Original Paper *Forma*, **15**, 267–272, 2000

Sierpinski Gaskets in Excitable Media

Yumino HAYASE

Graduate School of Humanities and Sciences, Ochanomizu University, Tokyo 112-8610, Japan
E-mail: yumino@newtopia.phys.ocha.ac.jp

(Received April 19, 2000; Accepted May 19, 2000)

Keywords: Sierpinski Gasket, Excitable Media, Reaction-Diffusion System, Spatio-Temporal Pattern

Abstract. In our previous papers, we have shown by computer simulations that a Sierpinski gasket pattern appears in a Bonhoffer-van der Pol type reaction diffusion system. In this paper, we show another class of regular self-similar structure which is found in four different excitable reaction diffusion systems. This result strongly implies that the existence of the self-similar spatio-temporal evolution is universal in excitable reaction-diffusion media.

1. Introduction

Recently a rich variety of pulse dynamics has been found in nonlinear open systems. Especially, collision and self-replication of pulses has been investigated in reaction diffusion systems. Computer simulations of several reaction-diffusion equations have revealed that a propagating pulse is stable upon collision with another pulse. That is, a pair of counter-propagating pulse undergoes an elastic-like collision (PETOV *et al.*, 1994; OHTA *et al.*, 1997). It is also possible that a pulse pair is deformed during collision but survives again just like a soliton in an integrable non-dissipative system (KOSEK and MAREK, 1995; HAYASE, 1997). Self-replication of pulses has also been found by simulations. In the Gray-Scott model, a motionless pulse splits into two pulses which grow and repeat self-replication untill the density of pulses is sufficiently large (PETOV *et al.*, 1994; NISHIURA *et al.*, 1995). A propagating pulse can also self-replicate in which a pulse is emitted as a back-firing (PETOV *et al.*, 1994; MIMURA and NAGAYAMA, 1997).

It is important to note that three basic properties of pulses, pair annihilation and preservation upon collision and self-replication can coexist in a small but finite parameter region. In this situation, we have shown in previous papers (HAYASE, 1997; HAYASE and OHTA, 1998) that the interplay among the three properties causes a regular self-similar spatio-temporal evolution of a trajectory of pulses. An extinction of pulses except for the edges occurs every three generations. This is isomorphic to a Sierpinski gasket (SG) generated by cellular automaton

$$a^{t+1}(i) = a^t(i-1) + a^t(i+1), \ \ \text{mod}k \tag{1}$$

where $a^t(i) = 0, 1, ..., k - 1$ defined on a one-dimensional lattice. Actually the pattern in previous papers (HAYASE, 1997; HAYASE and OHTA, 1998) corresponds to the case $k = 3$.

The purpose of this paper is to explore how generic the formation of a regular self-similar pattern is. We will show that the SG is not an exceptional one for a particular set of reaction-diffusion equations. In four different model systems, a self-similar pattern equivalent with Eq. (1) with $k = 2$ will be obtained. The results definitely indicates that the existence of the self-similar spatio-temporal evolution is universal in excitable reaction-diffusion media.

In Secs. 2–5, we introduce four different excitable reaction-diffusion systems which SG can be produced. Discussion is given in Sec. 6.

2. Bvp Model with a Cubic Nonlinear Term

In our previous papers (HAYASE, 1997; HAYASE and OHTA, 1998), we have reported that a regular self-similar spatio-temporal pattern like an SG appears in a Bonhoffer-van der Pol (BvP) type reaction-diffusion equations,

$$\tau \frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u) - v, \tag{2}$$

$$\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} + u - \gamma v + I, \tag{3}$$

where positive constants D_u and D_v are the diffusion rate of u and v, respectively. The parameters τ, γ and I are positive constants. In this section we shall explore the pulse dynamics of BvP model (2) and (3) with a cubic nonlinearity

$$f(u) = a\, u\, (u + 1)(1 - u), \tag{4}$$

where a is a positive constant. The parameters are chosen such that the system Eqs. (2) and (3) with (4) is excitable. Throughout this section, we set $D_u = 1$, $D_v = 10$, $a = 5$, $\gamma = 0.25$ and $I = 0.1$. We examine the behavior of pulses by changing the values of τ.

Equations (2)–(4) have been studied in detail theoretically (RINZEL and KELLER, 1973; KOGA and KURAMOTO, 1980; ITO and OHTA, 1992). As shown in Fig. 1, Eqs. (2)–(4) have a stable traveling pulse when $\tau < \tau_p$, whereas a stable motionless pulse exists when $\tau > \tau_m$.

In the region $\tau_b < \tau < \tau_m$, a motionless pulse looses stability and the breathing motion appears. It should be noted that there is an interval $\tau_p < \tau < \tau_b$ where neither motionless pulse nor traveling pulse exist. This is the very region where self-replication of pulses is observed and hence rich varieties of spatio-temporal patterns appear.

When the value of τ is in the middle of the region, a regular self-similar pattern like a Sierpinski gasket appears as in Fig. 2. Note that the SG in Fig. 2 differs from that reported

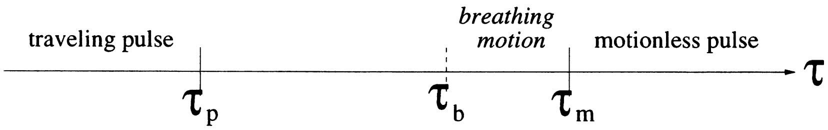

Fig. 1. Phase diagram of Eqs. (2) and (3) with (4) with the parameter τ.

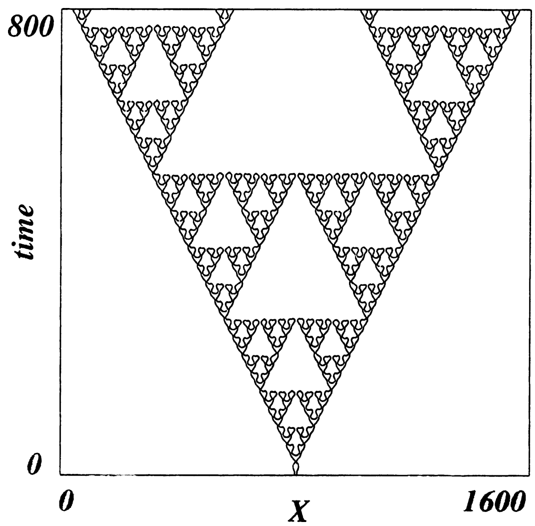

Fig. 2. Sierpinski gasket pattern for Eqs. (2), (3) and (4). The parameters is τ = 0.50. The lines indicate the contour line of u = 0.

previously (HAYASE, 1997; HAYASE and OHTA, 1998). Most crucial property is that preservation of pulses does not occur in Fig. 2. All of the pulses undergo pair-annihilation so that extinction of pulses except for the edge pulses occurs every two generations. This SG is equivalent with (1) with k = 2.

3. The BvP Model with a Hyperbolic Tangent Nonlinear Term

In the preceding section, we have obtained an SG with k = 2 in the excitable system (2)–(4). This is quite contrast to our previous result that an SG with k = 3 was obtained in Eqs. (2) and (3) with a hyperbolic tangent nonlinearity,

$$f(u) = \frac{1}{2}\left[\tanh\frac{u-\alpha}{\delta} + \tanh\frac{\alpha}{\delta}\right] - u \tag{5}$$

where α and δ are positive constants. The essential difference is that Eqs. (2) and (3) with (5) is bistable in a sence that a uniform stable solution and a stable limit cycle solution coexists.

In this section, we will show that an SG with k = 2 can be realized even for the hyperbolic tangent nonlinearity. We have carried out numerical simulations of Eqs. (2) and (3) with (5) in the parameter region where γ = 0.21 and other parameters are chosen to be almost same as of the SG with k = 3. An SG with k = 2 is really obtained as shown in Fig. 3.

4. The Gray-Scott Model

In Sec. 2 and Sec. 3, we have shown that the SG with k = 2 appears in the BvP model with two different nonlinear terms. In this section, it will be shown that the SG emerges the Gray-Scott model given by the following set of equations.

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} - uv^2 + F(1-u) \tag{6}$$

$$\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} + uv^2 - (F+k)v \tag{7}$$

where D_u and D_v are the diffusion coefficients. F and k are positive constants.

Self-replication of a pulse in the Gray-Scott model has been studied both numerically and analytically (PETOV *et al*., 1994; NISHIURA and UEYAMA, 1999; PEARSON, 1993). However, the spatio-temporal evolution of the interacting pulses has not been attracted much attention. We carry out simulations of (6) and (7) and obtain an SG with k = 2 as shown in Fig. 4.

5. The Prague Model

The fourth example of excitable systems where an SG appears is the following two-component reaction-diffusion model

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{\varepsilon} u[c - u + (1-c)v]\left(u - \frac{v+b}{a}\right), \tag{8}$$

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + u - v \tag{9}$$

where D is diffusion coefficient and a, b, c and ε are positive constants.

KASTANEK *et al*. (1995) have introduced (8) and (9) and studied numerically to simulate splitting of a reduction wave in Belousov-Zhabotinski reaction. We call this model (8) and (9) the Prague Model. They have found that self-replication of pulses occurs for a = 0.99, b = 0.01, c = 0.2, D = 1 and ε = 0.01. Here, we have investigated the behavior of the Prague model by changing the value of D and ε. An SG with k = 2 appears for D = 1.2 and ε = 0.01 as shown in Fig. 5.

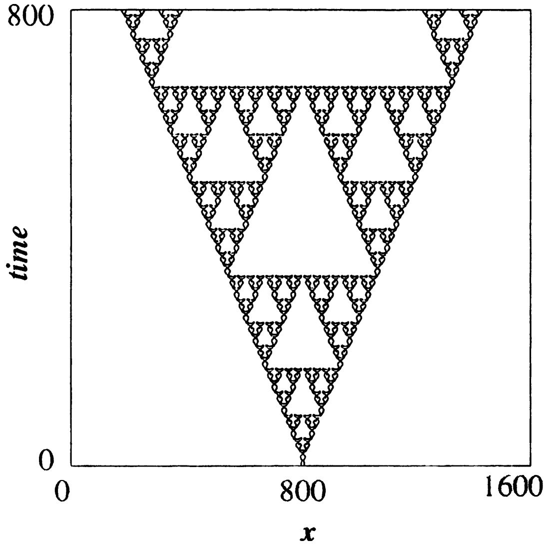

Fig. 3. SG for Eqs. (2), (3) and (5). The parameters are $\alpha = 0.105$, $\gamma = 0.21$, $\delta = 0.05$, $\tau = 0.4$, $D_v = 10.5$ and $D_u = 1$. The lines indicate the contour line of $u = 0.2$.

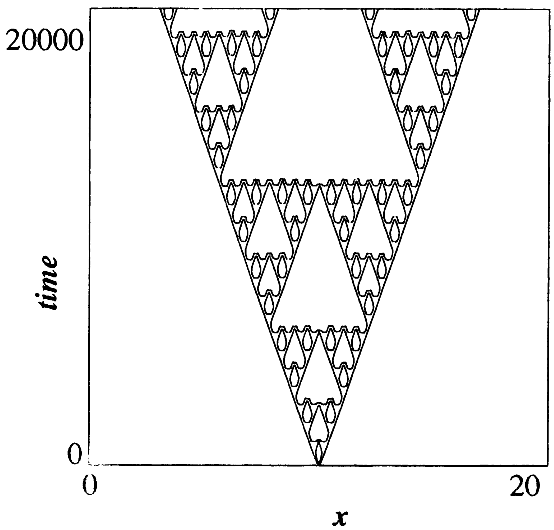

Fig. 4. SG for the Gray-Scott model. The parameters are $F = 0.0253$, $k = 0.0525$, $D_u = 1.15 \times 10^{-5}$ and $D_v = 1.0 \times 10^{-5}$. The lines indicate the contour line of $u = 0.5$.

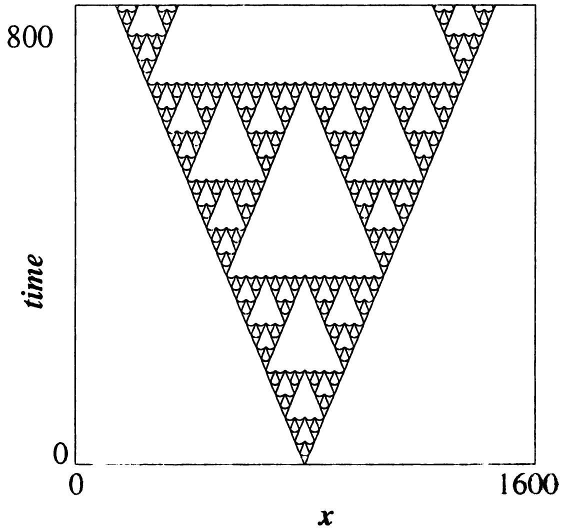

Fig. 5. SG for the Prague model. $a = 0.99$, $b = 0.01$, $c = 0.20$, $D = 1.2$ and $\varepsilon = 0.009$. The lines indicate the contour line of $u = 0.2$.

6. Discussion

In this paper, we have shown by numerical simulations, that Sierpinski gasket pattern with $k = 2$ can be produced by four different excitable reaction-diffusion systems, (i) the BvP model with a cubic nonlinear term, (ii) the BvP model with a hyperbolic nonlinear term, (iii) the Gray-Scott model and (iv) the Prague model. We expect from these results that the SG is very common, characterizing pulse dynamics in excitable media.

We are grateful to Professor T. Ohta for valuable discussion. We thank Professor Y. Nishiura for many suggestions of a self-replicating pulse in the Gray-Scott model.

REFERENCES

HAYASE, Y. (1997) *J. Phys. Soc. Jpn.*, **66**, 2584.
HAYASE, Y. and OHTA, T. (1998) *Phys. Rev. Lett.*, **81**, 1726.
ITO, A. and OHTA, T (1992) *Phys. Rev. A*, **45**, 8374.
KASTANEK, P., KOSEK, J., SNITA, D., SCHEIBER, I. and MAREK, M. (1995) *Physica D*, **84**, 79.
KOGA, S. and KURAMOTO, Y. (1980) *Prog. Theor. Phys.*, **63**, 105.
KOSEK, J. and MAREK, M. (1995) *Phys. Rev. Lett.*, **74**, 2134.
MIMURA, M. and NAGAYAMA, M. (1997) *Chaos*, **7** (4), 817.
NISHIURA, Y. and UEYAMA, D. (1999) *Physica D*, **130**, 73.
OHTA, T., KIYOSE, J. and MIMURA, M. (1997) *J. Phys. Soc. Jpn.*, **66**, 1551.
PEARSON, J. E. (1993) *Science*, **261**, 189.
PETOV, V., SCOTT, K., and SHOWALTE, K. (1994) *Philos. Trans. R. Soc. London Ser. A*, **347**, 631.
RINZEL, J. and KELLER, J. B. (1973) *Biophys. J.*, **13**, 1313.