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Phase Transitions of Route Patterns in the Steiner's Problem with Four Cities

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Abstract. The route patterns of the simplified Steiner's problem with four cities are investigated by the experiment using soap film and by numerical calculation. We show that the change of route patterns are understandable by the concept of phase transition of thermodynamics. The change of patterns is categorized into two cases. The first one shows the first order phase transition and the other shows the second order phase transition. The third order phase transition occurs if the particular path in phase plane is chosen.

1. Introduction

There exist several shortest path problems. We are interested in the Steiner's problem (COURANT and ROBBINS, 1941; HILDEBRANDT and TROMBA, 1985) since the technical method to solve this problem is very unique and is instructive. This problem is summarized as follows:

Steiner's Problem

Set n cities on the plane. Define the route pattern connecting all cities whose total length is the shortest compared with other patterns.

In this paper, we study the route patterns of the simplified Steiner's problem with four cities on the plane (abbreviated as the simplified four-city problem) by the experiment using soap film and by the numerical calculation, jointly. We shall give a new view of this problem.

First we show the simplified four-city problem. Three cities named as B (=(–1, –1)), C (=(1, –1)) and D (=(1, 1)) are fixed at three corners of square whose length of one edge is 2 and the fourth city A is settled at any place on the plane except B, C and D (see Fig. 1). We say $\triangle$BCD *the basic pattern*. By the experiment using soap film, we can determine the most appropriate pattern which means that the total length of roads connecting four cities has the shortest length compared with other patterns. This is guaranteed by the fact that the surface tension of soap film gives a minimum surface. In our experiment, the

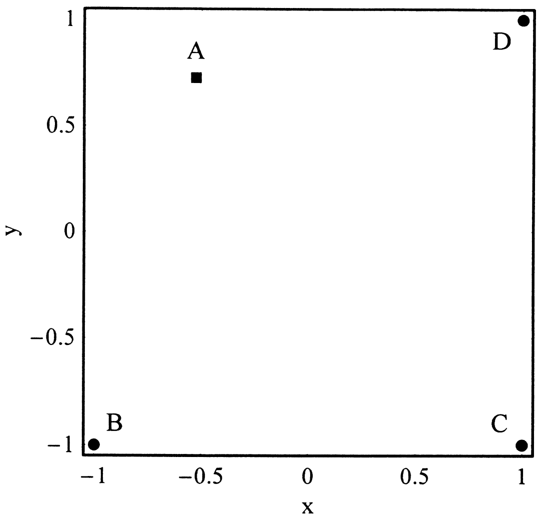

Fig. 1. Three cities B, C and D are fixed at the corners of square. The fourth city A = (x, y) is settled at any position.

distance d between two parallel plates is 2.5 cm and the length between fixed two poles showing two cities (B and C, and C and D) is 7.3 cm and a synthetic detergent with glycerin is used to prepare for soapy water. Our results do not depend on the property of soapy water and other conditions (for example, temperature, humidity and so on). In this problem, the total length L plays a role of free energy F in thermodynamics since $F \propto L \times d$. Note that Ld is the total surface area of films. Hereafter we regard the total length as the free energy if necessary. The pattern of soap film has a beautiful structure. Let P be a bifurcation point (not on the cities) of three soap films. Three angles at P have the same value $2\pi/3$ (so called the $2\pi/3$-*law*). To specify the boundary curves, we use this law and execute the numerical calculation under the experimental results. Patterns and the phase diagram are obtained in the next section. We give the detailed properties of phase diagram.

Next we consider the change of route patterns induced by the movement of the city A. The change of patterns is characterized by the concept of the phase transition of thermodynamics (STANLEY, 1971). The discussion of phase transition is obtained in Sec. 3. In Sec. 4, we give two remarks.

2. Phase Diagram

Figure 2 shows the phase diagram where the horizontal axis is the x-coordinate of A and the longitudinal axis is the y-coordinate of A. The phase diagram has the symmetrical structure with respect to the line $y = -x$ due to the symmetry of the isosceles triangle $\triangle$BCD. The symbols in Fig. 2 means the name of patterns. For example, the numeral 5 of symbol (5a-1) shows the number of roads. Examples of all kinds of route patterns are illustrated in Figs. 3–5.

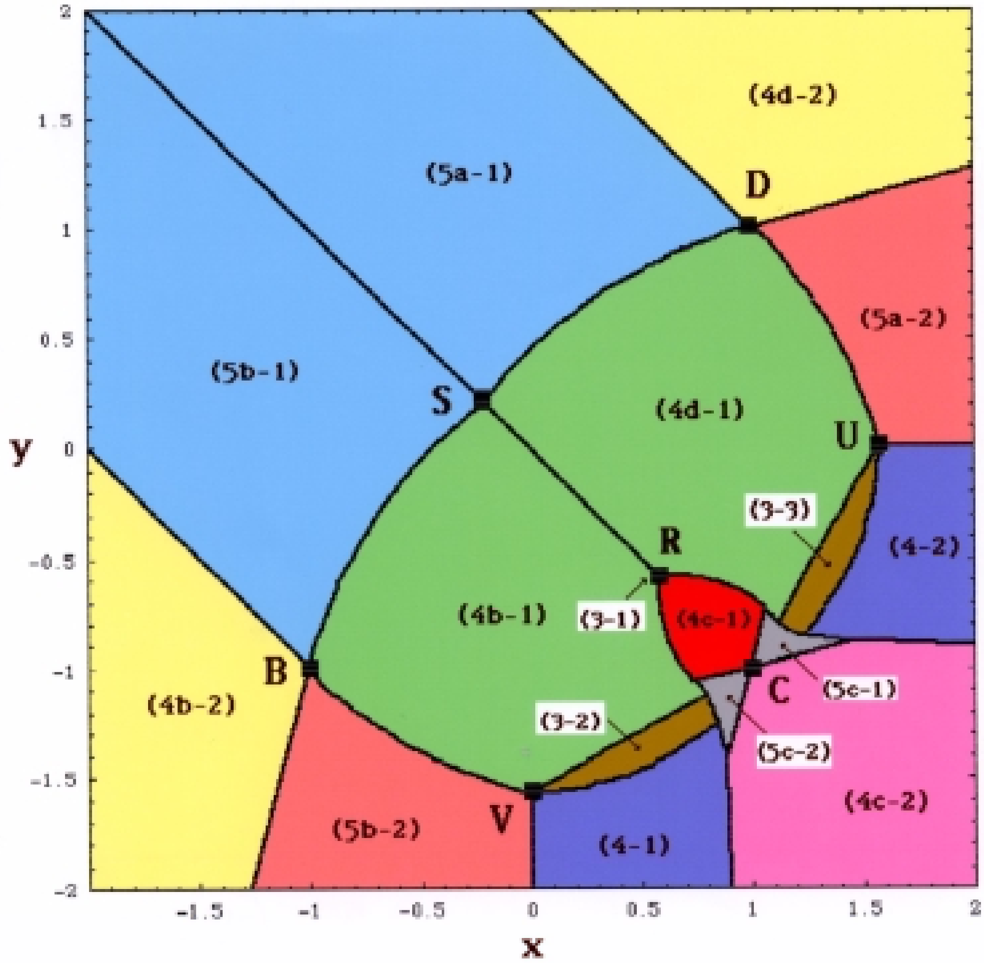

Fig. 2. Phase diagram ($-2 < x, y < 2$). For example, (5a-1) is a name of pattern.

We can determine particular positions and several boundary lines in the phase diagram in terms of the $2\pi/3$-law. This gives that three angles $\angle$BRC, $\angle$CRD and $\angle$DRB of the pattern (3-1) are $2\pi/3$. Using the information of this structure, the position of R is determined. Using the same method, we can determine other positions and several boundary lines. In Tables 1 and 2, the results are listed.

We can obtain the phase diagram by using the $2\pi/3$-law in principle. But it is very complicate work to draw the boundary curves. Then we carry out the experiment with soap film and excute the numerical calculation to specify the boundaries. When we study the change of route patterns, we need the information of total length of pattern. The numerical calculation using MATHEMATICA (copyrighted software sold by Wolfram Research Inc.) is a powerful method to analyze the structure change of patterns.

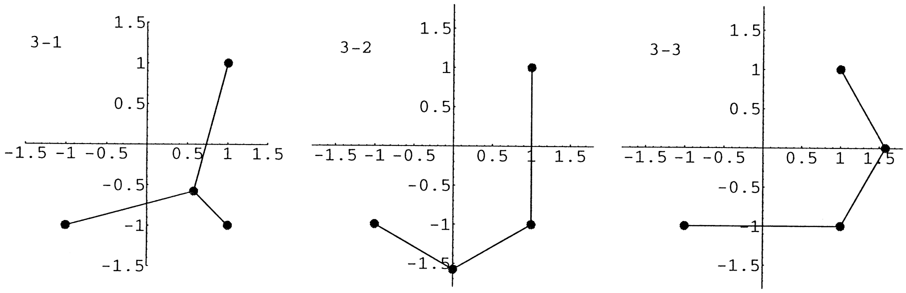

Fig. 3. Patterns with three roads.

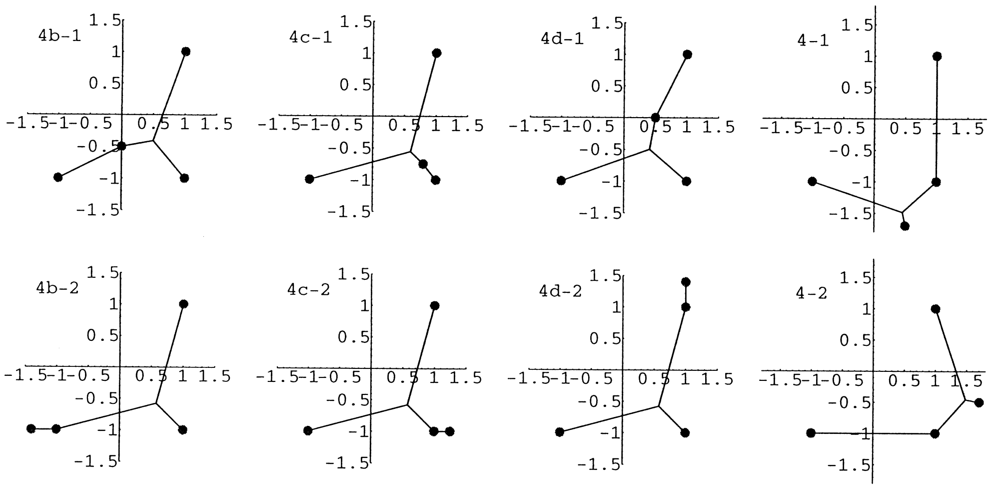

Fig. 4. Patterns with four roads.

Table 1. Several positions in the phase diagram.

Name of point	Position
R	$(1/\sqrt{3}, -1/\sqrt{3})$
S	$(-\alpha, \alpha)$*
U	$(1/\sqrt{3}, 0)$
V	$(0, -1/\sqrt{3})$
Intersection of curve SD and $x = 0$	$(0, 1 - 1/\sqrt{3})$
Intersection of curve SB and $y = 0$	$(-1 + 1/\sqrt{3}, 0)$

$$*\alpha = \frac{\sqrt{3}+1}{2} - \sqrt{1 + \frac{5\sqrt{3}}{6}} = 0.1971...$$

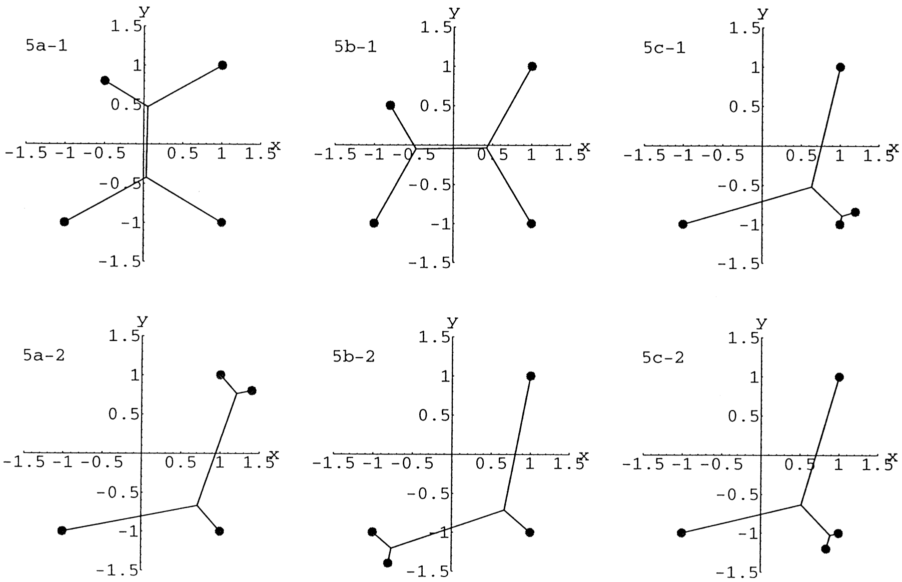

Fig. 5. Patterns with five roads.

Table 2. Expressions of several boundary lines.

Boundary line	Expression
(5a-1)–(5b-1)	$y = -x$ $(x < -\alpha)$
(4d-1)–(4b-1)	$y = -x$ $(\alpha < x < 1/\sqrt{3})$
(5a-1)–(4d-2)	$y = -x + 2$ $(x < 1)$
(5b-1)–(4b-2)	$y = -x - 2$ $(x < -1)$
(4d-2)–(5a-2)	$y = \tan(\pi/12)(x - 1) + 1$ $(x > 1)$
(4b-2)–(5b-2)	$y = \tan(5\pi/12)(x + 1) - 1$ $(x < -1)$
(5c-1)–(4c-2)	$y = \tan(\pi/12)(x - 1) - 1$ $(x > 1)$
(5c-2)–(4c-2)	$y = \tan(5\pi/12)$ $(x - 1) - 1$ $(x < 1)$
(5a-2)–(4-2)	$y = 0$ $(x > 1/\sqrt{3})$
(5b-2)–(4-1)	$x = 0$ $(y < -1/\sqrt{3})$

3. Phase Transition

We study the change of patterns when we move the fourth city A. The boundaries in the phase diagram are categorized into two. The first one has the property that the first order phase transition occurs at each boundary. The other one has the property that the second order phase transition occurs at each boundary.

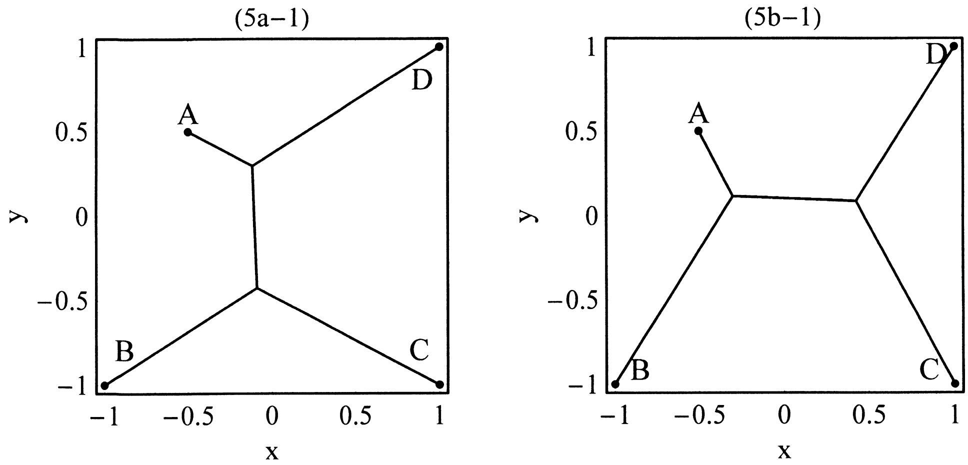

Fig. 6. Coexistence of (5a-1) and (5b-1).

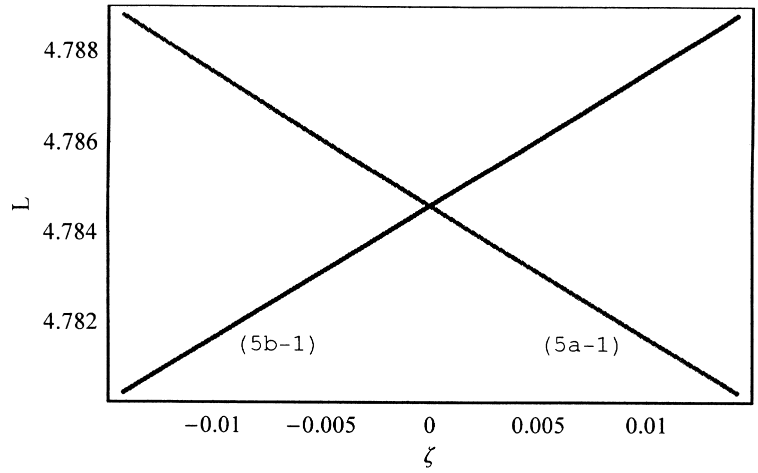

Fig. 7. Change of total lengths corresponding to Fig. 6.

3.1. The first order phase transition

One of two patterns (5a-1) and (5b-1) exists in the line segment $y = -x$ ($x < -\alpha$) (see Fig. 6). We move the city A from the region of (5b-1) to that of (5a-1). We calculate the change of total lengths of two patterns. The result is shown in Fig. 7. Here we use the coordinate ζ along the line $y = x + 1$, and the origin of new coordinate is (–0.5, 0.5). In the region $\zeta < \zeta_c$ (=0), the total length of (5b-1) is less than that of (5a-1). In the region $\zeta > \zeta_c$, the opposite relation holds. This fact implies that the transition from (5b-1) to (5a-1) is

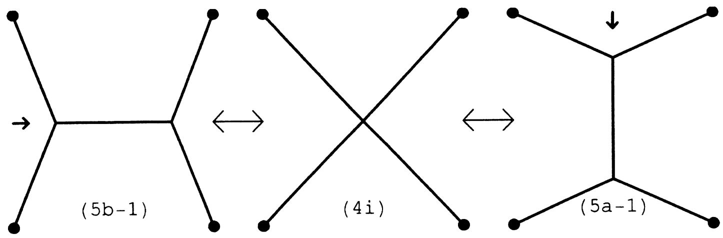

Fig. 8. Blowing on the pattern (5a-1) toward the right direction, the transition from (5a-1) to (5b-1) via (4i) occurs. If we blow on the pattern (5a-1) toward downward, the inverse process occurs.

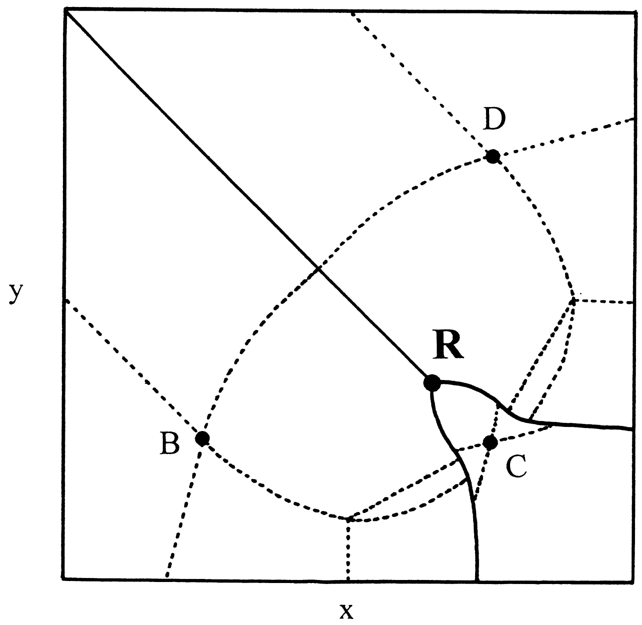

Fig. 9. Classification of boundary curves. The solid thick line shows the boundaries of the first order phase transition and the broken one shows the boundaries of the second order phase transition.

considered as the first order phase transition. The movement of A does not gives rise to the smooth transition from (5b-1) to (5a-1). The unstable pattern (5b-1) may exist in the area of (5a-1). This corresponds to the metastable state of thermodynamics. If the metastable state is destroyed by the external perturbation, the transition to stable state occurs. But the destruction of metastable state means that of soap film. As a result, the transition to stable state does not occur in the soap film. If we add the external force (for example, blowing on the soap film), the transition from (5b-1) to (5a-1) may occur (see Fig. 8). The pattern (4i) can appear at the intermeadiate state, only. The transition of Fig. 8 is not the pattern change

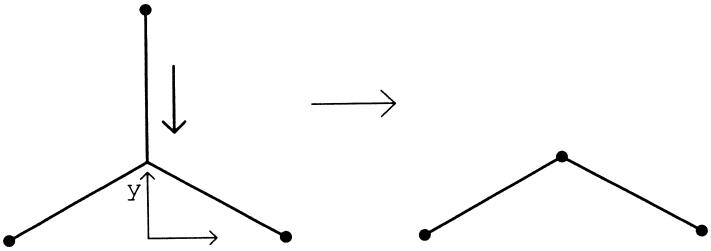

Fig. 10. Basic transition of the second order phase transition.

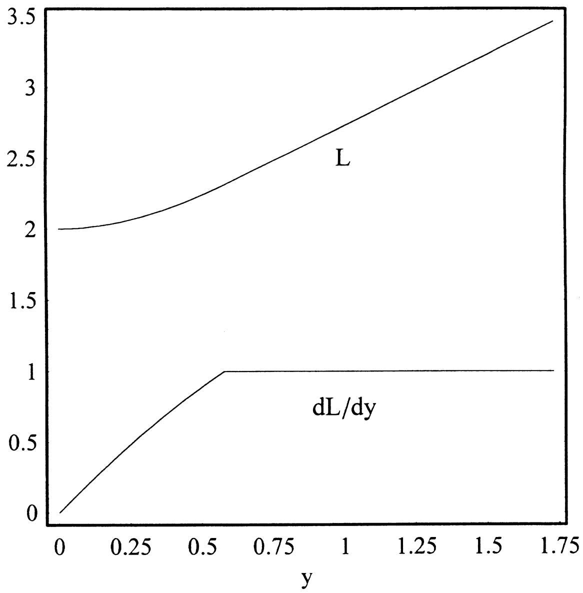

Fig. 11. Change of total length and its derivative corresponding to Fig. 10.

induced by the smooth movement of A. It is noted that the transitions mentioned above appear in the case of soap film, and do not occur in the Steiner's problem since the metastable state is not the appropriate solution for the Steiner's problem.

In Fig. 9, the thick solid curves means the boundaries showing the first order phase transition, and the dotted curves implies the boundaries showing the second order phase transition discussed in the next subsection. Three curves starting from R separate the phase plane into three areas.

3.2. The second order phase transition

We show the example of the second order phase transition. The second order phase

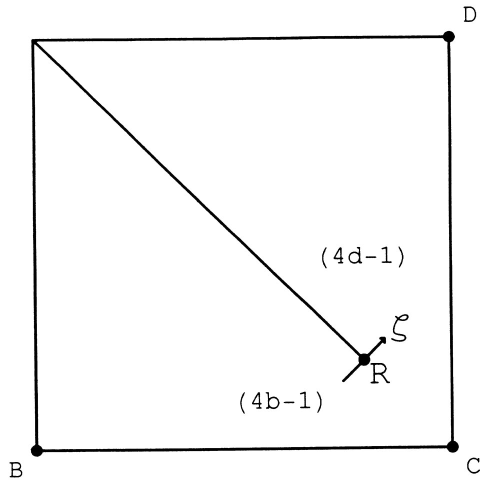

Fig. 12. Particular bypath passing through R giving rise to the third order phase transition.

transition occurs when we move A from the area of (5a-1) to that of (4d-1). This pattern change occurs smoothly. This pattern change is characterized by the process illustrated in Fig. 10. We say it the basic transition of the second order phase transition. In Fig. 11, the change of total length and its derivative are shown. The total length decreases smoothly, but the derivative has a cusp at the transition point. Then this fact implies that the second order phase transition occurs at the transition point. The boundaries showing the second order phase transition have been already shown in Fig. 9. We can move the city A over the boundaries illustrated in dotted line. Whenever the city A moves over the dotted boundary, the pattern change occurs smoothly.

Although we cannot directly move the city A from the area surrounded by three boundaries of the first order phase transition to another area, we can do it if we take the path in the phase space passing through the point R. Then we say it the *bypath.*

3.3. The third order phase transition

We choose the symmetrical bypath. Typical path is shown in Fig. 12. Along this bypath, we plot the change of total length, the first order derivative and the second order derivative in Figs. 13a–c. In Fig. 13c, there exists a cusp at the transition point. This means the evidence of the third order phase transition. The origin of the third order phase transition is the symmetrical structure of the basic pattern. If the relation BC = CD of the basic pattern does not hold, we think that there is no paths showing the third order phase transition. This fact is not proved. Along the path shown in Fig. 12, the pattern (4b-1) can appear as a metastable state in the region of (4d-1). The existence of metastable state is the characteristic property of the odd order phase transition. However, the third order phase transition is a rare phenomena. As another example of the third order phase transition, the Bose-Einstein condensation is well known.

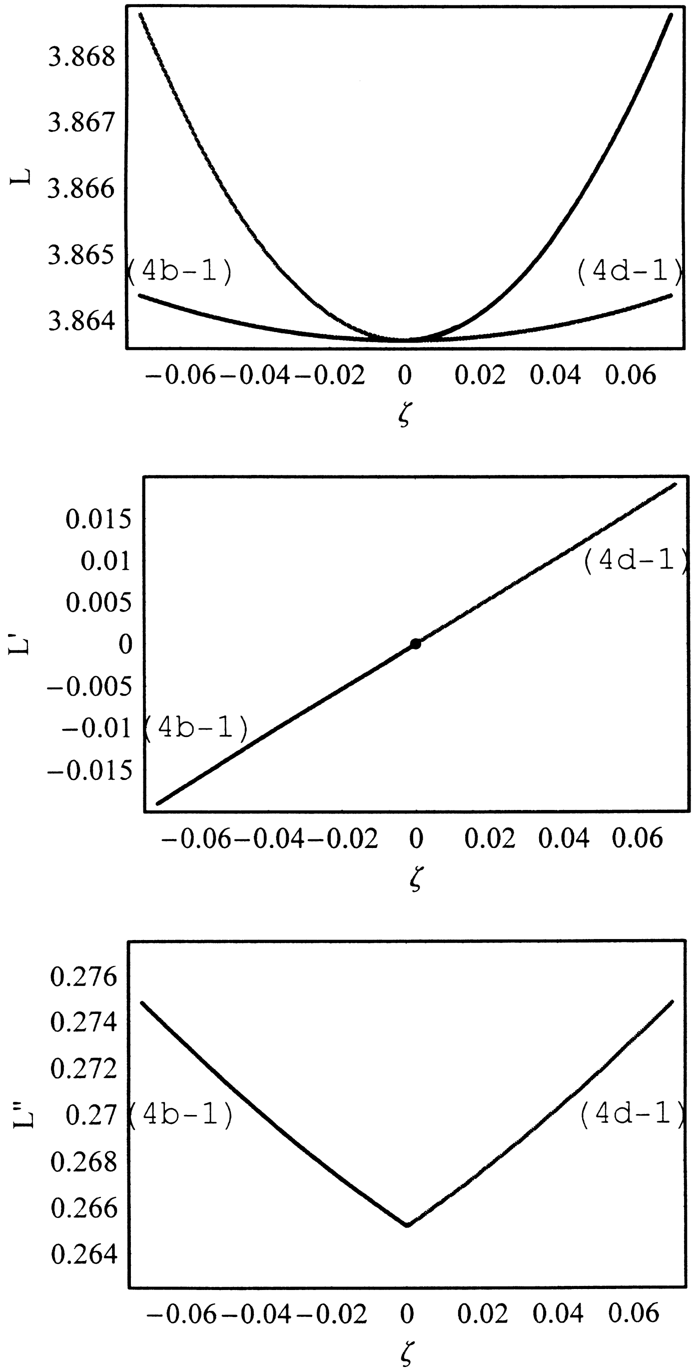

Fig. 13. Change of total length, its derivative and the second order derivative for the bypath shown in Fig. 12.

4. Remarks

The number of patterns $P(n)$ for the n-city problem is given in Table 3 and the basic patterns ([1], [3], [5] and [7])are illustrated in Fig. 14. For example, an abbreviation 2[1] means [1] ∪ [1] in Table 3 and (4b-1) is classified as [1] ∪ [3]. The patterns shown in Fig. 3 are classified in the same pattern 3[1]. The algorithm to determine $P(n)$ is obtained in Appendix B. Note that $P(n)$ is equal to the number of partitions $p(n–1)$ without restriction of positive integer $n – 1$ (see Table 4) (SCHROEDER, 1984). However the number of patterns is three in the four-city problem, but one pattern includes many variations (see Figs. 3–5). As a result, it is difficult to predict the appearance of realized pattern without the experiment or the numerical calculation.

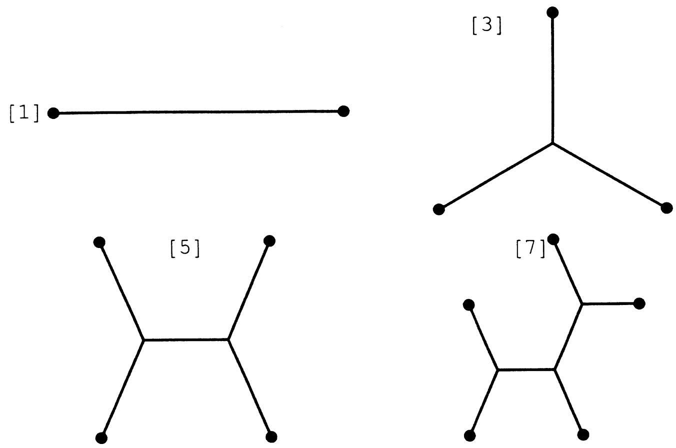

Fig. 14. The basic pattern of [$2n + 1$] ($n \le 3$).

Table 3.

n	$P(n)$	Patterns
2	1	[1]
3	2	2[1], [3]
4	3	3[1], [1] ∪ [3], [5]
5	5	4[1], 2[1] ∪ [3], [1] ∪ [5], 2[3], [7]

Table 4.

n	2	3	4	5	6	7	8	9	10
$P(n) = p(\mathrm{n} - 1)$	1	2	3	5	7	11	15	22	30

We show that the thermodynamic interpretaion is useful to understand the change of patterns for this problem. In the two-city problem, there is no transitions of patterns. In the three-city problem, the second order phase transition occurs and the first order phase transition does not occur. In the n (≥4)-city problem, both the first and second order phase transitions can occur. The occurrence of the third order phase transition in the problem with n (≥4) cities depends on the symmetrical structure of basic pattern and on how to choose the path.

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APPENDIX A: Program to Determine the Pattern

We give a sample program to determine the pattern (5b-1) (see Fig. 15) written by MATHEMATICA. We use *FindMinimum* of MATHEMATICA, which determines two bifurcation points P = (s, t) and Q = (u, v), and the total length.

```
(* Program to Draw Pattern (5b-1) by MATHEMATICA ver.3 *)
$DefaultFont = {"Times-Roman", 16};
(* Input Position of A *)
x = -0.8; y = 0.4;
(* Positions of Fixed Three Cites: B, C and D *)
xb = -1; yb = -1; xc = 1; yc = -1; xd = 1; yd = 1;
(* Lengths of 5 rouds *)
t1 = Sqrt[ (x - s)^2 + (y - t)^2];
t2 = Sqrt[ (xb - s)^2 + (yb - t)^2];
t3 = Sqrt[ (u - s)^2 + (v - t)^2];
t4 = Sqrt[ (xc - u)^2 + (yc - v)^2];
t5 = Sqrt[ (xd - u)^2 + (yd - v)^2];
(* Total Length *)
tt = t1 + t2 + t3 + t4 + t5;
(* Determination of Pattern *)
ans = FindMinimum[tt, {s, -0.5}, {t, 0.5}, {u, 0.5}, {v, 0}];
(* Drawing of Pattern *)
ss = Part[Part[Part[ans, 2], 1], 2];
tt = Part[Part[Part[ans, 2], 2], 2];
uu = Part[Part[Part[ans, 2], 3], 2];
vv = Part[Part[Part[ans, 2], 4], 2];
g1 = {AbsoluteThickness[2], Line[{{x, y}, {ss, tt}}]};
g2 = {AbsoluteThickness[2], Line[{{xb, yb}, {ss, tt}}]};
```

```
g3 = {AbsoluteThickness[2], Line[{{ss, tt}, {uu, vv}}]};
g4 = {AbsoluteThickness[2], Line[{{xc, yc}, {uu, vv}}]};
g5 = {AbsoluteThickness[2], Line[{{xd, yd}, {uu, vv}}]};
ta = Text[StyleForm["A", FontSize -> 18], {x, y + 0.1}];
tb = Text[StyleForm["B", FontSize -> 18], {xb + 0.15, yb + 0.05}];
tc = Text[StyleForm["C", FontSize -> 18], {xc - 0.15, yc + 0.05}];
td = Text[StyleForm["D", FontSize -> 18], {xd - 0.1, yd - 0.05}];
tp = Text[StyleForm["P (s,t)"], {ss - 0.25, tt}];
tq = Text[StyleForm["Q (u,v)"], {uu + 0.35, vv}];
pp = {Point[{x, y}], Point[{xb, yb}], Point[{xc, yc}], Point[{xd, yd}]};
Show[Graphics[pp],
Graphics[g1], Graphics[g2], Graphics[g3], Graphics[g4], Graphics[g5],
Graphics[ta], Graphics[tb], Graphics[tc], Graphics[td], Graphics[tp], Graphics[tq],
Prolog -> AbsolutePointSize[6], AspectRatio -> 1, Frame -> True,
FrameStyle -> {{AbsoluteThickness[2]}, {AbsoluteThickness[2]},
               {AbsoluteThickness[2]}, {AbsoluteThickness[2]}},
FrameLabel -> {"x", "y", "(5b-1)", ""}];
```

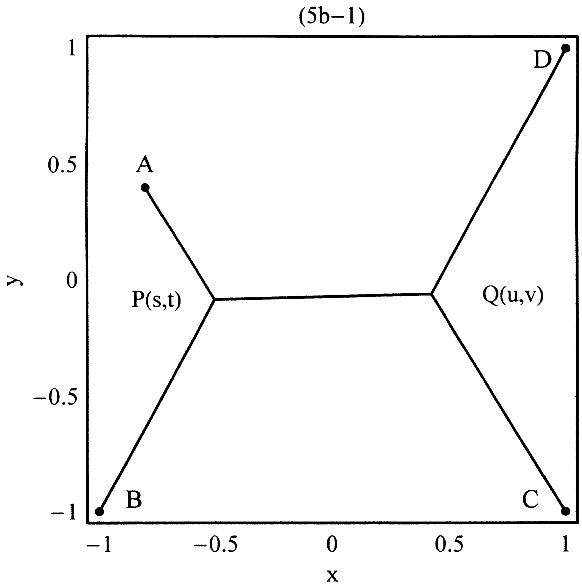

Fig. 15. Pattern (5b-1) where P = (s, t) and Q = (u, v) are determined by *FindMinimum* of MATHEMATICA. This figure is the output of sample program in Appendix A.

APPENDIX B: Algorithm to Determine $P(n)$

Using the $2\pi/3$-law, the maximum number $M(n)$ of roads of the n-city problem is obtained.

$$M(n) = 2n - 3. \tag{1}$$

Let us consider the three-city problem with $M(3) = 3$. There exist the pattern [3] in Fig. 14. Moving one city to the bifucation point, the right pattern (expressed by 2[1]) in Fig. 10 appears. We write this process as the following expression.

$$[M(3)] = [3] \to [1] \cup [1]. \tag{2}$$

Next we consider the four-city problem with $M(4) = 5$. Moving one city of [5] to the bifurcation point, we have the pattern $[1] \cup [3]$. This process is written by

$$[M(4)] = [5] \to [1] \cup [3]. \tag{3}$$

Using the decomposition of [3], we rewrite Eq. (3) as

$$[5] \to [1] \cup [3] \to [1] \cup [1] \cup [1]. \tag{4}$$

This decomposition means that there exist three patterns in the four-city problem. This procedure is considered as the partition of even number into two odd numbers. Repeating this procedure and omitting the same patterns, we have $P(n)$. For the six-city problem, we have the following decomposition and then have $P(6) = 7$.

$$\begin{aligned}[M(6)] = [9] &\to [1] \cup [7], [3] \cup [5] \to [1] \cup [1] \cup [5], [1] \cup [3] \cup [3] \\ &\to [1] \cup [1] \cup [1] \cup [3] \to [1] \cup [1] \cup [1] \cup [1] \cup [1].\end{aligned} \tag{5}$$

Here we consider the partition of 5.

$$5 \to 1 + 4, 2 + 3 \to 1 + 1 + 3, 1 + 2 + 2 \to 1 + 1 + 1 + 2 \to 1 + 1 + 1 + 1 + 1. \tag{6}$$

At the first stage of partition into two integers, there exist two ways of decomposition. This corresponds to the first decomposition of Eq. (5). The second partition of Eq. (6) corresponds to the second decomposition of Eq. (5), and so on. This fact implies the relation $P(n) = p(n - 1)$. The number of partition $p(n)$ is obtained by the following program of MAPLE (copyrighted software sold by Waterloo Maple Inc.).

```
>with(combinat):
>seq(numbpart(n), n = 1 ... 20);
```

REFERENCES

COURANT, R. and ROBBINS, H. (1941) *What is Mathematics?*, Interscience.
HILDEBRANDT, S. and TROMBA, A. (1985) *Mathematics and Optimal Form*, W.H.Freeman.
SCHROEDER, M. R. (1984) *Number Theory in Science and Communication*, Springer-Verlag.
STANLEY, H. E. (1971) *Introduction to Phase Transitions and Critical Phenomena*, Clarendon Press.