Original Paper *Forma*, **15**, 75–79, 2000

Tiling Problem of Convex Pentagon

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(Received November 19, 1999; Accepted January 10, 2000)

Keyword: Convex Pentagon, Tiling, Tessellation, Tessellating Pentagon, Pattern

Abstract. We discovered the new tiling patterns each of which is composed of a single kind of convex pentagon. Moreover, we propose a new concept of classification method of the tessellating convex pentagons, which is the only unsolved case among the corresponding tessellating convex polygon problems.

1. Introduction

The problem of plane tiling by a single kind of convex pentagon is studied, which is the only unsolved case among the corresponding convex polygon problems. Any triangle and quadrilateral can tile the plane. The convex hexagons that can tile the plane with single element are classified into three types. Tiling of the plane using a single kind of convex polygons of more than six sides are proved to be impossible. Only the case of convex pentagon awaits settlement.

Now, the pentagonal tiling patterns with single convex element are conventionally classified into fourteen types (Figs. 1 and 2). The classification, however, is not so systematic and merely a kind of list.

So far the classification and exhaustive studies of tiling patterns are not very much noticed to the present. These studies don't include to the exhaustive study of the tiling figures, but mean a relation between them.

We are aiming at the solution of an otherwise exhaustive study of tiling convex pentagons by the following: through a systematic classification of them and through the examination of the possibility of tiling by respective convex pentagons.

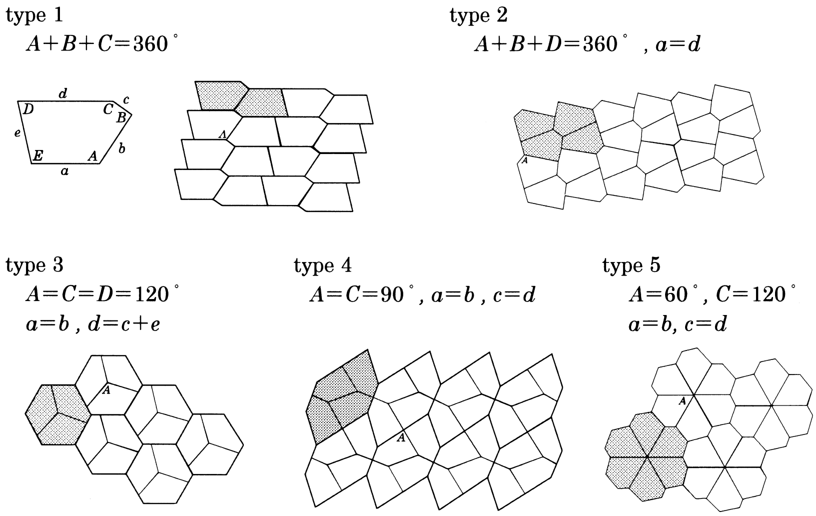

Fig. 1. Examples of type 1 to type 5. Colored regions outline a fundamental regions (unit cells) that cover the plane by translation. Only type 2 requires reflection. In all types of pentagons, arrangement of angles *A*, *B*, *C*, *D* and *E* and edge *a*, *b*, *c*, *d* and *e* is indicated common in the figure of type 1. Only the position of the angle *A* is indicated in every types.

2. Relation of Vertex and Edge in *Edge-to-Edge* Tiling

We first classify tiling of convex pentagon into *edge-to-edge** tiling and *non-edge-to-edge*** tiling that should be separated from the topological point of view. In the former, an edge of a tile can't be shared with plural edges and can be in the latter. Though the former tiling is still pentagonal even in topological point of view, the latter is not. Our interest lies more in the *edge-to-edge* case since it is more essential.

They are classified into twelve types depending on variety and arrangement of five edge-lengths. Then the possibility of tiling with a certain type of tile is examined.

In every *edge-to-edge* tiling by identical convex pentagons contain an infinite number of tiles each of which has at most six adjacent. Therefore a possible *valence**** of every vertex in tiling is 3, 4, 5 or 6. And in the case of tiling by convex pentagons, it can be shown that the average valence is 10/3 from the following Euler's formula.

The Euler's formula in an infinite plane is expressed as $V = E - F$ (V: the number of vertices, E: the number of edges, F: the number of faces).

Define V_3, V_4, V_5 and V_6 for the number of the vertices of valence 3, 4, 5 and 6, respectively, in *edge-to-edge* tiling of convex pentagon. Then the following relations hold for the above variables V_3, V_4, V_5 and V_6.

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*Example of *edge-to-edge* tiling are type 4, 5, 6, etc in Fig. 1.

**Example of *non-edge-to-edge* tiling are type 10, 11, 12, etc in Fig. 2.

***The number of edges which emanates a vertex is called a *valence* of the vertex.

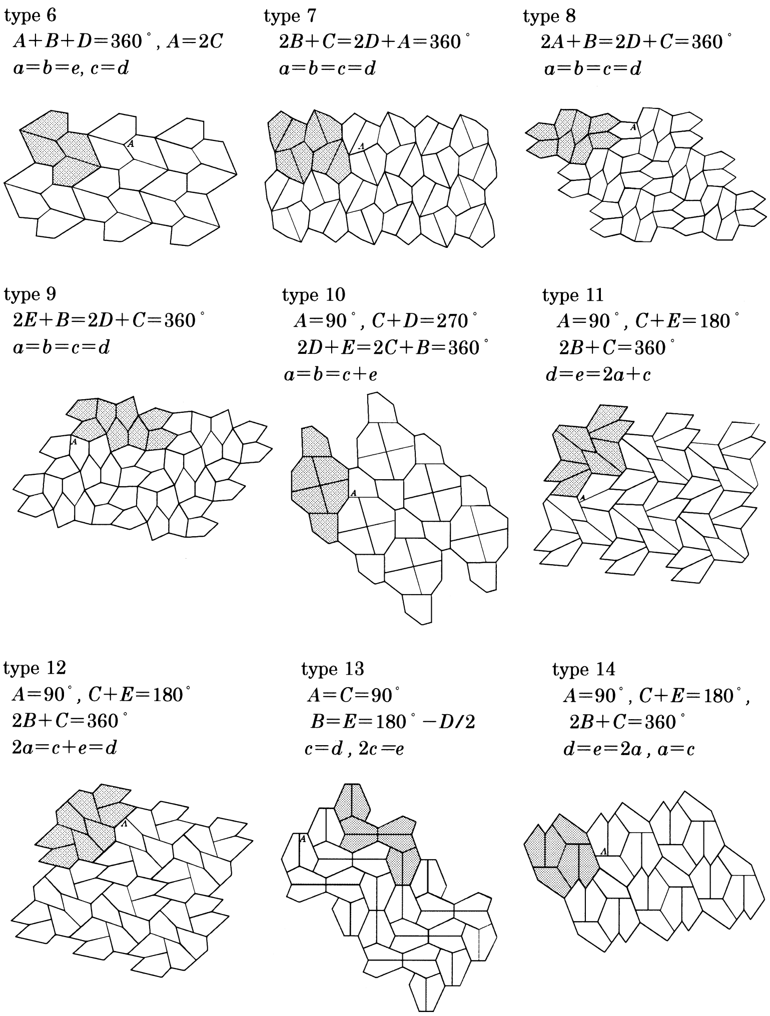

Fig. 2. Examples of type 6 to type 14. Colored regions outline as in Fig. 1 a fundamental regions (unit cells) that cover the plane by translation. Types 7, 8, 9, 10, 11, 12, 13 and 14 requires reflection. For the arrangement of angles and edges see the caption in Fig. 1.

(1) The tiling which contains only vertices of valences 3 and 4.

$$V_3: V_4 = 2: 1.$$

(2) The tiling which contains vertices of valences 3, 4 and 5.

$$V_3: V_4: V_5 = (1 + k): \left(\frac{1}{2} - 2k\right): k, \quad \text{where } V_5 = kF.$$

(3) The tiling which contains vertices of valences 3, 4 and 6.

$$V_3\colon V_4\colon V_6 = (1 + 2k)\colon \left(\frac{1}{2} - 3k\right)\colon k, \quad \text{where } V_6 = kF.$$

In the case (1), it can be seen that the number of possible combinations of angles is limited to three, for the tiling which uses one kind of combination of angles comprising the vertex of valence 4. Furthermore, in the above case there are several tilings where more than two kinds of combinations of angles of valence 4 are used. It is known that the tilings just mentioned includes those tiles with high degree of symmetry and those tiles with special angles. Therefore, we will conclude that it is able to find all possible tilings and tiles, by inspecting the convex pentagons which are included in the case (1).

Similarly, in the tiling which contains vertices of valence 5 or 6, it might be able to find all possible tilings by inspecting the convex pentagons which obey the relation (2) and (3), respectively.

3. Conclusion

We will find all *edge-to-edge* tilings of convex pentagon, by adopting the relations between vertices and edges which we have derived.

In fact, we obtained partially the type of convex pentagons which satisfy the relations between vertices and angles and which can form a tiling. We have also shown that convex pentagons with five different edge length are unable to form a tiling.

Further we discovered a set of solutions, which contains infinite kinds of tiling patterns. Huge variety of tiling patterns is included in it; periodic with various periods, quasi-periodic with various type, and irregular (Fig. 3).

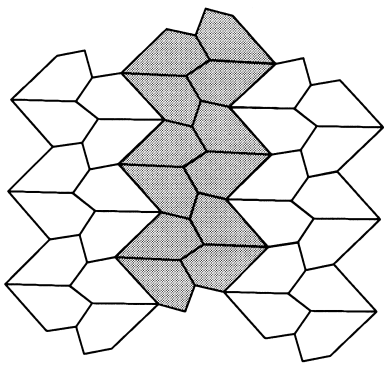

Fig. 3. An example of the new tiling pattern by the pentagon of type 6. In this tiling pattern, reflection tiles are used. In figure, colored tiles are reflection tiles. In the tilling by pentagon of type 6, it is possible to make infinite kinds of tiling patterns by using reflection tiles. Among these tilings, periodic patterns in two-dimension and quasi-periodic and irregular arrangements in one-dimension are contained.

The authors would like to thank Tanemura Masaharu for editical reading manuscripts.

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