

## Basic Consideration of Structures with Fractal Properties and Their Mechanical Characteristics

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**Abstract.** The purpose of this study is to clarify the mechanical characteristics of structures with fractal properties. This clarification is necessary, in order to apply fractal notion to artificial structures. Very few studies have been made from the viewpoint of such attempts. First, fractal systems are formulated for engineering applications, and an example of 3 dimensional structures with fractal properties is presented. Then the model and method of the analysis are introduced. Results of the analysis indicate that both how to change forms against applied forces and how to divide their forces reflect geometrical shapes. Especially in the case of the structures with fractal properties, the levels of internal forces are not much different from the regular structures. And it can be said that the structures with fractal properties can absorb stress unbalance caused by the change of local stiffness.

### 1. Introduction

There are many shapes in the natural world. They look so complex that we could not express them by several simple functions. But after B. B. Mandelbrot devised the notion of ‘fractal’ (MANDELBROT, 1977), we are able to express many natural shapes by some simple algorithms. From that time, hard- and soft-ware of computer systems have been improved day by day. And various researches on how to construct fractal shapes or how to imitate natural creatures have been developed (BARNESLEY, 1988). Recently fractal notion has been used for image compression, analysis on non-linear dynamics, and so on. But they are not sufficient in order to study why we can see fractal shapes widely in the natural world and to study how to apply fractal notion to our actual engineering structures, because they usually apply fractal notion only as some measures of shapes.

“Fractal” is geometrically defined by its self-similarity or self-affinity, which means that one partial shape can be mapped to the whole, or vice versa. The fractal systems

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provide the following useful characters for the actual engineering structures.

1. Fractal systems can be expressed with some simple algorithms not with complex functions.

2. Fractal systems consist of some unit elements, and form some stratified systems. The former is effective especially for programming on computers, and the latter may be useful to construct large structures. In this paper, behaviors of the structures with fractal properties are analyzed based on these characters of fractal systems.

## 2. Definition of Fractal Systems

First of all, it is helpful to formulate fractal systems. It has a connection with how to construct fractal systems. Here we adopt the method using “generator”. Let one unit consist of  $n$  elements,  $P_1, P_2, \dots, P_n$ . Using assembling rule  $A$ , a first generation  $G^1$  (equal to the first unit, called a “generator”) is denoted by  $G^1 = A(P_1, P_2, \dots, P_n)$ . Next generation  $G^2$  consists of  $n$   $G^1$ 's assembled according to the rule  $A$ . Also a certain generation consists of  $n$  previous generations according to the same rule  $A$ . This formulation can be expressed as follows:

$$\begin{aligned}
 G^1 &= A(P_1, P_2, \dots, P_n) \\
 G^2 &= A(G_{P_1}^1, G_{P_2}^1, \dots, G_{P_n}^1) \\
 &= A(A_{P_1}(P_1, P_2, \dots, P_n), \dots, A_{P_n}(P_1, P_2, \dots, P_n)) \\
 &\quad \vdots \\
 G^k &= A(G_{P_1}^{k-1}, G_{P_2}^{k-1}, \dots, G_{P_n}^{k-1}) \\
 &= A(A_{P_1}(G_{P_1}^{k-2}, \dots, G_{P_n}^{k-2}), \dots, A_{P_n}(G_{P_1}^{k-2}, \dots, G_{P_n}^{k-2})) \\
 &= \dots,
 \end{aligned} \tag{1}$$

where  $G_{P_n}^{k-1}$  is a  $(k-1)$ -th generation  $G^{k-1}$  on an element  $P_n$  of a  $k$ -th generation  $G^k$ , and  $A_{P_n}$  is assembling rule on an element  $P_n$ . It should be added that fractal systems are limited in this paper.

1. Fractal systems have strict the self-similarity, which means that a part of some generation takes the similar shape to the previous generation.

2. The number of mappings is finite.

3. Self-contact, self-intersection and self-overlap are allowed.

The first is useful to construct the structures made of same units. The second is inevitable to the actual structures. The third is also inevitable to avoid unstable states.

## 3. 3-D Spatial Structures with Fractal Properties

### 3.1. Construction of structures with fractal properties

According to the definition stated above, one of famous fractals, which is well known as a Koch curve, can be constructed using generators. In this case, elements of a generator

are 1-d (1-dimensional) lines and a generator itself is a curve in 2-d plane. In general the total length of all elements becomes longer according to the number of generation. But let the total length of each generation be the same. It can be seen that one line is folded like a string (Fig. 1 (a)). Other kinds of Koch curves, which correspond to some different shapes of generators, are shown in Fig. 1 (b). “Fractal dimension” of each curve is appended in the figure. In brief, it can represent how complex the fractal shape is. As the fractal dimension of Koch curve becomes larger, it can be seen that one line is folded like a string in the different way from in Fig. 1 (a).

3.2. 3-D Spatial structure with fractal properties

Considering the possibility of actual structures with fractal properties, we propose an example of 3-d spatial structures constructed by 3-d generators with 2-d elements (Fig. 2). In this case an element of a generator is 2-d regular triangle, and a generator consists of the 6 elements. Some generator is on each element of the next generation. Clearly many edges overlap each other. 3-d spatial truss structures with fractal properties can be constructed considering that all edges are some kind of truss members. Besides it forms a part of octahedral trusses which are conventional truss structures (NATORI *et al.*, 1985).

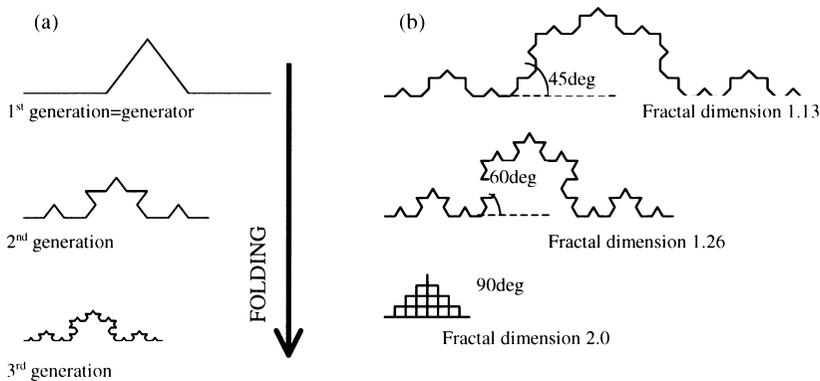


Fig. 1. 2-d Koch curves with constant total lengths.

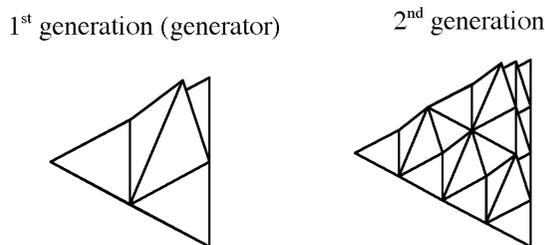


Fig. 2. 3-d spatial structure with fractal properties.

#### 4. Mechanical Analysis on Structures with Fractal Properties

##### 4.1. Method and models for analysis

Mechanical analysis is discussed in this section. Sub-structuring methods are suitable for analysis on structures with fractal properties, since these structures consist of stratified sub-structures. In this research the direct flexibility method (PARK and FELIPPA, 1998) is used, which is not only one of sub-structuring methods but also suitable for parallel computing. Parallel computing was not used this time, since models are so simple.

The model of the structure with fractal properties is the 2-d truss shaped Sierpinski-Gasket made by regular triangles (Fig. 3). The model shown in Fig. 3 (a) is the fourth generation shape of Sierpinski-Gasket, and we call this model the Sierpinski-Gasket truss. In contrast, the complete planer truss shown in Fig. 3 (b) is analyzed by the same method.

The quantities of elements are assumed to be those of steel shown in Table 1. And three cases of applied forces and four cases of boundary conditions are considered (Fig. 4). Let all applied forces be 1N step loads.

##### 4.2. Results of the analysis

The representative results are shown in the following figures for the cases a and c, and cases 3 and 4. The results of other cases have been shown in KISHIMOTO (1999). The displacements of nodes are shown in Fig. 5. The scale indicates the unit value of arrows. The unit length of arrows is equal to the length of an element of a triangle in each figure. It can be seen that displacements of the structures with fractal properties are larger than displacements of the complete planer truss. But both levels are not so much different that

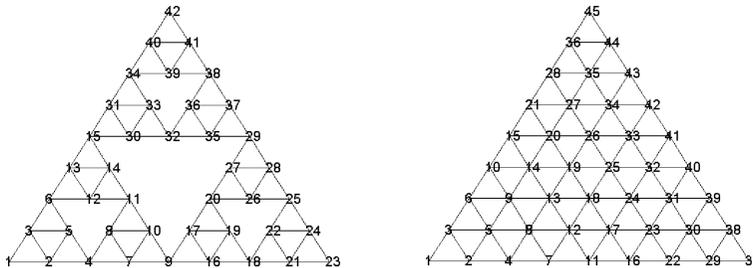


Fig. 3. Models of a Sierpinski-Gasket truss and a complete planer truss.

Table 1. Quantities of models.

Young modulus; $E$	$2.16 \times 10^{11} \text{ N/m}^2$
Section area of element; $A$	$\pi \times 2^2 \times 10^{-6} \text{ m}^2$
Length of element; $l$	0.1 m
Density of element; $\rho$	$7.86 \times 10^3 \text{ kg/m}^3$

they can be expressed on the same scale.

The distributions of internal forces of truss members are shown in Fig. 6. Levels of internal forces are not so different among these models, in spite that Sierpinski-Gasket truss has fewer elements. It should be noted that some part of Sierpinski-Gasket truss (circled part) shows the similar force distribution even under different cases.

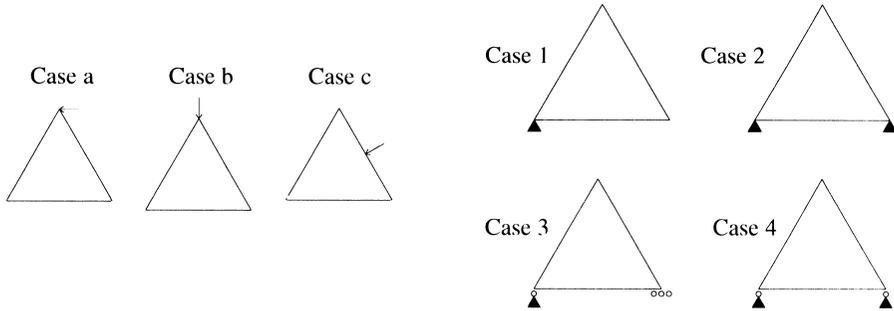


Fig. 4. Applied forces and boundary conditions.

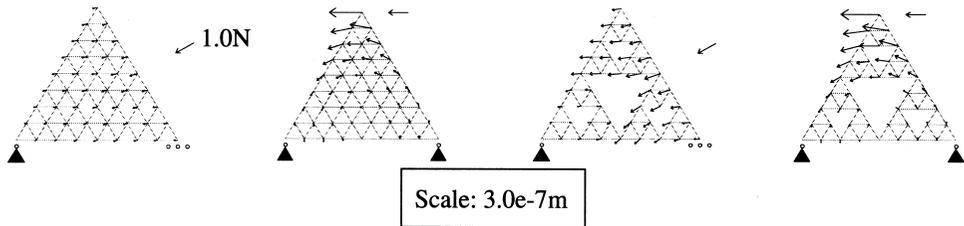


Fig. 5. Displacements; Case 3c and Case 4a.

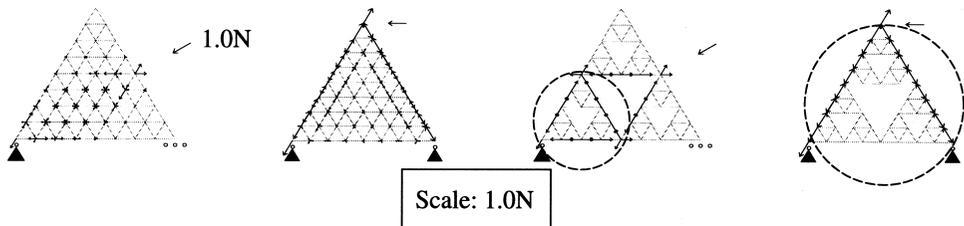


Fig. 6. Internal forces; Case 3c and Case 4a.

4.3. Results of the models with random errors on elements' lengths

Errors of elements' lengths seem to be avoidable in actual structures. Random errors with normal distribution  $N(0, 0.05^2)$  are assumed on all elements. The differences of displacements of nodes between the models with no error and those with random errors are shown in Fig. 7. The differences of Sierpinski-Gasket trusses are larger than those of complete planar trusses. But they can be denoted on the same scale.

The differences of internal forces among these models are shown in Fig. 8. The differences of complete planar trusses are 100 times larger than those of Sierpinski-Gasket trusses. And they can not be denoted on the same scale.

4.4. Consideration to the results of analysis

The whole stiffness of Sierpinski-Gasket trusses is less than that of complete planar trusses. Displacements from initial to equilibrium positions of the former trusses are larger than those of the latter. But there is no so much difference on levels of internal forces between two models. Levels of internal forces are important to design the elements of structures. Because Sierpinski-Gasket trusses have fewer elements, it can lighten the total weight that levels of internal forces of both models are nearly the same. This characteristic comes from their gaps; it is the most typical feature of fractal.

Other feature of fractal is the scaling, which means that its characteristics are independent of its size. From this feature the way to divide applied forces is different between the model with fractal properties and the model of conventional trusses. In

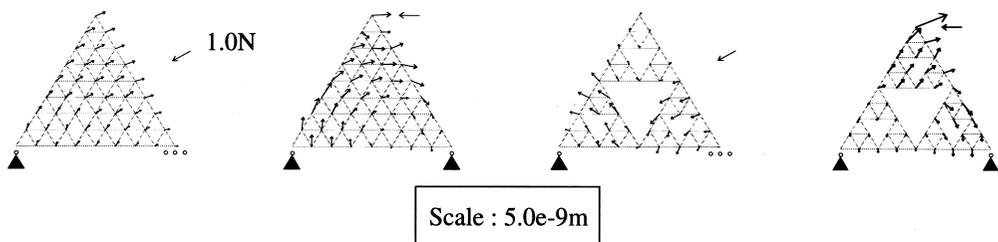


Fig. 7. Differences of displacements between the models with no errors and those with random errors.

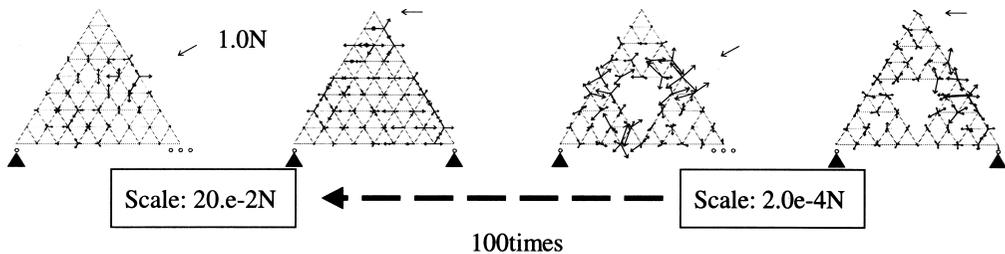


Fig. 8. Differences of internal forces between the models with no errors and those with random errors.

Sierpinski-Gasket trusses the elements transmitting forces are limited (Fig. 6). This characteristic seems to come from its self-similarity. And it may be suitable for decentralized control for its shape, because the elements under control can be limited.

The degree of stability of complete planar trusses is far more than that of Sierpinski-Gasket trusses. Sierpinski-Gasket trusses are almost statically determinate. From these results, it can be said that they easily absorb stress unbalance under a few changes of deformations. This properties remarkably appears for the models with random errors and with local defects (KISHIMOTO, 1999).

## 5. Conclusion

The characteristics of planar truss structures with fractal properties are clarified. They have some ability of averaging stress unbalances, and their internal force distributions reflect the self-similarity nature. This might be a first attempt to analyze the mechanical behavior of structures with fractal properties and a first step to apply useful characters of the natural world to our artifacts.

## REFERENCES

- BARNESLEY, M. F. (1988) *Fractals Everywhere*, Academic Press.
- KISHIMOTO, N. (1999) A study on the construction and the mechanical characters of structures with fractal properties, Master Thesis, The Univ. of Tokyo (Dept. Aeronautics and Astronautics).
- MANDELBROT, B. B. (1977) *The Fractal Geometry of Nature*, W. H. Freeman and Company.
- NATORI, M. and MIURA, K (1985) Deployable structures for space applications, AIAA85-0727, AIAA/ASME/ASCE/AHS 26th Structures, Structural Dynamics and Materials Conf., Orlando.
- PARK, K. C. and FELIPPA, C. A. (1998) A variational framework for solution method developments in structural mechanics, *Trans. of the ASME*, Vol. 65.