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Dense Packing of Equal Circles on a Sphere by the Minimum-Zenith Method: Symmetrical Arrangement

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Abstract. Dense packing of equal circles on a sphere is investigated. A systematic algorithm, the Minimum-Zenith Method (MZM), is proposed in this report. Started from a proper initial configuration, a circle is sequentially packed one by one so that the zenith angle is as small as possible. It is necessary to fix the size of the circle and some initial configuration. Circle configurations we examined have three- to six-fold rotational symmetry. The densest one among them for a specified circle number is the desired configuration of the method. All the cases up to N = 150 are studied in this paper. The obtained packing densities are equal to or slightly smaller than those by other methods (exact solutions, Monte-Carlo method, etc.) in spite of simplicity of the method.

1. Introduction

How do we locate non-overlapping N equal circles on a sphere so as to make the size of the circles as large as possible? It is equivalent to the problem of maximizing the minimum distance between N points on a sphere. It is called the Tammes problem after a botanist who studied the distribution of hollows on the surface of spherical pollen grain (TAMMES, 1930).

Many approaches to find the circle configuration and resulting packing density were performed to the Tammes problem. Mathematically proved solutions were found for N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and 24. The solutions for N = 2, 3, 4, 5, 6, 7, 8 and 9 were derived and proved by Schütte and Van Der Waerden (SCHÜTTE and VAN DER WAERDEN, 1951; VAN DER WAERDEN, 1952). DANZER (1963) proved the conjectures for N = 10 and 11. FEJES TÓTH (1943) proved the conjecture for N = 12. ROBINSON (1961) proved the conjecture for N = 24.

Goldberg presented putative solutions for some numbers under N = 42 by the axially symmetric packing (GOLDBERG, 1967, 1969). He investigated arrangements which have three- to five-fold symmetry about only one axis. Tarnai and Gáspár studied the multi-symmetric arrangements which have tetrahedral, octahedral and icosahedral rotational symmetry and used same dense arrangements on each face of the polyhedra (TARNAI, 1984;

TARNAI and GÁSPÁR, 1987; GÁSPÁR, 1989).

Recently, some people applied a Monte-Carlo method by introducing the repulsive interaction. KOTTWITZ (1991) calculated the arrangement up to 90 circles and HARDIN *et al*. (1997) pushed it up to 130. Though their results are quite good, there is no mathematical proof that each of their results yields the highest packing density.

We propose a new algorithm, the Minimum-Zenith method (MZM), for the packing problem on a spherical surface. The method is that started from a point whose zenith angle is zero, the circles are packed sequentially with the zenith angle as small as possible. The method does not always give currently known best configurations. However the MZM has advantages that the algorithm is simple and the time for calculation is short. Circle configurations we examined have three- to six-fold rotational symmetry. The total number of circles calculated is up to 150. The general tendency of the packing density by the MZM is discussed.

2. Method

2.1. The minimum-zenith method (MZM)

In the MZM, the circles are sequentially packed onto the space whose zenith angle is minimum. The detailed procedure of the packing is as follows. (The words *North Pole* and *South Pole* are introduced for the convenience of the explanation; the zenith angle is zero on the North Pole and π [rad] on the South Pole. The word *point* refers to the center of the spherical cap*.)

1. Fix the size of the circle and give some points as the initial configuration around the North Pole.
2. Search for several points which has the possibility to be packed by a new circle.
3. Put the new circle on a point whose zenith angle is the minimum among all the points found in step 2.
4. Repeat step 2 and step 3.
5. Stop the packing if there is no vacancy to add one more circle in the vicinity of the South Pole.

Location of new points in step 2 are calculated by using the following equation (Eq. (1)). When we know the coordinates of two points (**A** and **B**) on a sphere and the size (angular diameter d [rad]) of the circle, we can calculate a new point (**C**) which is at d [rad] from both two points. If we use a unit sphere, Eq. (1) can be confirmed. Derivation of Eq. (1) is given in Appendix A.

$$\mathbf{C} = \frac{\cos d}{1+\cos l}(\mathbf{A}+\mathbf{B}) \pm \frac{1}{1+\cos l}\sqrt{\frac{1+\cos l - 2\cos^2 d}{1-\cos d}}(\mathbf{A}\times\mathbf{B}), \tag{1}$$

$$\cos l = (\mathbf{A},\mathbf{B}),$$

$$\cos d = (\mathbf{A},\mathbf{C}) = (\mathbf{B},\mathbf{C}),$$

*A spherical cap is the inside of a circumference of the circle on the spherical surface.

where **A**, **B** and **C** are unit vectors and l is the angular distance between **A** and **B**.

Our interest is a maximal d which produces the densest structure of N circles by the MZM. We include the MZM in bisection method in order to find such a d_{max}. In the bisection method, calculation begins with d_1 and d_2 which satisfy $d_1 < d_{max}$ and $d_2 > d_{max}$, and $d_3 = (d_1 + d_2)/2$ takes the place of d_1 if the MZM with d_3 produces a structure not less than N, otherwise d_2 is replaced by d_3. An arbitrary precision of d_{max} is attained by repeating this procedure. Appropriate d_1 and d_2 for each N are found by a preliminary calculation, in which the MZM is carried out for roughly changing d.

2.2. Further assumptions

In the MZM, there is no constraint about the initial configuration. So any initial configuration is possible if only the condition of non-overlapping among circles is satisfied. The examined structures are restricted to three- to six-fold rotational symmetry in the present paper. We use 7 initial configurations which have such symmetry and they are shown in Fig. 1.

Each resultant structure also has the same symmetry as its initial configuration. The axis of rotation is the earth's axis. Though some structures in Sec. 3 have imperfect rotational symmetry, they are made from complete symmetrical structures by removing a few circles.

As concerns n-fold rotational symmetry, N is restricted to the integer of the types: nk, $nk + 1$, and $nk + 2$ (k is a natural number), and calculation is not carried out for other N. It means that the finish of packing around the South Pole is restricted to two types which have n-fold rotational symmetry. For example, if the initial configuration is 03-type, the finish is either 03- or 13-type.

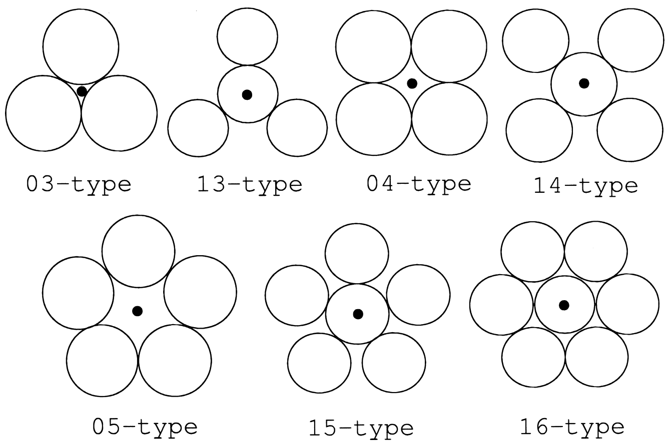

Fig. 1. Initial configurations. The solid circle is the North Pole. Concerning 16-type, the center circle does not contact with other 6 circles because of the curvature of the spherical surface.

3. Results

Calculation of d and the coordinates* of all circles were done for every initial configuration within N = 3–150. All values of d, which was rounded off to the seventh decimal places, are in Tables 1–3. The execution time for N = 150 of 03-type by a SONY PCG-Z505JX (with Pentium II 400 MHz) was 6 minutes. The program for the MZM was written in *Mathematica* Version 4.

About each N, some diameters made from different initial configurations are given in Tables 1–3. The biggest value in them is the candidacy for the final result. The final value of d should decrease monotonously when N increases. However the biggest value for N = 29 in Table 1 is d = 38.2028692 and the biggest value for N = 30 is d = 38.5873360, which are contrary to the monotonous decease. A structure made by removing one circle from the structure of N = 30 is denser than the structure of N = 29. Therefore d = 38.5873360 calculated for N = 30 is adopted as the final result for N = 29. Such revises are done for some other numbers, and we get Tables 4–6 as the final result.

The biggest value of d, the packing density D of its structure and the type of the initial configuration are in Tables 4–6. The packing density is defined by $D = (N/2)(1 - \cos(d/2))$, which is derived in Appendix B. Tables 4–6 also have d and D of the accomplished solutions. The word "accomplished solution" means the exact solution, the conjectured solution or the currently best solution. Numerical values of d of the accomplished solutions are cited from HARDIN *et al*. (1997). Some diameters in Tables 4–6 have an asterisk(*), where the revises mentioned in last paragraph were done.

Tables 7–9 are supplementary Tables to Tables 4–6. In the third column of Tables 7–9, numbers of circles on the same zenith angle are written in the braces such as $\{\cdots\}$, numbers of circles N_{north} and N_{south} in the Northern and the Southern hemisphere are in the parenthesis such as (N_{north}/N_{south}), and a note where necessary.

Both the first and the last number in the braces are 0 or 1, which shows respectively the non-existence or the existence of a circle at the North Pole and the South Pole. The numbers in the braces are divided into two parts by a slash mark. The numbers of the left part are the circles in the Northern hemisphere and the right part are those in the Southern hemisphere. It often happens that a circle is crossed by the equator and exists on both hemisphere. For such cases, the center of the spherical cap is the criterion to judge which hemisphere the circle belongs. Sometimes the center of the spherical cap is just on the equator. On such cases, the numbers of the circle on the equator are sandwiched between two slash marks. Generally, the numbers in the braces correspond to the degree of the rotational symmetry of the initial configuration except for both ends in the braces. For example, the braces for 04-type and 14-type are lined with 4 except for both ends. There is a possibility that a multiple of the degree of the rotational symmetry comes in the brace. Such cases are seen for 13-type, where 6, a multiple of 3 occasionally appears.

In the fourth column of Tables 7–9, D_{north} and D_{south}, the packing density in the Northern and the Southern hemisphere are shown. The left is the Northern density, the right is the Southern density and they are distinguished by a slash mark. Derivation of equations

*The coordinates data of circles are omitted.

Table 1. Results of calculation of the MZM (N: 1–50).

N	03-type	13-type	04-type	14-type	05-type	15-type	16-type
1	—	—	—	—	—	—	—
2	—	—	—	—	—	—	—
3	120.0000000	—	—	—	—	—	—
4	109.4712206	109.4712206	90.0000000	—	—	—	—
5	—	90.0000000	90.0000000	90.0000000	72.0000000	—	—
6	90.0000000	—	—	90.0000000	72.0000000	72.0000000	—
7	77.8695421	77.8695421	—	—	—	72.0000000	—
8	—	70.5287794	74.8584922	—	—	—	60.0000000
9	70.5287794	—	70.0746957	70.0746957	—	—	—
10	66.0855635	66.0855635	—	65.5301995	63.4349488	—	—
11	—	60.2867706	—	—	63.4349488	63.4349488	—
12	63.4349488	—	60.0000000	—	—	63.4349488	—
13	56.5754591	56.4696228	57.1367031	57.1367031	—	—	—
14	—	55.3739522	—	54.7356103	—	—	54.7356103
15	53.6578501	—	—	—	52.5001329	—	—
16	51.4264327	51.9153522	52.2443958	—	51.7770431	51.7770431	—
17	—	48.7672621	50.7598800	50.7598800	—	51.0265527	—
18	49.4973220	—	—	49.5515904	—	—	—
19	46.6664554	47.4226185	—	—	—	—	—
20	—	47.4146707	46.5585052	—	45.5239984	—	46.6746201
21	45.5572900	—	45.2180560	45.1915559	44.9559895	44.9559895	—
22	44.4026793	44.3008472	—	44.0150592	—	44.4031267	—
23	—	41.5538952	—	—	—	—	—
24	43.6907671	—	43.6907671	—	—	—	—
25	41.1559947	40.4341526	41.4709700	41.2740766	41.3930609	—	—
26	—	40.3477621	—	40.4266144	41.0261415	41.0261415	40.7247910
27	40.2436110	—	—	—	—	40.6775806	—
28	39.1767874	39.0123428	38.8186439	—	—	—	—
29	—	37.4165969	38.2028692	38.0179963	—	—	—
30	38.5873360	—	—	37.4786219	38.1608932	—	—
31	36.6883674	36.7619756	—	—	37.7090249	37.7098291	—
32	—	36.7619756	37.1407154	—	—	37.4308238	36.7944299
33	36.0270676	—	35.9810429	35.7179506	—	—	—
34	35.1862485	35.5660107	—	35.3400804	—	—	—
35	—	34.3637642	—	—	34.7424671	—	—
36	34.8675633	—	34.4120592	—	34.3573825	34.3939571	—
37	33.5997903	33.9079263	34.0105999	34.1127402	—	34.0280886	—
38	—	33.9079263	—	33.7339548	—	—	34.2506607
39	33.1576064	—	—	—	—	—	—
40	32.1406217	32.9471036	32.9711929	—	32.8907914	—	—
41	—	31.8398237	32.2415763	32.5991358	32.6007976	32.6429088	—
42	31.7918479	—	—	32.3356860	—	32.5063863	—
43	31.2557653	31.3789080	—	—	—	—	—
44	—	31.2110028	31.3269105	—	—	—	31.8524379
45	30.8702382	—	31.1540286	31.2395058	30.4994776	—	—
46	30.2583189	30.6687447	—	30.7224694	30.2763267	30.2625981	—
47	—	29.9232761	—	—	—	30.0388485	—
48	29.8070466	—	30.7627856	—	—	—	—
49	29.1305497	29.6354607	29.7751470	29.6690225	—	—	—
50	—	29.4692102	—	29.4281698	29.5492902	—	29.7529564

The diameters were calculated for 7 initial configurations by using bisection method and were rounded off to the seventh decimal places.

Table 2. Results of calculation of the MZM (N: 51–100).

N	03-type	13-type	04-type	14-type	05-type	15-type	16-type
51	28.9149508	—	—	—	29.3038144	29.2409948	—
52	28.6589755	28.7860576	28.7240016	—	—	29.0570155	—
53	—	28.1597710	28.5315199	28.6154248	—	—	—
54	28.3372868	—	—	28.2218810	—	—	—
55	27.9806275	27.9846035	—	—	27.7316071	—	—
56	—	27.5244851	28.0328553	—	27.5649886	27.5613942	27.7912023
57	27.5178515	—	27.5231868	27.3967549	—	27.3970889	—
58	26.8866563	27.2707899	—	27.0223300	—	—	—
59	—	27.0214228	—	—	—	—	—
60	26.6819848	—	26.6238660	—	27.0957727	—	—
61	26.4884612	26.5924420	26.4398154	26.3482006	26.7825525	26.7890645	—
62	—	26.0669625	—	26.1258600	—	26.6630773	26.4428471
63	26.2229594	—	—	—	—	—	—
64	25.8580897	25.8818529	25.8237219	—	—	—	—
65	—	25.6348832	25.5118499	25.6990905	25.5510655	—	—
66	25.5133394	—	—	25.5307403	25.4019066	25.5399275	—
67	25.0509323	25.2566219	—	—	—	25.3972721	—
68	—	24.9695949	25.0153957	—	—	—	25.0522983
69	24.9080432	—	24.9087606	24.8354524	—	—	—
70	24.7482837	24.5761966	—	24.4672636	24.8738502	—	—
71	—	24.1016042	—	—	24.6938308	24.7697553	—
72	24.5219162	—	24.4722456	—	—	24.6464783	—
73	24.2762368	23.9657903	24.1081653	24.0495454	—	—	—
74	—	23.8693935	—	23.9576561	—	—	24.2329882
75	23.9743019	—	—	—	23.8260000	—	—
76	23.6044441	23.7479711	23.6050144	—	23.7189429	23.7699069	—
77	—	23.4062819	23.5134479	23.5636803	—	23.6438765	—
78	23.5017552	—	—	23.3180259	—	—	—
79	23.3474424	23.1854413	—	—	—	—	—
80	—	22.7330068	23.2165522	—	23.3096378	—	23.2845983
81	23.1441548	—	22.9086668	22.7935831	23.0804012	23.0834686	—
82	23.0066249	22.5992291	—	22.6407631	—	22.9767068	—
83	—	22.5419589	—	—	—	—	—
84	22.6046477	—	22.4240658	—	—	—	—
85	22.2810637	22.3033212	22.3465100	22.3245450	22.3947732	—	—
86	—	22.2004175	—	22.2136674	22.3141534	22.3145128	22.6077519
87	22.1884679	—	—	—	—	22.2427482	—
88	22.0422899	22.0383441	22.0357712	—	—	—	—
89	—	21.8993945	21.9016293	21.9490008	—	—	—
90	21.8423094	—	—	21.7624332	22.0661684	—	—
91	21.7127199	21.7334707	—	—	21.9319342	21.9821440	—
92	—	21.4742746	21.5768735	—	—	21.9229884	21.8065783
93	21.3779120	—	21.4548326	21.3524402	—	—	—
94	21.1256318	21.2960409	—	21.2362523	—	—	—
95	—	21.0872931	—	—	21.2296104	—	—
96	21.0592116	—	21.0978716	—	21.1443837	21.1333118	—
97	20.8835011	20.9568795	20.9768047	21.0175725	—	21.0595592	—
98	—	20.8930562	—	20.9082133	—	—	21.2851672
99	20.7483534	—	—	—	—	—	—
100	20.6235988	20.7596424	20.7717411	—	20.8733274	—	—

Table 3. Results of calculation of the MZM (N: 101–150).

N	03-type	13-type	04-type	14-type	05-type	15-type	16-type
101	—	20.5936339	20.6881347	20.7025288	20.7272872	20.7914661	—
102	20.4352840	—	—	20.5694760	—	20.6939996	—
103	20.2255280	20.5293017	—	—	—	—	—
104	—	20.2911483	20.4235489	—	—	—	20.4675310
105	20.1581303	—	20.3488939	20.1853993	20.1231965	—	—
106	19.9751552	20.1248250	—	20.0736722	20.0568560	20.0819460	—
107	—	19.9618695	—	—	—	19.9917557	—
108	19.9092797	—	20.0115567	—	—	—	—
109	19.8717401	19.7970862	19.8825806	19.8897034	—	—	—
110	—	19.6983876	—	19.7986109	19.7386184	—	20.0920311
111	19.5870529	—	—	—	19.6714784	19.5611536	—
112	19.3762547	19.6232236	19.6920869	—	—	19.5030108	—
113	—	19.4283413	19.6020811	19.5880021	—	—	—
114	19.3209526	—	—	19.5554518	—	—	—
115	19.2664953	19.3809077	—	—	19.2888005	—	—
116	—	19.0784712	19.3853978	—	19.2505461	19.1350114	19.3410301
117	19.1667588	—	19.3437944	19.3104798	—	19.1032727	—
118	19.0905751	18.9613007	—	19.1306627	—	—	—
119	—	18.7949225	—	—	—	—	—
120	18.9203907	—	19.1159978	—	18.9972021	—	—
121	18.7163278	18.6776818	18.9248872	18.9504836	18.9156676	18.8830391	—
122	—	18.5724875	—	18.9051462	—	18.8267317	19.0192862
123	18.6638432	—	—	—	—	—	—
124	18.5749472	18.5061748	18.7415454	—	—	—	—
125	—	18.3618961	18.6870591	18.6621669	18.4307073	—	—
126	18.4841219	—	—	18.6006261	18.3848350	18.4055446	—
127	18.3591133	18.3105925	—	—	—	18.3666667	—
128	—	18.1613917	18.4504020	—	—	—	18.3843935
129	18.2845934	—	18.3737450	18.3813904	—	—	—
130	18.1857447	18.0631634	—	18.2198193	18.1523036	—	—
131	—	17.8941344	—	—	18.1225812	18.1954044	—
132	18.0848921	—	18.2831860	—	—	18.1679650	—
133	17.9216669	17.8136514	18.1164476	18.0410611	—	—	—
134	—	17.7653965	—	18.0104086	—	—	18.0352521
135	17.8211070	—	—	—	17.8617281	—	—
136	17.7102536	17.6765303	17.8995373	—	17.8132267	17.7247533	—
137	—	17.6382918	17.8597825	17.7756227	—	17.6737649	—
138	17.6473645	—	—	17.6717639	—	—	—
139	17.5756744	17.5944791	—	—	—	—	—
140	—	17.4076440	17.6355493	—	17.4939552	—	17.4819291
141	17.5481852	—	17.5598327	17.4464944	17.4645664	17.4013185	—
142	17.4217485	17.3141554	—	17.3710050	—	17.3674829	—
143	—	17.2387159	—	—	—	—	—
144	17.3390412	—	17.4136350	—	—	—	—
145	17.2419108	17.1199714	17.2468707	17.2274582	17.3587143	—	—
146	—	17.0014986	—	17.1879344	17.3312211	17.1918675	17.2607005
147	17.1801361	—	—	—	—	17.1744775	—
148	17.0450209	16.9721720	17.0320373	—	—	—	—
149	—	16.9493732	17.0020507	16.9783988	—	—	—
150	16.9743217	—	—	16.9311068	16.9823967	—	—

Table 4. The results of the MZM and the accomplished solutions (N: 1–50).

	Minimum-Zenith method			Accomplished solution	
N	Diameter:d[deg]	Density:D	Type	Diameter:d[deg]	Density:D
1	360.0000000	1.0000000	(trivial)	360.0000000	1.0000000
2	180.0000000	1.0000000	(trivial)	180.0000000	1.0000000
3	120.0000000	0.7500000	03	120.0000000	0.7500000
4	109.4712206	0.8452995	03,13	109.4712206	0.8452995
5	*90.0000000	0.7322330	13,04,03,14	90.0000000	0.7322330
6	90.0000000	0.8786797	03,14	90.0000000	0.8786797
7	77.8695421	0.7774833	03,13	77.8695421	0.7774833
8	74.8584922	0.8235820	04	74.8584922	0.8235820
9	70.5287794	0.8257654	03	70.5287794	0.8257654
10	66.0855635	0.8086817	03,13	66.1468220	0.8101397
11	*63.4349488	0.8214206	05,03,15	63.4349488	0.8214206
12	63.4349488	0.8960951	03,15	63.4349488	0.8960951
13	57.1367031	0.7913928	04,14	57.1367031	0.7913928
14	55.3739522	0.8015051	13	55.6705700	0.8099449
15	53.6578501	0.8073144	03	53.6578501	0.8073144
16	52.2443957	0.8171435	04	52.2443957	0.8171435
17	51.0265527	0.8288732	15	51.0903285	0.8309120
18	49.5515904	0.8284086	14	49.5566548	0.8285753
19	47.4226185	0.8019592	13	47.6919141	0.8109610
20	47.4146707	0.8438887	13	47.4310362	0.8444630
21	45.5572900	0.8189210	03	45.6132231	0.8209065
22	44.4031267	0.8155370	15	44.7401612	0.8278061
23	*43.6907671	0.8257992	03,04	43.7099642	0.8265162
24	43.6907671	0.8617035	03,04	43.6907671	0.8617035
25	41.4709700	0.8096883	04	41.6344612	0.8160145
26	41.0261415	0.8243005	05,15	41.0376616	0.8247585
27	40.6775806	0.8416732	15	40.6776007	0.8416740
28	39.1767874	0.8102446	03	39.3551436	0.8175660
29	*38.5873360	0.8143567	03	38.7136512	0.8196461
30	38.5873360	0.8424379	03	38.5971159	0.8428610
31	37.7098291	0.8317307	15	37.7098291	0.8317307
32	37.4308238	0.8460160	15	37.4752140	0.8480059
33	36.0270676	0.8087723	03	36.2545530	0.8189326
34	35.5660107	0.8122596	13	35.8077844	0.8232501
35	*34.8675633	0.8038827	03	35.3198076	0.8247048
36	34.8675633	0.8268507	03	35.1897322	0.8420804
37	*34.2506607	0.8202359	16	34.4224080	0.8284204
38	34.2506607	0.8424044	16	34.2506607	0.8424044
39	33.1576064	0.8106501	03	33.4890466	0.8268214
40	32.9711929	0.8221779	04	33.1583563	0.8314733
41	32.6429088	0.8261474	15	32.7290944	0.8304859
42	32.5063863	0.8392806	15	32.5063863	0.8392806
43	*31.8524379	0.8252594	16	32.0906244	0.8375667
44	31.8524379	0.8444515	16	31.9834230	0.8513657
45	31.2395058	0.8309295	14	31.3230814	0.8353537
46	*30.7627856	0.8238233	04	30.9591635	0.8343106
47	*30.7627856	0.8417325	04	30.7818159	0.8427680
48	30.7627855	0.8596417	04	30.7627855	0.8596417
49	29.7751470	0.8224203	04	29.9235851	0.8305940
50	29.7529564	0.8379610	16	29.7529564	0.8379610

The word "accomplished solution" means the exact solution, the conjectured solution or the finest solution at the present time. Numerical values of the accompolished solutions is cited from HARDIN *et al*. (1997). Numerical values of the MZM are the best values in Tables 1–3.

Table 5. The results of the MZM and the accomplished solutions (N: 51–100).

	Minimum-Zenith method			Accomplished solution	
N	Diameter:d[deg]	Density:D	Type	Diameter:d[deg]	Density:D
51	29.3038144	0.8292495	05	29.7529564	0.8379610
52	29.0570155	0.8314035	15	29.1947579	0.8392618
53	28.6154248	0.8219644	14	28.8138972	0.8333457
54	28.3372868	0.8213547	03	28.7169205	0.8433934
55	*28.0328553	0.8187761	04	28.2627914	0.8321946
56	28.0328553	0.8336629	04	28.1480466	0.8404938
57	27.5231868	0.8181223	04	27.8266759	0.8361748
58	27.2707899	0.8173489	13	27.5564159	0.8344769
59	*27.0957727	0.8208530	05	27.3949757	0.8389947
60	27.0957727	0.8347658	05	27.1928300	0.8407287
61	26.7890645	0.8296613	15	26.8732779	0.8348618
62	26.6630773	0.8353851	15	26.6839970	0.8366906
63	26.2229594	0.8211881	03	26.4869225	0.8377296
64	25.8818529	0.8127527	13	26.2350433	0.8349885
65	25.6990905	0.8138842	14	26.0698299	0.8374339
66	25.5399275	0.8162430	15	25.9474437	0.8423866
67	25.3972721	0.8194175	15	25.6839813	0.8379447
68	25.0522983	0.8092978	16	25.4638245	0.8359939
69	24.9087606	0.8118530	04	25.3336364	0.8396713
70	24.8738502	0.8213210	05	25.1709200	0.8409769
71	24.7697553	0.8261233	15	24.9879381	0.8406832
72	24.6464783	0.8294728	15	24.9264861	0.8483523
73	24.2762368	0.8160100	03	24.5537792	0.8347031
74	24.2329882	0.8242545	16	24.4209398	0.8370413
75	23.9743019	0.8177174	03	24.3017225	0.8401210
76	23.7699069	0.8146021	15	24.1281944	0.8392529
77	23.6438765	0.8166229	15	24.0012837	0.8414070
78	23.5017552	0.8173482	03	23.9310254	0.8473698
79	23.3474424	0.8170292	03	23.6239917	0.8364302
80	23.3096378	0.8247034	05	23.5530672	0.8419575
81	23.1441548	0.8232384	03	23.3476377	0.8377275
82	23.0066249	0.8235597	03	23.1946074	0.8370268
83	*22.6077519	0.8050419	16	23.0829976	0.8391280
84	*22.6077519	0.8147412	16	23.0517306	0.8469467
85	*22.6077519	0.8244405	16	22.7791621	0.8369482
86	22.6077519	0.8341398	16	22.6743694	0.8390468
87	22.2427482	0.8168963	15	22.5466574	0.8392992
88	*22.0661684	0.8132590	05	22.4678810	0.8430433
89	*22.0661684	0.8225006	05	22.3166023	0.8412166
90	22.0661684	0.8317421	05	22.1540232	0.8383577
91	21.9821440	0.8346109	15	22.0517963	0.8398920
92	21.9229884	0.8392610	15	22.0275815	0.8472635
93	21.4548326	0.8126414	04	21.8103801	0.8397165
94	21.2960409	0.8093009	13	21.7237135	0.8420341
95	*21.2851672	0.8170779	16	21.5945501	0.8409323
96	*21.2851672	0.8256787	16	21.5206099	0.8439919
97	*21.2851672	0.8342795	16	21.4006197	0.8433281
98	21.2851672	0.8428803	16	21.3710607	0.8496770
99	*20.8733274	0.8189401	05	21.1359674	0.8396197
100	20.8733274	0.8272123	05	21.0312020	0.8397375

Table 6. The results of the MZM and the accomplished solutions (N: 101–150).

	Minimum-Zenith method			Accomplished solution	
N	Diameter:d[deg]	Density:D	Type	Diameter:d[deg]	Density:D
101	20.7914661	0.8289620	15	20.9286834	0.8399094
102	20.6939996	0.8293602	15	20.8556887	0.8423350
103	20.5293017	0.8242490	13	20.7382700	0.8410685
104	20.4675310	0.8272640	16	20.6566210	0.8425784
105	20.3488939	0.8255894	04	20.5388524	0.8410338
106	20.1248250	0.8152453	13	20.4394089	0.8408636
107	*20.0920311	0.8202634	16	20.3612035	0.8423304
108	*20.0920311	0.8279294	16	20.3044447	0.8454816
109	*20.0920311	0.8355954	16	20.1493196	0.8403550
110	20.0920311	0.8432614	16	20.1113276	0.8448778
111	*19.6920869	0.8174707	04	19.9824769	0.8416966
112	19.6920869	0.8248353	04	19.8913044	0.8415667
113	19.6020811	0.8246284	04	19.8056013	0.8417980
114	19.5554518	0.8279823	14	19.7450093	0.8440720
115	*19.3853978	0.8208164	04	19.6239931	0.8410963
116	19.3853978	0.8279540	04	19.5497969	0.8420223
117	19.3437944	0.8315194	04	19.4612911	0.8416273
118	19.1306627	0.8202908	14	19.3893497	0.8425717
119	*19.1159978	0.8259776	04	19.3257514	0.8441602
120	19.1159978	0.8329186	04	19.3240201	0.8511018
121	*19.0192862	0.8314025	16	19.1357298	0.8415903
122	19.0192862	0.8382736	16	19.0700369	0.8427429
123	*18.7415454	0.8206961	04	19.0063891	0.8440015
124	18.7415454	0.8273684	04	18.9539116	0.8461820
125	18.6870591	0.8292090	04	18.8448151	0.8432368
126	18.6006261	0.8281455	14	18.7815856	0.8443011
127	*18.4504020	0.8213187	04	18.6900568	0.8427461
128	18.4504020	0.8277858	04	18.6349726	0.8443936
129	18.3813904	0.8280370	14	18.5634726	0.8444869
130	*18.2831860	0.8255823	04	18.5103522	0.8461803
131	*18.2831860	0.8319329	04	———	———
132	18.2831860	0.8382835	04	———	———
133	18.1164476	0.8293307	04	———	———
134	18.0352521	0.8281086	16	———	———
135	*17.8995373	0.8218052	04	———	———
136	17.8995373	0.8278927	04	———	———
137	17.8597825	0.8302872	04	———	———
138	17.6717639	0.8188659	14	———	———
139	*17.6355493	0.8214293	04	———	———
140	17.6355493	0.8273388	04	———	———
141	17.5598327	0.8261228	04	———	———
142	17.4217485	0.8189736	03	———	———
143	*17.4136350	0.8239745	04	———	———
144	17.4136350	0.8297365	04	———	———
145	17.3587143	0.8302468	05	———	———
146	17.3312211	0.8333317	05	———	———
147	17.1801361	0.8245019	03	———	———
148	17.0450209	0.8171291	03	———	———
149	17.0020507	0.8185153	04	———	———
150	16.9823967	0.8221082	05	———	———

The accomplished solutions are up to 130 (HARDIN *et al.*, 1997).

Table 7. Supplementary table to Table 4 (N: 1–50).

N	Type	{Numbers of circles on the same zenith angle}(N_{north}/N_{south}) and Notes	D_{north}/D_{south}
1	(trivial)	(trivial)	1.000/1.000
2	(trivial)	(trivial)	1.000/1.000
3	03	{0/3/0}(0/3/0)	0.750/0.750
4	03,13	{0,3/1}(3/1) or {1/3,0}(1/3) These two are congruent.	0.896/0.794, 0.794/0.896
5	13,04,03,14	{1/3/1}(1/3/1) or {0/4/1}(1/4/0) or Remove one circle from the strucure of N=6.	Omitted
6	03,14	{0,3/3,0}(3/3) or {1/4/1}(1/4/1) These two are congruent.	0.879/0.879, 0.879/0.879
7	03,13	{0,3/3,1}(3/4) or {1,3/3,0}(4/3) These two are congruent.	0.874/0.681, 0.681/0.874
8	04	{0,4/4,0}(4/4)	0.824/0.824
9	03	{0,3/3/3,0}(3/3/3)	0.826/0.826
10	03,13	{0,3,3/3,1}(6/4) or {1,3/3,3,0}(4/6) These two are congruent.	0.832/0.785, 0.785/0.832
11	05,03,15	{0,5/5,1}(5/6) or Remove a circle from the structure of N=12.	Omitted
12	03,15	{0,3,3/3,3,0}(6/6) or {1,5/5,1}(6/6) These two are congruent.	0.896/0.896, 0.896/0.896
13	04,14	{0,4,4/4,1}(8/5) or {1,4/4,4,0}(5/8) These two are congruent.	0.785/0.798, 0.798/0.785
14	13	{1,3,3/3,3,1}(7/7)	0.802/0.802
15	03	{0,3,3,3/3,3,0}(9/6)	0.847/0.768
16	04	{0,4,4/4,4,0}(8/8)	0.817/0.817
17	15	{1,5/5/5,1}(6/5/6)	0.829/0.829
18	14	{1,4,4/4,4,1}(9/9)	0.828/0.828
19	13	{1,3,3,3/3,3,3,0}(10/9)	0.844/0.760
20	13	{1,3,3,3/3,3,3,1}(10/10)	0.844/0.844
21	03	{0,3,3,3/3,3,3,3,0}(9/12)	0.843/0.795
22	15	{1,5,5/5,5,1}(11/11)	0.829/0.829
23	03,04	Remove one circle from the strucure of N=24.	Omitted
24	03,04	{0,3,3,3,3/3,3,3,3,0}(12/12) or {0,4,4,4/4,4,4,0}(12/12) These two are conguent.	0.862/0.862, 0.862/0.862
25	04	{0,4,4,4/4,4,4,1}(12/13)	0.840/0.779
26	05,15	{0,5,5,5/5,5,1}(15/11) or {1,5,5/5,5,5,0}(11/15) These two are conguent.	0.806/0.843, 0.843/0.806
27	15	{1,5,5/5/5,5,1}(11/5/11)	0.842/0.842
28	03	{0,3,3,3,3,3/3,3,3,3,1}(15/13)	0.846/0.774
29	03	Remove one circle from the strucure of N=30.	Omitted
30	03	{0,3,3,3,3,3/3,3,3,3,3,0}(15/15)	0.845/0.840
31	15	{1,5,5,5/5,5,5,0}(16/15)	0.846/0.818
32	15	{1,5,5,5/5,5,5,1}(16/16)	0.846/0.846
33	03	{0,3,3,3,3,3,3/,3,3,3,3,0}(18/15)	0.833/0.784
34	13	{1,3,3,6,3,3/6,3,3,3,0}(19/15)	0.824/0.801
35	03	Remove one circle from the strucure of N=36.	Omitted
36	03	{0,3,3,3,3,3,3/3,3,3,3,3,3,0}(18/18)	0.829/0.825
37	16	Remove one circle from the strucure of N=38.	Omitted
38	16	{1,6,6,6/6,6,6,1}(19/19)	0.842/0.842
39	03	{0,3,3,3,3,3,3,3/3,3,3,3,3,3,0}(21/18)	0.822/0.800
40	04	{0,4,4,4,4,4/4,4,4,4,4,0}(20/20)	0.858/0.786
41	15	{1,5,5,5,5/5,5,5,5,0}(21/20)	0.840/0.813
42	15	{1,5,5,5,5/5,5,5,5,1}(21/21)	0.839/0.839
43	16	Remove one circle from the strucure of N=44.	Omitted
44	16	{1,6,6,6,6/6,6,6,1}(25/19)	0.850/0.839
45	14	{1,4,4,4,4,4/4,4,4,4,4,4,0}(21/24)	0.832/0.830
46	04	Remove two circles from the strucure of N=48.	Omitted
47	04	Remove one circle from the strucure of N=48.	Omitted
48	04	{0,4,4,4,4,4,4/4,4,4,4,4,4,0}(24/24)	0.860/0.860
49	04	{0,4,4,4,4,4,4/4,4,4,4,4,4,1}(24/25)	0.843/0.802
50	16	{1,6,6,6,6/6,6,6,6,1}(25/25)	0.838/0.838

In the third column of Tables 7–9, numbers of circles on the same zenith angle are written in the braces, numbers of circles in the Northern and the Southern hemisphere are in the parenthesis, and a note where necessary. In the fourth column of Tables 7–9, the packing density in the Northern and the Southern hemisphere are shown. See Sec. 3 for details.

Table 8. Supplementary table to Table 5 (N: 51–100).

N	Type	{Numbers of circles on the same zenith angle}(N_{north}/N_{south}) and Notes	D_{north}/D_{south}
51	05	{0,5,5,5,5,5/5,5,5,5,5,1}(25/26)	0.829/0.830
52	15	{1,5,5,5,5,5/5,5,5,5,5,1}(26/26)	0.831/0.831
53	14	{1,4,4,4,4,4,4/4,4,4,4,4,4,4,0}(25/28)	0.835/0.809
54	03	{0,3,3,3,3,3,3,3,3,3,3/3,3,3,3,3,3,3,3,0}(30/24)	0.833/0.810
55	04	Remove one circle from the strucure of N=56.	Omitted
56	04	{0,4,4,4,4,4,4,4/4,4,4,4,4,4,4,0}(28/28)	0.848/0.819
57	04	{0,4,4,4,4,4,4,4/4,4,4,4,4,4,4,1}(28/29)	0.850/0.786
58	13	{1,3,3,6,3,3,6,3,3/3,3,3,3,3,3,3,3,3,0}(31/27)	0.835/0.800
59	05	Remove one circle from the strucure of N=60.	Omitted
60	05	{0,5,5,5,5,5,5/5,5,5,5,5,5,0}(30/30)	0.835/0.835
61	15	{1,5,5,5,5,5,5/5,5,5,5,5,5,0}(31/30)	0.836/0.823
62	15	{1,5,5,5,5,5,5/5,5,5,5,5,5,1}(31/31)	0.835/0.835
63	03	{0,3,3,3,3,3,3,3,3,3,3,3/3,3,3,3,3,3,3,3,3,3,0}(33/30)	0.846/0.797
64	13	{1,3,3,6,3,3,6,3,3,3/3,3,3,3,3,3,3,3,3,3,0}(34/30)	0.849/0.777
65	14	{1,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,0}(33/32)	0.821/0.807
66	15	{1,5,5,5,5,5,5,5/5,5,5,5,5,5,0}(36/30)	0.830/0.802
67	15	{1,5,5,5,5,5,5,5/5,5,5,5,5,5,1}(36/31)	0.831/0.808
68	16	{1,6,6,6,6,6/6,6,6,6,6,6,1}(31/37)	0.823/0.796
69	04	{0,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,1}(36/33)	0.830/0.794
70	05	{0,5,5,5,5,5,5,5/5,5,5,5,5,5,5,0}(35/35)	0.832/0.811
71	15	{1,5,5,5,5,5,5,5/5,5,5,5,5,5,5,0}(36/35)	0.836/0.816
72	15	{1,5,5,5,5,5,5,5/5,5,5,5,5,5,5,1}(36/36)	0.836/0.822
73	03	{0,3,3,3,3,3,3,3,3,3,3,3,3,3/3,3,3,3,3,3,3,3,3,3,3,1}(39/34)	0.834/0.798
74	16	{1,6,6,6,6,6,6/6,6,6,6,6,6,1}(37/37)	0.824/0.824
75	03	{0,3,3,3,3,3,3,3,3,3,3,3,3,3/3,3,3,3,3,3,3,3,3,3,3,3,0}(39/36)	0.834/0.801
76	15	{1,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,0}(41/35)	0.844/0.785
77	15	{1,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,1}(41/36)	0.845/0.788
78	03	{0,3,3,3,3,3,3,3,3,3,3,3,3,3/3,3,3,3,3,3,3,3,3,3,3,3,3,0}(39/39)	0.837/0.797
79	03	{0,3,3,3,3,3,3,3,3,3,3,3,3,3/3,3,3,3,3,3,3,3,3,3,3,3,3,1}(39/40)	0.838/0.796
80	05	{0,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,0}(45/35)	0.839/0.810
81	03	{0,3,3,3,3,3,3,3,3,3,3,3,3,3,3/3,3,3,3,3,3,3,3,3,3,3,3,3,0}(42/39)	0.839/0.808
82	03	{0,3,3,3,3,3,3,3,3,3,3,3,3,3,3/3,3,3,3,3,3,3,3,3,3,3,3,3,1}(42/40)	0.838/0.809
83	16	Remove three circles from the strucure of N=86.	Omitted
84	16	Remove two circles from the strucure of N=86.	Omitted
85	16	Remove one circle from the strucure of N=86.	Omitted
86	16	{1,6,6,6,6,6,6,6/6,6,6,6,6,6,6,1}(43/43)	0.847/0.822
87	15	{1,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,1}(46/41)	0.844/0.790
88	05	Remove two circles from the strucure of N=90.	Omitted
89	05	Remove one circle from the strucure of N=90.	Omitted
90	05	{0,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,5,0}(45/45)	0.839/0.825
91	15	{1,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,5,0}(46/45)	0.845/0.824
92	15	{1,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,5,1}(46/46)	0.845/0.833
93	04	{0,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,1}(48/45)	0.831/0.794
94	13	{1,3,3,6,3,3,6,6,3,6,3,3/3,6,3,3,3,3,3,3,3,3,3,3,3,3,3,0}(46/48)	0.827/0.791
95	16	Remove three circles from the strucure of N=98.	Omitted
96	16	Remove two circles from the strucure of N=98.	Omitted
97	16	Remove one circle from the strucure of N=98.	Omitted
98	16	{1,6,6,6,6,6,6,6,6/6,6,6,6,6,6,6,6,1}(49/49)	0.843/0.843
99	05	Remove one circle from the strucure of N=100.	Omitted
100	05	{0,5,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,5,5,0}(50/50)	0.835/0.819

for the density on the hemisphere is given in Appendix C. Each density was rounded off to the third decimal places.

4. Discussion

For N = 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 24, 31, 38, 42, 48 and 50, the MZM gave the same d-value as the best value by HARDIN *et al*. (1997). Tables 4–6 show that the MZM yields packing densities equal to or slightly smaller than those of other methods (exact

Table 9. Supplementary table to Table 6 (N: 101–150).

N	Type	{Numbers of circles on the same zenith angle}(N_{north}/N_{south}) and Notes	D_{north}/D_{south}
101	15	{1,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,5,5,5,0}(46/55)	0.836/0.822
102	15	{1,5,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,5,5,1}(51/51)	0.835/0.824
103	13	{1,3,3,6,3,3,6,6,3,6,3,3,3,6/3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,0}(55/48)	0.826/0.822
104	16	{1,6,6,6,6,6,6,6,6,6/6,6,6,6,6,6,6,6,1}(55/49)	0.841/0.814
105	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,1}(56/49)	0.838/0.813
106	13	{1,3,3,6,3,3,6,6,3,6,3,3,3,6/3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,0}(55/51)	0.833/0.798
107	16	Remove three circles from the strucure of N=110.	Omitted
108	16	Remove two circles from the strucure of N=110.	Omitted
109	16	Remove one circle from the strucure of N=110.	Omitted
110	16	{1,6,6,6,6,6,6,6,6,6/6,6,6,6,6,6,6,6,6,1}(55/55)	0.844/0.842
111	04	Remove one circle from the strucure of N=112.	Omitted
112	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(56/56)	0.842/0.808
113	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(56/57)	0.842/0.807
114	14	{1,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(57/57)	0.843/0.813
115	04	Remove one circle from the strucure of N=116.	Omitted
116	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(56/60)	0.844/0.812
117	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(56/61)	0.845/0.818
118	14	{1,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(61/57)	0.843/0.798
119	04	Remove one circle from the strucure of N=120.	Omitted
120	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(60/60)	0.846/0.820
121	16	Remove one circle from the strucure of N=122.	Omitted
122	16	{1,6,6,6,6,6,6,6,6,6,6/6,6,6,6,6,6,6,6,6,6,1}(61/61)	0.839/0.837
123	04	Remove one circle from the strucure of N=124.	Omitted
124	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(64/60)	0.843/0.811
125	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(64/61)	0.843/0.816
126	14	{1,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(65/61)	0.845/0.811
127	04	Remove one circle from the strucure of N=128.	Omitted
128	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(64/64)	0.841/0.815
129	14	{1,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(69/60)	0.846/0.810
130	04	Remove two circles from the strucure of N=132.	Omitted
131	04	Remove one circle from the strucure of N=132.	Omitted
132	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(68/64)	0.845/0.832
133	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(68/65)	0.844/0.815
134	16	{1,6,6,6,6,6,6,6,6,6,6,6/6,6,6,6,6,6,6,6,6,6,6,1 }(67/67)	0.834/0.822
135	04	Remove one circle from the strucure of N=136.	Omitted
136	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(68/68)	0.842/0.814
137	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(68/69)	0.842/0.819
138	14	{1,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(69/69)	0.844/0.793
139	04	Remove one circle from the strucure of N=140.	Omitted
140	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(72/68)	0.839/0.816
141	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(72/69)	0.838/0.814
142	03	{0,3,3,3,···,3,3,3/3,3,3,···,3,3,3,1}(72/70)	0.839/0.799
143	04	Remove one circle from the strucure of N=144.	Omitted
144	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,0}(72/72)	0.837/0.823
145	05	{0,5,5,5,5,5,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,5,5,5,5,5,5,0}(75/70)	0.836/0.825
146	05	{0,5,5,5,5,5,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,5,5,5,5,5,5,1}(75/71)	0.836/0.831
147	03	{0,3,3,3,···,3,3,3/3,3,3,···,3,3,3,0}(75/72)	0.842/0.807
148	03	{0,3,3,3,···,3,3,3/3,3,3,···,3,3,3,1}(75/73)	0.844/0.790
149	04	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4/4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,1}(76/73)	0.831/0.806
150	05	{0,5,5,5,5,5,5,5,5,5,5,5,5,5,5/5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,0}(75/75)	0.839/0.805

solutions, Monte-Carlo method, etc.) in spite of the simplicity of our algorithm.

Structures of $N = 3k + 1$ are made of both 03- and 13-type initial configurations. Structures of $N = 4k + 1$ are made of 04- and 14-type and structures of $N = 5k + 1$ are made of 05- and 15-type. Both structures have the same N and the same rotational symmetry, as an example, the structure of $N = 13$ in 03-type and the structure of $N = 13$ in 13-type. They are generally different structures and occasionally congruent, which are found in Tables 1–3. For example, most of the structures of $N = 3k + 1$ made of 03- and 13-type are different each other, except for congruent cases: $N = 4, 7, 10$. This fact suggests a characteristic that

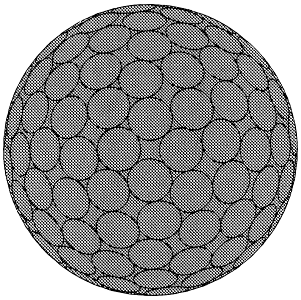

Fig. 2.

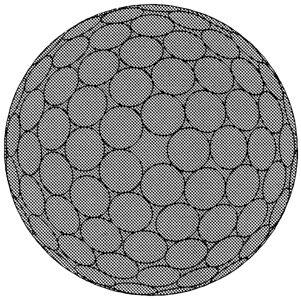

Fig. 3.

Fig. 2. A packing structure of $N = 149$, started from 04-type initial configuration. (The northern hemisphere.)
Fig. 3. A packing structure of $N = 150$, started from 05-type initial configuration. (The northern hemisphere.)

the MZM has a difference of the packing efficiency between the Northern and the Southern hemisphere. Therefore we note N_{north}, N_{south}, D_{north}, and D_{south}.

In Tables 7–9, the $D_{north} < D_{south}$ are found for only N = 4, 7, 10, 13, 26 and 51 in spite of the cases of $N_{north} < N_{south}$ are found for N = 4, 7, 10, 11, 13, 21, 25, 26, 45, 49, 51, 53, 57, 68, 79, 94, 101, 113, 116, 117 and 137. Thus, D_{north} and D_{south} cannot be expected by N_{north} and N_{south}. The cases $D_{north} > D_{south}$ when $N_{north} < N_{south}$ are not mysterious but possible if the centers of some circles which are counted as N_{south} are close to the equator. Anyway, it is suggested by Tables 7–9 that there is a tendency that D_{north} is larger than D_{south}. Though the tendency is strong, the explicit reason is hard to state. The cases of $D_{north} = D_{south}$ are found on N = 1, 2, 3, 6, 8, 9, 12, 14, 16, 17, 18, 20, 22, 24, 27, 32, 38, 42, 48, 50, 52, 60, 62, 74 and 98. These structures have the symmetry of reflection or the symmetry of rotatory reflection and the equatorial plane becomes the reflection surface or the rotatory reflection surface of them.

Which kind of the rotational symmetry gives the more efficient packing than any other types? Judging from Tables 4–6, there seems to be no special type which is clearly efficient. It is suggested that even if the local packing density around the initial configuration is small, the whole packing density is not always small. For example, 03- and 13-type obviously have the bigger local densities on around the North Pole than 04- and 14-type, but the appearances of 04- and 14-type in Tables 4–6 are not few.

One of the merits of using the symmetrical initial configuration is to make calculation time short. Authors examined symmetrical arrangements as the first target of the MZM to check its validity. The comparison between symmetrical arrangements and asymmetrical ones is a further problem.

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APPENDIX A: The Position of a Circle Which is in Contact with Given Two Circles

Derivation of Eq. (1) is given here. The point **C** is on the perpendicular surface at midpoint of **A** and **B**. So **C** is expressed as a linear combination with (**A** + **B**) and (**A** × **B**):

$$\mathbf{C} = k_1 (\mathbf{A} + \mathbf{B}) + k_2 (\mathbf{A} \times \mathbf{B}). \tag{2}$$

Two letters k_1 and k_2 are constant coefficients of the linear combination and they are decided as follows. By taking inner products of ***A*** and the both sides of Eq. (2), we have

$$(\mathbf{C}, \mathbf{A}) = k_1 (1 + (\mathbf{B}, \mathbf{A})). \tag{3}$$

Hence

$$k_1 = \frac{(\mathbf{C},\mathbf{A})}{1+(\mathbf{B},\mathbf{A})} = \frac{\cos d}{1+\cos l}. \tag{4}$$

And, by normalizing **C**, we have

$$k_2 = \pm\frac{1}{1+\cos l}\sqrt{\frac{1+\cos l - 2\cos^2 d}{1-\cos d}}. \tag{5}$$

APPENDIX B: The Area of a Circle and the Packing Density of Circles

The packing density is defined with the ratio of the area of all circles to the whole area of a spherical surface. The area of a circle is calculated as the area of a spherical cap. On the unit sphere, the area S of a circle having the radius r [rad] is

$$S = 2\pi\int_0^r \sin r' dr' = 2\pi(1-\cos r). \tag{6}$$

Therefore, if there are non-overlapping N circles having the diameter d (=2r) on the unit sphere, the packing density D is

$$D = \frac{NS}{4\pi} = \frac{N\cdot 2\pi(1-\cos(d/2))}{4\pi} = \frac{N}{2}(1-\cos(d/2)). \tag{7}$$

APPENDIX C: Division of a Circle by a Great Circle

Suppose that a circle and a great circle of a unit sphere cross each other. The center of the circle of radius r is A and two cross points, which are the both ends of a spherical chord, are B and C. The circle is divided into two segments S_1 and S_2 (Fig. 5a). The major part S_1,

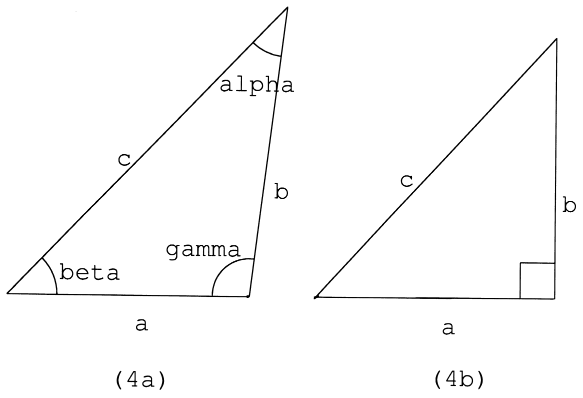

Fig. 4. Two spherical triangles.

in which the center lies, consists of a spherical sector K_1 and a spherical triangle K_2 (Fig. 5b). Let us derive the expression of the area of the segment.

At first, three fundamental formulas: spherical cosine theorem Eq. (8), area *P* of a spherical triangle Eq. (9), and the spherical version of Pythagorean theorem Eq. (10) are introduced, where *a*, *b*, and *c* are the edge length expressed by the central angle, and α, β, and γ are the inner angle (Fig. 4).

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma, \tag{8}$$

$$P = \alpha + \beta + \gamma - \pi, \tag{9}$$

$$\cos c = \cos a \cos b. \tag{10}$$

Equation (11) for a spherical right triangle in Fig. 5c is given by Eq. (10), where $a = f$, $b = e$, $c = r = d/2$.

$$\cos(d/2) = \cos e \cos f. \tag{11}$$

By using Eq. (8), we have

$$\cos g = \frac{\cos f - \cos e \cos(d/2)}{\sin e \sin(d/2)}, \quad \cos h = \frac{\cos e - \cos f \cos(d/2)}{\sin f \sin(d/2)}. \tag{12}$$

By using Eq. (6), we have

$$K_1 = 2\pi\left(1 - \cos(d/2)\right)\left(\frac{2\pi - 2g}{2\pi}\right) \tag{13}$$

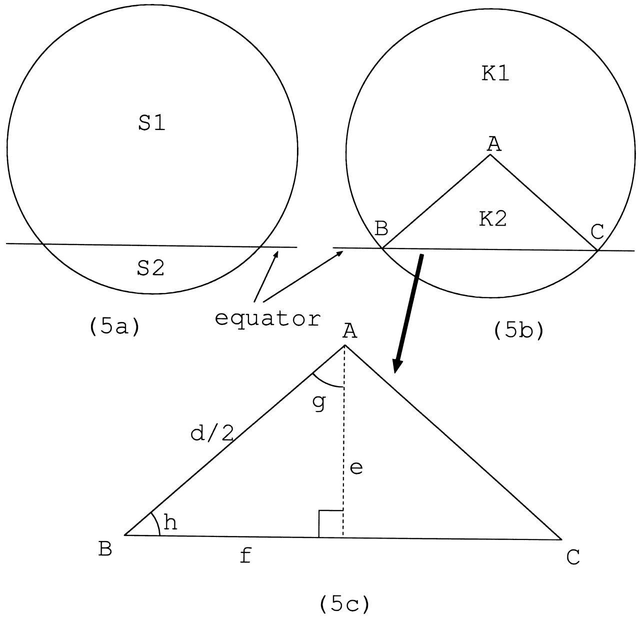

Fig. 5. Division of a spherical cap by a great circle.

$$= (2\pi - 2g)(1 - \cos(d/2)). \tag{14}$$

And by Eq. (9), we have

$$K_2 = 2g + h + h - \pi \tag{15}$$

$$= 2\,(g + h - \pi/2). \tag{16}$$

Hence

$$S_1 = K_1 + K_2 = \pi + 2h + 2\,(g - \pi)\cos(d/2), \tag{17}$$

$$S_2 = 2\pi\,(1 - \cos(d/2)) - S_1 = \pi - 2h - 2g\cos(d/2). \tag{18}$$

The net packing densities on the Northern hemisphere and the Southern hemisphere can be calculated with Eqs. (17) and (18), where of course a great circle is the equator.

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