. From Eqs. (1)–(4), we can obtain $Y_1 \sim Y_4$ as follows:

A: $Y_1 = P \cdot \sin x_3 \sin x_2 \sin x_1 + Px_3 \cdot \cos x_3 \sin x_2 \sin x_1 + Px_2 \cdot \cos x_2 \sin x_1 / \sin x_3 + Px_1 \cdot \cos x_1 / (\sin x_3 \sin x_2)$,

$Y_2 = P \cdot \sin x_3 \sin x_2 \cos x_1 + Px_3 \cdot \cos x_3 \sin x_2 \cos x_1 + Px_2 \cos x_2 \cos x_1 / \sin x_3 - Px_1 \cdot \sin x_1 / (\sin x_3 \sin x_2)$,

$Y_3 = P \cdot \sin x_3 \cos x_2 + Px_3 \cdot \cos x_3 \cos x_2 - Px_2 \cdot \sin x_2 / \sin x_3$,

$Y_4 = P \cdot \cos x_3 - Px_3 \cdot \sin x_3$.

Above formula is equal to that of three-dimensional ovaloid if $x_3 = \pi/2$, i.e.

B: $Y_1 = P \cdot \sin x_2 \sin x_1 + Px_2 \cos x_2 \sin x_1 + Px_1 \cdot \cos x_1 / \sin x_2$,

$Y_2 = P \cdot \sin x_2 \cos x_1 + Px_2 \cos x_2 \cos x_1 - Px_1 \cdot \sin x_1 / \sin x_2$,

$Y_3 = P \cdot \cos x_2 - Px_2 \cdot \sin x_2$.

In the same way we have that of two-dimensional oval by putting $x_2 = \pi/2$, *i.e.*

C: $Y_1 = P \cdot \sin x_1 + Px_1 \cdot \cos x_1$,

$Y_2 = P \cdot \cos x_1 - Px_1 \cdot \sin x_1$.

2.2. Supporting functions of sphere and ellipse

Here, we actually give the Supporting function to the formula of the above-parametric expression to perform the calculation. To this end, we had to wait for the development of recent formula manipulation software programs like Maple V software.

However, the Supporting function was intuitionally found by adding such conditions as $P > 0$. The direction cosine N toward the Supporting hyperplane of a four-dimensional ovoid is given as follows:

$$N = (n_1, n_2, n_3, n_4) = (\sin x_3 \sin x_2 \sin x_1, \sin x_3 \sin x_2 \cos x_1, \sin x_3 \cos x_2, \cos x_3).$$

Theorem 1. Generally, $P = an_1 + bn_2 + cn_3 + dn_4 + h$ is the Supporting function of four-dimensional sphere.

It is easily seen by substituting this into fomula A, as follows:

$Y_1 = a + h \sin x_3 \sin x_2 \sin x_1,$
$Y_2 = b + h \sin x_3 \sin x_2 \cos x_1,$
$Y_3 = c + h \sin x_3 \cos x_2,$
$Y_4 = d + h \cos x_3.$

The above equations indicate a four-dimensional sphere $(Y_1 - a)^2 + (Y_2 - b)^2 + (Y_3 - c)^2 + (Y_4 - d)^2 = h^2$. This is the generalized formula of three-dimensional spheric surface and two-dimensional circumference.

Theorem 2. $P = \mathrm{sqrt}(k^2 - c^2 \sin^2(\pi/2 - x_1))/2 + c \cos(\pi/2 - x_1)/2$

This is the Supporting function of an ellipse. It is seen by substituting this into fomula C,

$x = (1/2) \sin(x_1)(k^2)/\mathrm{sqrt}(k^2 - c^2 \cos^2(x_1)) + c/2,$
$y = (-1/2) \cos(x_1)(-k^2 + c^2)/\mathrm{sqrt}(k^2 - c^2 \cos^2(x_1)).$

This is an ellipse, i.e. $4(x-c/2)^2/k^2 + 4y^2/(k^2-c^2) = 1$.

2.3. Supporting functions of ovaloid and their CGs

We give Eqs. (1)–(3) as the Supporting function of the three-dimensional ovaloid to derive the parametric expression from formula B and draw CGs with functional graphic software:

(1) $P = n_1^3 + 2n_1^2 + 3n_1 + 5$ (Fig. 2–1, Fig. 2–2),
(2) $P = n_1^2 + n_2^2 n_3 + 3$ (Fig. 3),
(3) $P = n_2 + 3$ (Fig. 4),

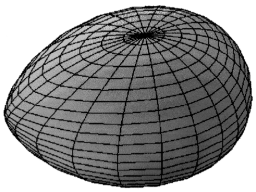

Fig. 2–1. Ovaloid 1.

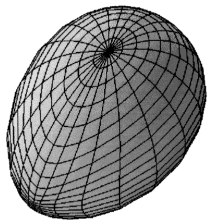

Fig. 2–2. Ovaloid 1.

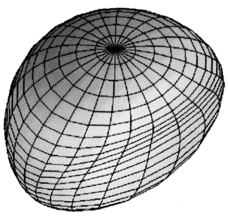

Fig. 3. Ovaloid 2.

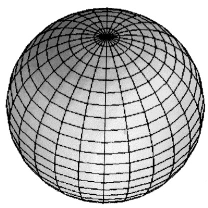

Fig. 4. Sphere.

where

$$N = (n_1, n_2, n_3) = (\sin x_2 \cdot \sin x_1, \sin x_2 \cdot \cos x_1, \cos x_2).$$

These Ovaloids have different views depending on the viewing direction:

2.4. Supporting functions of pseudo-forms and their CGs

We show some forms defined by Supporting function $P1$, $P2$, $P3$ and calculated with fomula B, where

$P1 = \sin(5\sin^3(x_2)\cos(x_1) + 6) + 2\sin(x_2)\cos^2(x_1) + 3$ (Fig. 5),
$P2 = \sin^2(x_2)\cos^2(x_1)$ (Fig. 6),
$P3 = \sin(12\cos^2(x_2)) + 3$ (Fig. 7).

3. Summary

The examples shown above reveal a relation between the Supporting function and the parametric expression of the original graphic. As concrete examples, we gave the Supporting functions of apparent ovaloids to calculate the parametric expressions of their original graphics (ovaloids) with Maple V, a formula manipulation application, and show the results in figures. In the same way, we can consider Supporting functions and their parametric equations of two-dimensional Oval Curves and four-dimensional Ovoids.

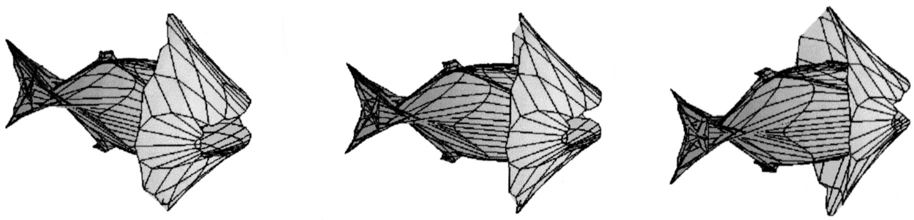

Fig. 5. Fish.

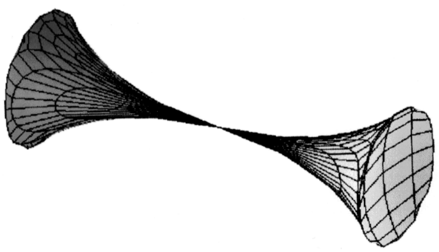

Fig. 6. Bugle.

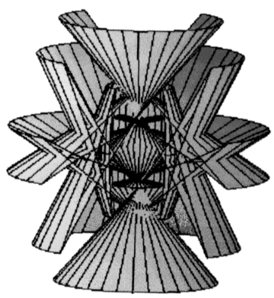

Fig. 7. Engine.

For three-dimensional ovaloids, we have examined whether these figures are convex graphics or not by checking the condition (see KOBAYASHI (1995)), $LN-M^2 > 0$, $K > 0$. Under this condition the second basic differential equation, $Ldx_1{}^2 + 2Mdx_1dx_2 + Ndx_2{}^2$, is a positive definite in the three-dimensional graph. For ovoids etc., we should perform a strict investigation. For circles, spheres and four-dimensional spheres as convex figures, the relation between Supporting functions and parametric expressions was confirmed.

Giving any Supporting function, we can generate various CGs by the use of formulae A; B; C. Furthermore, the present work will contribute to a development of technological applications.

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