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Animation of Some Truncated Polyhedrons

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Abstract. A rhombic enneacontahedron consists of two kinds of equilateral rhombuses, fat and thin. There are thirty thin rhombuses whose normals are the same as those of a triacontahedron. This shows that the enneacontahedron is a polyhedron truncated by a triacontahedron and it can be transformed into a triacontahedron by increasing the depth of truncation. This change from an enneacontahedron to a triacontahedron is shown by animation using a 3D viewer called Geomview available from the Geometry Center of the University of Minnesota. The software called Qhull, also available from the Geometry Center, which finds a convex hull for given points is used to find the truncated polyhedron. Also shown are a rhombic dodecahedron truncated by a hexahedron with different depths of truncation, and 3D sections of the 4D test polytope with different sectioning positions for 3D Beenker pattern.

1. Introduction

It is known that 3D n-star generates a polyhedron by projection of nD unit cube and that the mixing of 3D stars with different orientation and mixing ratio produces a star and generates another polyhedron. As a novel combination of 3D star the mixing of stars generating a truncated enneacontahedron is presented (SOMA and WATANABE, 1999) together with that generating a truncated rhombic dodecahedron. Also presented is a novel polyhedron generated as a section of 4D polytope, the test polytope for generating 3D Beenker pattern by cut-and-project method (KATZ and DUNEAU, 1986).

2. Truncated Polyhedron by Mixing 3D Stars

A 3D n-star is a projection of n basis vectors in nD space $\mathbf{R}^n$ to 3D space $\mathbf{R}^3$ and defines a 3 by n projection matrix P_n from nD to 3D space. If $\{\varepsilon_i\}$ denotes the standard orthonormal basis of $\mathbf{R}^n$, the star vectors are the projection $e_i = P_n(\varepsilon_i)$ in $\mathbf{R}^3$ and P_n is a matrix consisting of n star vectors as its column vectors.

An nD unit cube $\gamma_n = \{\Sigma\xi_i\varepsilon_i \mid 0 < \xi_i < 1\}$ projected by the matrix is a convex polyhedron with pairs of parallel facets defined by the two star vectors e_i and e_j. The number of pairs is at most the number of combination two out of n. This polyhedron can be obtained by projecting the vertex points of nD unit cube $\{\Sigma\xi_i\varepsilon_i \mid \xi_i = \{0, 1\}\}$ to 3D space and by finding the convex hull for these points. A program called Qhull, available from the Geometry Center (BARBER *et al.*, 1996), which finds the convex hull for given points can be used. It finds the vertex points, facet normals and offsets, facet loop indices, and facet colors, among other things.

Examples of 3D stars are vectors from the center to vertices of regular polyhedrons; a hexahedral star, an octahedral star, a dodecahedral star, and an icosahedral star. By the projection of a unit hypercube, the hexahedral star gives a rhombic dodecahedron, the octahedral star a rhombic hexahedron (cube), the dodecahedral star a rhombic enneacontahedron, and the icosahedral star a rhombic triacontahedron, respectively.

Consider a mixing of two 3D stars defined by a matrix obtained by concatenating two projection matrices P_m and P_n column-wise with factors λ and $(1 - \lambda)$ as $P_{m+n}(\lambda) = (\lambda P_m \; (1-\lambda)P_n)$, where λ is a mixing parameter taking the value in the range [0,1]. It can be shown that the mixed star vectors make a $(m + n)$-star and that the projection of $(n + m)$D unit cube by $P_{m+n}(\lambda)$ gives a polyhedron by λP_m truncated by another polyhedron by $(1-\lambda)P_n$ or vice versa. For $\lambda = 1$ it gives a polyhedron by P_m and for $\lambda = 0$ that by P_n. By changing the value of parameter λ in the range [0,1], the polyhedrons with different depths of truncation are obtained and viewed by animation using appropriate 3D viewer.

2.1. *Mixing of a hexahedral star and an octahedral star*

Consider the mixing of a hexahedral star and an octahedral star as

$$P_7(\lambda) = \frac{1}{l(\lambda)} \begin{pmatrix} \lambda/\sqrt{2} & \lambda/\sqrt{2} & -\lambda/\sqrt{2} & \lambda/\sqrt{2} & 1-\lambda & 0 & 0 \\ \lambda/\sqrt{2} & \lambda/\sqrt{2} & \lambda/\sqrt{2} & -\lambda/\sqrt{2} & 0 & 1-\lambda & 0 \\ \lambda/\sqrt{2} & -\lambda/\sqrt{2} & \lambda/\sqrt{2} & \lambda/\sqrt{2} & 0 & 0 & 1-\lambda \end{pmatrix}, \tag{1}$$

where $l(\lambda) = \sqrt{3\lambda^2 - 2\lambda + 1}$. The first four colums represent the four vectors of a hexahedral star with a factor λ and the last three colums the three vectors of an octahedral star with a factor $(1 - \lambda)$. Figure 1 shows the mixed star for $\lambda = 1/2$ with orientation of each star and Fig. 2 shows the truncated polyhedrons for $\lambda = \{1, 7/8, 6/8, 5/8, 4/8, 3/8, 2/8, 1/8, 0\}$.

2.2. *Mixing of a dodecahedral star and an icosahedral star*

Next consider the mixing of a dodecahedral star and an icosahedral star as

$$P_{16}(\lambda) = \frac{1}{m(\lambda)} \begin{pmatrix} \lambda S & \lambda\tau S & -(1-\lambda)S & 0 \\ \lambda C & \lambda\tau C & -(1-\lambda)C & 0 \\ \lambda(\tau+1)H & \lambda(\tau-1)H & (1-\lambda)H & (1-\lambda)\sqrt{5}/2 \end{pmatrix}, \tag{2}$$

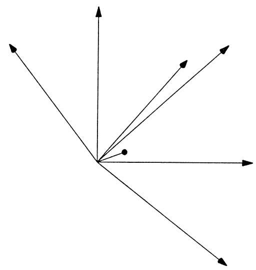


Fig. 1. A mixed star defined by Eq. (1) with $\lambda = 1/2$.

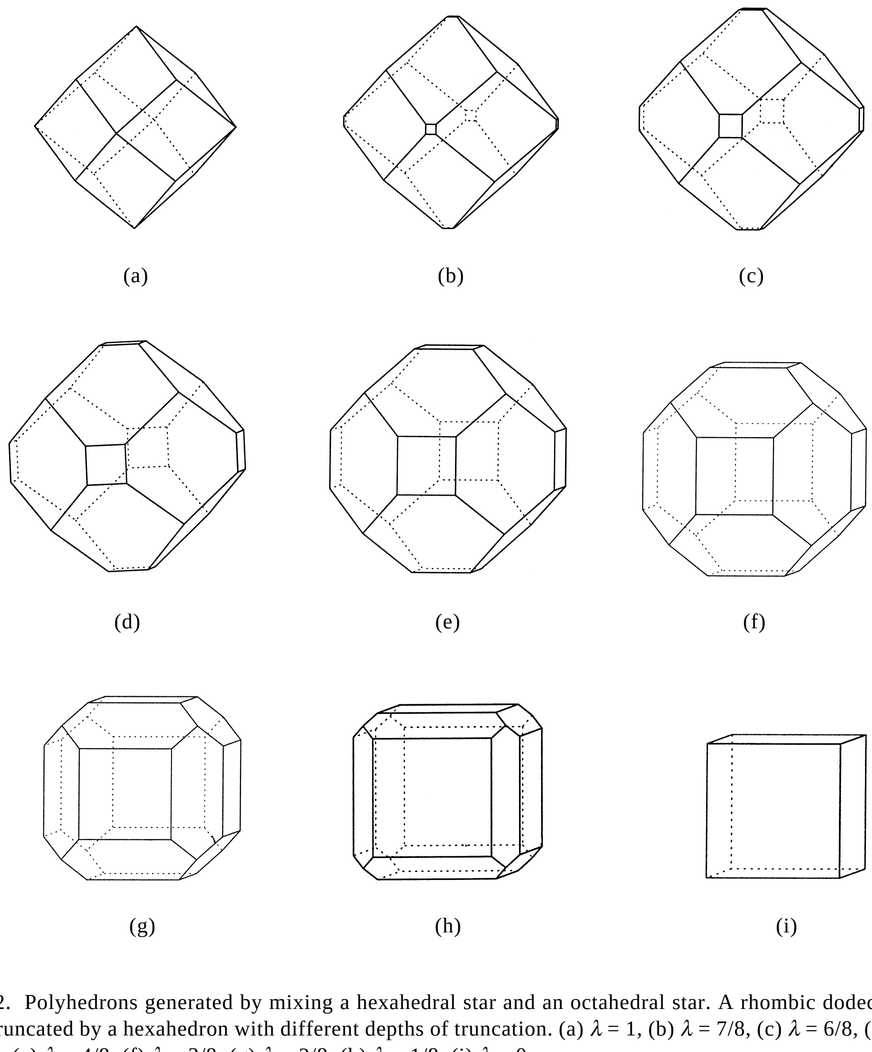


Fig. 2. Polyhedrons generated by mixing a hexahedral star and an octahedral star. A rhombic dodecahedron truncated by a hexahedron with different depths of truncation. (a) $\lambda = 1$, (b) $\lambda = 7/8$, (c) $\lambda = 6/8$, (d) $\lambda = 5/8$, (e) $\lambda = 4/8$, (f) $\lambda = 3/8$, (g) $\lambda = 2/8$, (h) $\lambda = 1/8$, (i) $\lambda = 0$.

where $m(\lambda) = \sqrt{5((2\tau^2+3)\lambda^2-2\lambda+1)/2}$, S = (0 sinα sin2α sin3α sin4α), C = (1 cosα cos2α cos3α cos4α), H = (1/2 1/2 1/2 1/2 1/2), $\tau = (1+\sqrt{5})/2$ and $\alpha = 2\pi/5$. The first ten columns represent the ten vectors of a dodecahedral star with a factor λ and the last six columns the six vectors of an icosahedral star with a factor $(1-\lambda)$. Figure 3 shows the mixed star for $\lambda = 1/2$ with orientation of each star and Fig. 4 shows the truncated polyhedrons for λ = {1, 7/8, 6/8, 5/8, 4/8, 3/8, 2/8, 1/8, 0}.

3. Truncated Polyhedron as a 3D Section of 4D Polytope

Consider a 4D n-star and a polytope by the projection of nD unit cube to 4D space with coordinate (x', y', z', w'). The projection matrix P'_n consists of n star vectors as its column vectors. Since the 4D polytope is convex, the 3D section of the polytope at, say, $w' = W$ is also a convex polyhedron. It is obtained by projecting vertex points of nD unit cube to 4D space to find a polytope by using Qhull and then finding the intersection with a plane $w' = W$. The point $p_4(\mathrm{x}_i, \mathrm{y}_i, \mathrm{z}_i, W)$ at which three or more facets meet with the plane. These points are the vertex point $p_3(\mathrm{x}_i, \mathrm{y}_i, \mathrm{z}_i)$ of the polyhedron, if regarded as the point in 3D space. Qhull can also be used to find the facet loop index and facet color. By changing W, different polyhedrons are obtained and viewed by animation.

3.1. 3D Sections of a 4D test polytope for 3D Beenker pattern

Consider a 4D 7-star defined by the matrix

$$P'_7(\lambda) = \frac{1}{l(\lambda)}\begin{pmatrix} -(1-\lambda)/2 & -(1-\lambda)/2 & (1-\lambda)/2 & -(1-\lambda)/2 & \sqrt{2}\lambda & 0 & 0 \\ -(1-\lambda)/2 & -(1-\lambda)/2 & -(1-\lambda)/2 & (1-\lambda)/2 & 0 & \sqrt{2}\lambda & 0 \\ -(1-\lambda)/2 & (1-\lambda)/2 & -(1-\lambda)/2 & -(1-\lambda)/2 & 0 & 0 & \sqrt{2}\lambda \\ l(\lambda)/2 & -l(\lambda)/2 & -l(\lambda)/2 & -l(\lambda)/2 & 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

Figure 5 shows the 3D sections for w' = {–1, –3/4, –2/4 – 1/4, 0, 1/4, 2/4, 3/4, 1} with λ = 1/2. If origin is shifted to the center of gravity, the range of w' changes to [–1, 1].

It is interesting to note that the matrix obtained by concatenating P_7 and P'_7 row-wise constitutes an orthonormal seven by seven rotation matrix in 7D space and this matrix corresponds to the projection matrix from 7D lattice to pattern and test space generating the 3D Beenker pattern by cut-and-project method (SOMA and WATANABE, 1997).

4. 3D Viewer for Animation

There are many 3D viewer with animation capability available now. We used Geomview from the Geometry Center of the University of Minnesota (PHILLIPS *et al.*, 1993) because it is efficient and easy to use.

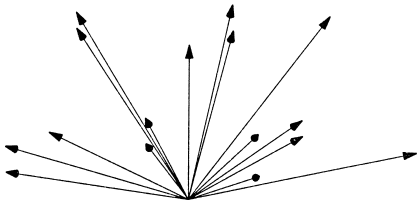
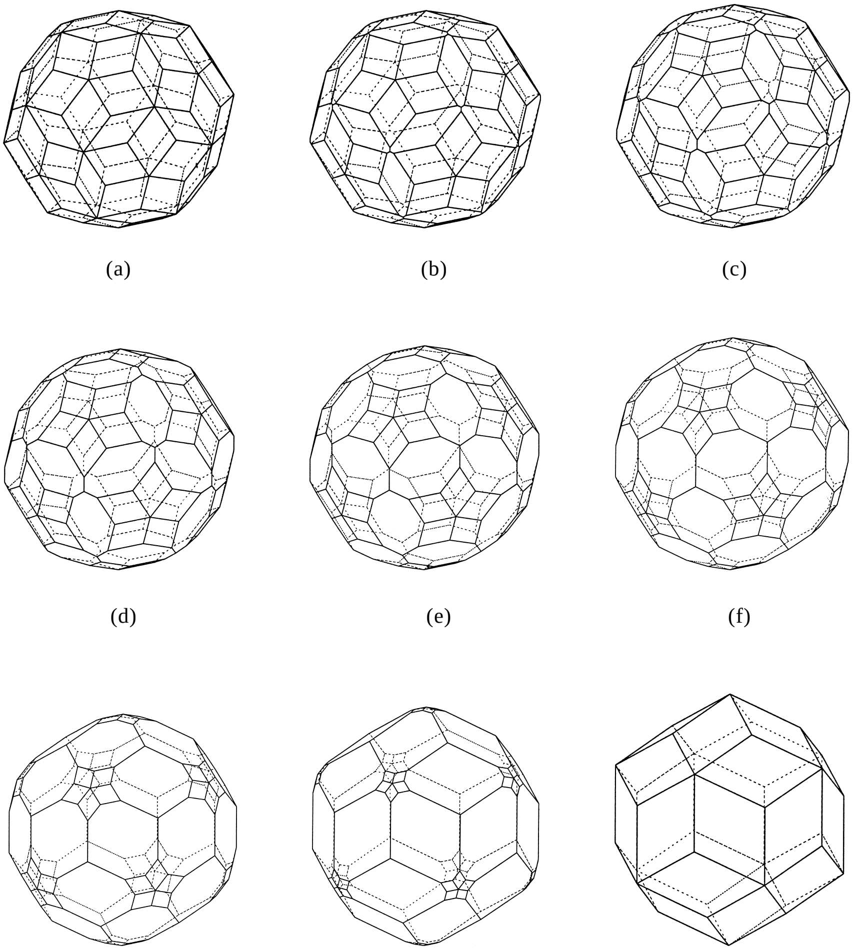


Fig. 3. A mixed star defined by Eq. (2) with $\lambda = 1/2$.



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Fig. 4. Polyhedrons generated by mixing a dodecahedral star and an icosahedral star. An enneacontahedron truncated by a triacontahedron with different depths of truncation. (a) $\lambda = 1$, (b) $\lambda = 7/8$, (c) $\lambda = 6/8$, (d) $\lambda = 5/8$, (e) $\lambda = 4/8$, (f) $\lambda = 3/8$, (g) $\lambda = 2/8$, (h) $\lambda = 1/8$, (i) $\lambda = 0$.

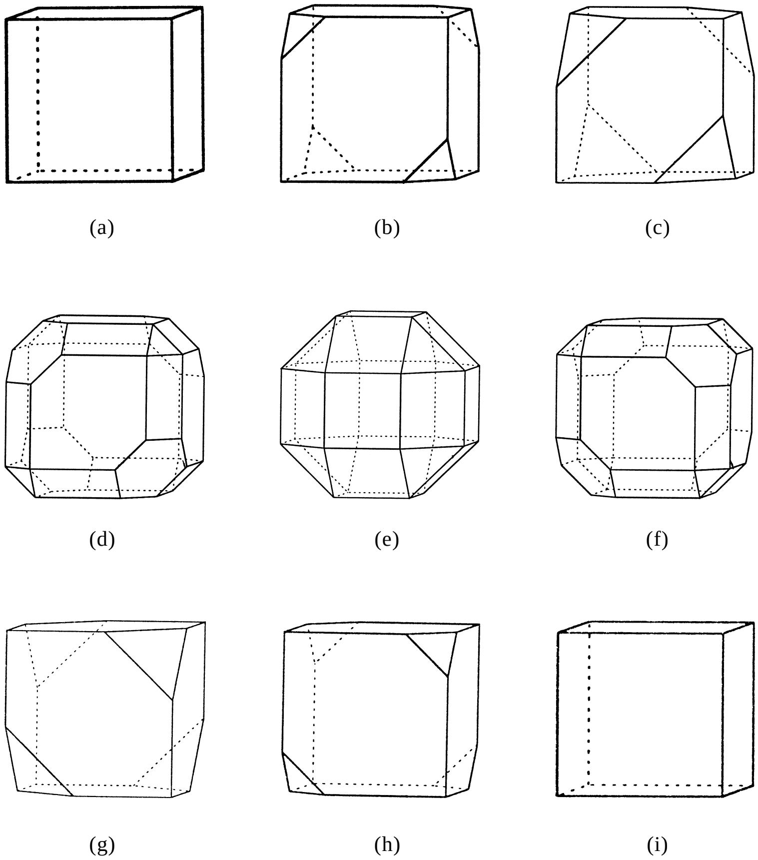


Fig. 5. 3D sections at different sectioning positions of a polytope by 4D star defined by Eq. (3) with $\lambda = 1/2$. (a) $w' = -1$, (b) $w' = -3/4$, (c) $w' = -2/4$, (d) $w' = -1/4$, (e) $w' = 0$, (f) $w' = 1/4$, (g) $w' = 2/4$, (h) $w' = 3/4$, (i) $w' = 1$.

4.1. Geomview and its external module

Geomview is an interactive program for viewing and manipulating geometric objects. It can be used as a stand alone viewer for static objects or as a display engine for other program which produces dynamically changing geometry, such as producing animation. The controlling program is called External Module of Geomview and can control Geomview through gcl (Geomview command language) passed by input/output stream.

4.2. External module for mixing of 3D stars

The following briefly describes the flow of the program for mixing.

```
loop
    set mixing parameter λ in the range [0, 1]
    define projection matrix for given λ
```

```
        project vertices of nD unit cube to 3D space, set origin as center of gravity
        find convex hull by Qhull; vertex point, facet index, facet color
        send polyhedron data to Geomview for viewing
    end loop
```

4.3. External module for sectioning of 4D polytope

The following briefly describes the flow of the program for sectioning.

```
    define projection matrix for given mixing parameter λ
    project vertices of nD unit cube to 4D space, set origin as center of gravity
    find 4D polytope by Qhull; facet normal and offset
    loop
        set w′ to W in the defined range
        find intersection of polytope with a plane w′ = W(3 or more facets + 1 plane)
        find convex hull by Qhull; facet index, facet color
        send polyhedron data to Geomview for viewing
    end loop
```

5. Concluding Remarks

The mixing is not restricted to two stars but any number of stars. The mixing concept can be applied for modelling of quasicrystal clusters for example. An interactive star mixing and polyhedron display tool may be useful for such purposes, which builds *n*D stars interactively, mixes with various orientations and sizes, and displays the projected polyhedron in animation.

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