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Pictures by Conformal Mapping

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Abstract. Conformal mapping is one of the basic properties of the regular function. One can easily understood of this mapping property by drawing a picture with the use of special regular function.

1. Introduction

It is said that mathematics is one of the natural sciences that is focused on the study of numbers, quantities, figures, spaces and so on. Naturally, it should include the learning of the forms. In this paper, I want to introduce how to understand the mapping property of regular function by drawing pictures. This property is called “Conformal mapping,” and is the most beautiful geometric property of regular functions.

When I was faced some time ago with a question of teaching contents for the first year medical students in my institution, fortunately and accidentally I encountered a mathematical book (OZAWA and KIMURA, 1985) entitled “Joyful world of the math illustrations.” It was about pictures that can be drawn with the use of the mapping property of simple linear transformations and conformal mappings. Later I found out that many students expressed concern and interest in this. In fact, there were many beautiful and impressive pictures presented in the book. And quickly I thought that the drawing pictures would become a powerful tool for the teaching of mathematics. Of course before you come to draw a picture, it is necessary to understand to some degree the theory or properties of mapping function. No doubt some knowledge of the theory would be a prerequisite in helping anybody draw pictures with relative ease.

In my first encounter, then, with such understanding off mathematics as presented in the book above, I felt strongly that what the book offered me was a powerful teaching method in helping medical students understand what I intended to teach.

In this paper, I will introduce how to draw a picture by simple conformal mapping.

2. Method of Drawing Pictures

Here a method of drawing the pictures is expounded with the use of conformal mapping of regular functions.

It is natural to associate the complex number $z = x + iy$ (x, y are real numbers) with a point in the plane. Each complex number corresponds to just one point. When used for the purpose of displaying the number $z = x + iy$ geometrically, the plane is called the complex plane, or the z plane. The following equation is known as Euler's formula;

$$\exp(i\theta) = \cos\theta + i\sin\theta,$$

where θ is a real number. It is well-known that the multiplication of z by $\exp(i\theta)$ amounts to rotating z counterclockwise through an angle θ about the origin. If only one value of w corresponds to each complex value z, we say that $w = f(z)$ is a single-valued complex function of z. We write it as below;

$$f(z) = u(x, y) + iv(x, y).$$

If $u(x, y)$, $v(x, y)$ belong to the class C^1 and $f(z)$ satisfies the Cauchy Riemann's differential equations near z,

$$u_x(x, y) = v_y(x, y), \quad u_y(x, y) = -v_x(x, y),$$

then the function $f(z)$ is said to be regular at z. The mapping given by $w = f(z)$ is conformal at all points where $f(z)$ is regular and $f'(z) \neq 0$. Namely, let C_1, C_2 be two smooth arcs passing through the point P and the angle of these intersection is α. Let two arcs Γ_1, Γ_2, which pass through the point $Q = f(P)$ be the images of C_1, C_2 under the mapping $w = f(z)$, respectively, and β be their intersection angle. Then it is well-known as the conformal mapping of the regular function $f(z)$ that the angle between two arcs is not changed, that is, the identity $\alpha = \beta$ holds, by the mapping $w = f(z)$, provided $f'(z) \neq 0$ at the point of intersection. Moreover, the sense of the angle is also preserved in this mapping. The mapping property is one of the most important properties of the regular function.

For example, since the function $f(z) = z^2$ satisfies $f'(z) = 2z$, this mapping is conformal except for $z = 0$. Describing this mapping in terms of polar coordinates, it is easy to show that a certain sector in the first quadrant on z plane is mapped in a one to one manner onto the sector in the second quadrant on w plane. In this mapping, a hyperbola $x^2 - y^2 = c$ is mapped onto the vertical line $u = c$, and a vertical line $x = a$ is mapped onto the parabola $v^2 = 4a^2(a^2 - u)$ and so on. Using these facts, one can draw many interesting pictures with this mapping. We, however, do not refer to anything about this matter here.

In the rest of this paper, we shall consider the two special cases of the mapping functions $f(z) = 1/z$ and $f(z) = \exp(-iz)$.

(1) At first, we consider the mapping properties of the function $f(z) = 1/z$. Since $f'(z) = -1/z^2$, the regular function $f(z) = 1/z$ is conformal at all points except for $z = 0$. This function can easily map the interior of the unit disk on z plane onto the exterior of the unit disk on w plane and vice versa. Since it is easy to get

$$f(z) = (x - iy)/(x^2 + y^2), \quad u = x/(x^2 + y^2), \quad v = -y/(x^2 + y^2),$$

if z moves in the first quadrant on z plane, then corresponding w moves in the fourth quadrant on w plane. In particular, the image of a circle under this mapping will again be a circle. To prove this, we note that all circles in the z plane have equations of the form

$$a(x^2 + y^2) + 2hx + 2gy + c = 0,$$

where a, c, g and h are real constants. By using $z = x + iy$ and $\bar{z} = x - iy$, the above equation becomes

$$a|z|^2 + 2\,\mathrm{Re}(\bar{\alpha}z) + c = 0, \quad \alpha = h + ig.$$

Hence, the image of the above circle under the mapping $w = 1/z$ has the form

$$c|w|^2 + 2\mathrm{Re}(\alpha w) + a = 0.$$

This, obviously, is again the equation of a circle. Particularly, the image of a circle passing through the origin is a straight line, and vice versa. If we regard the straight lines as special cases of circles, namely, circles that pass through the point at infinity, the mapping indeed maps circles onto circles. Considering these properties of mapping $w = 1/z$, we now can draw interesting pictures of this mapping. In Section 4, I shall show some beautiful pictures of the function $f(z) = 1/z$ drawn by some students of mine.

(2) Secondly, we consider the mapping function $f(z) = \exp(-iz)$. This function is expressed as a composition of the functions $t = -iz$ and $w = \exp(t)$. Since the transformation $t = -iz$ means a rotation through the angle $-\pi/2$ around the origin. We consider the only case of the function $f(z) = \exp(z)$. By reason of $f'(z) = \exp(z)$, the mapping by $w = f(z)$ is conformal mapping. Setting $z = x + iy$, $w = u + iv$, it is easy to show that the function has interesting mapping properties. For example, vertical and horizontal line segments are mapped onto portions of circles and rays respectively. And the images of various regions can be readily obtained from these observations. For example, seeing the images of the straight lines x = const., y = const., a rectangle in the z plane is mapped onto the sector in the w plane. Especially, our function maps a rectangle with height 2π onto the annuls. Considering these facts, we can draw the pictures with the use of the function $f(z) = \exp(z)$. In Section 4, I shall also show some pictures, using the function $f(z) = \exp(-iz)$ drawn by some students of mine.

3. General Theory

The Riemann mapping theorem asserts the existence of a function which maps a given simply connected domain onto a disk. We know how to construct the Riemann mapping function with great difficulty. There is a more simple method to know the Riemann mapping function from the theory of reproducing kernel. If we know the Green function $G(z, t)$ of D for the Laplace differential equation, or the complete orthnormal systems $\{\varphi_n(z)\}$ or $\{\psi_n(z)\}$ with respect to the following integrations

$$\iint_D \varphi_n(z)\overline{\varphi_m(z)}dxdy = \delta_{nm}, \quad \int_{\partial D} \psi_n(z)\overline{\psi_m(z)}|dz| = \delta_{nm},$$

then we have the Bergman kernel $K(z, \bar{t})$ and the Schiffer kernel $L(z, t)$ or the Szegö kernel $\hat{K}(z, \bar{t})$ and the Garabedian kernel $\hat{L}(z, t)$ of the domain D, respectively (NEHARI, 1952; BERGMAN, 1970);

$$K(z,\bar{t}) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_m(t)}, \quad \hat{K}(z,\bar{t}) = \sum_{n=0}^{\infty} \psi_n(z)\overline{\psi_m(t)}.$$

These two kernels $K(z, \bar{t})$ and $\hat{K}(z, \bar{t})$ for a simply connected domain D are concerned with the relation

$$\left(\hat{K}(z,\bar{t})/\hat{K}(t,\bar{t})\right)^2 = \left(K(z,\bar{t})/K(t,\bar{t})\right).$$

For the unit disk $\boldsymbol{U} = \{|z| < 1\}$, we have immediately

$$\varphi_n(z) = \sqrt{\frac{n+1}{\pi}}z^n, \quad \psi_n(z) = \frac{1}{\sqrt{2\pi}}z^n, \quad G(z,t) = \log\left|\frac{1-z\bar{t}}{z-t}\right|, \quad K(z,\bar{t}) = \frac{1}{\pi(1-z\bar{t})^2},$$

$$L(z,t) = \frac{1}{\pi(z-t)^2}, \quad \hat{K}(z,\bar{t}) = \frac{1}{2\pi(1-z\bar{t})} \quad \text{and} \quad \hat{L}(z,t) = \frac{1}{2\pi(z-t)}.$$

Under the normalizations $f(a) = 0$, $f'(a) > 0$ for a fixed point a of the simply connected domain D, the Riemann mapping function $f(z)$ will be given by using of the kernel functions as follows:

$$f(z) = c\int_a^z K(z,\bar{a})dz \quad (c \text{ is a constant}) \quad \text{or} \quad f(z) = \left(\hat{K}(z,\bar{a})/\hat{L}(z,a)\right).$$

For the multiply-connected domain D with smooth boundary components, without loss of generality, we can always define the above later function $f(z)$ called the Ahlfors function which is given as the quotient of the Szegö kernel and the Garabedian kernel of D. This function has many important and beautiful properties on the theory of functions. For the domain D, it is well known that there are analytic representation of the mapping functions from D onto the representative domains such as the parallel slit domain, the circular slit domain, the radial slit domain, the circle with concentric circular slits etc. (NEHARI, 1952; BERGMAN, 1970) To construct concretely these functions, we must need to know a complete orthnormal system or the Green function of the Laplace differential equation for the domain D. Therefore, any further details with this regards are discarded here.

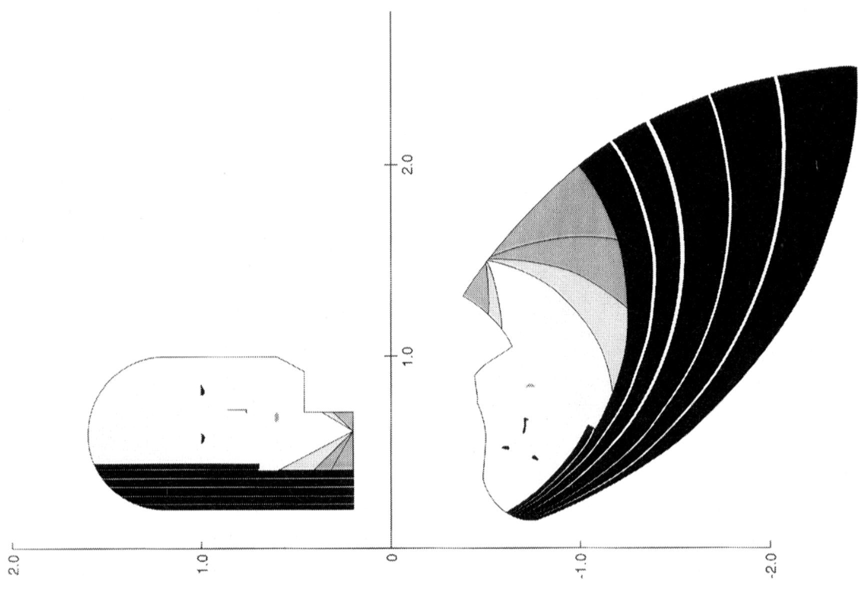

Fig. 1. “Beautiful heian woman” by K. Ikehata.

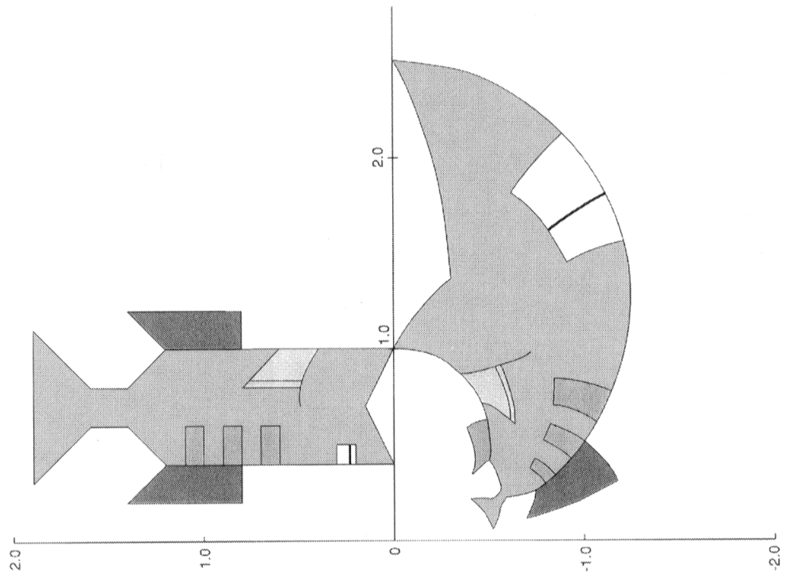

Fig. 2. “The fish with big mouth” by A. Furukawa.

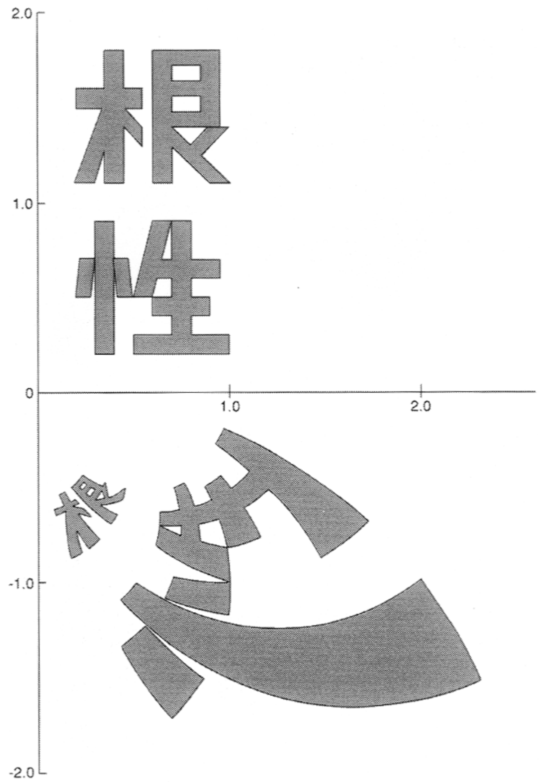

Fig. 3. “Bending Japanese spirits” by T. Moriue.

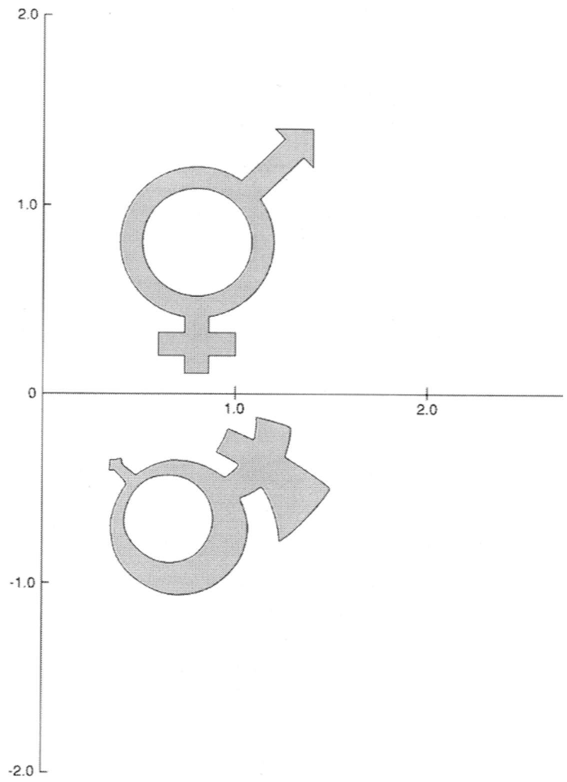

Fig. 4. “Prewar to postwar” by K. Sunabori.

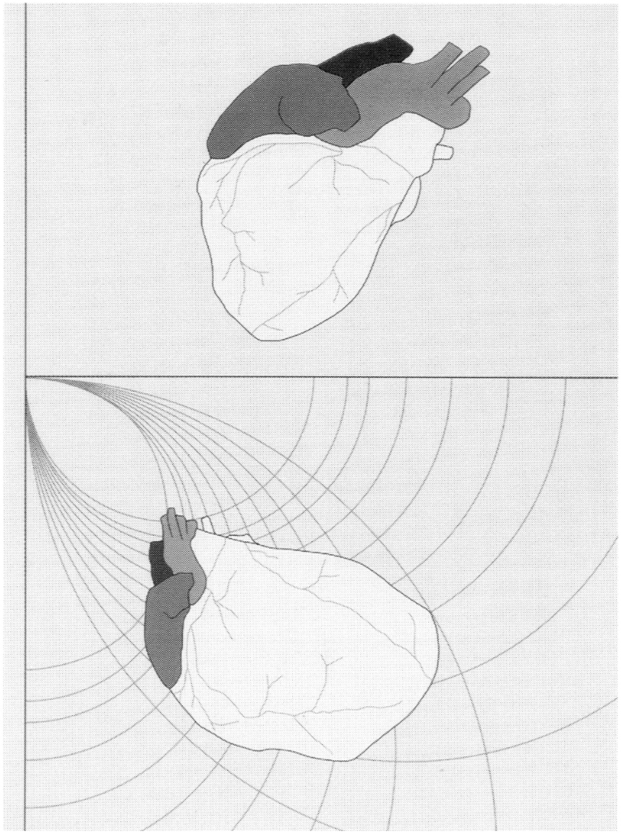

Fig. 5. “Shrunken heart” by T. Hirao.

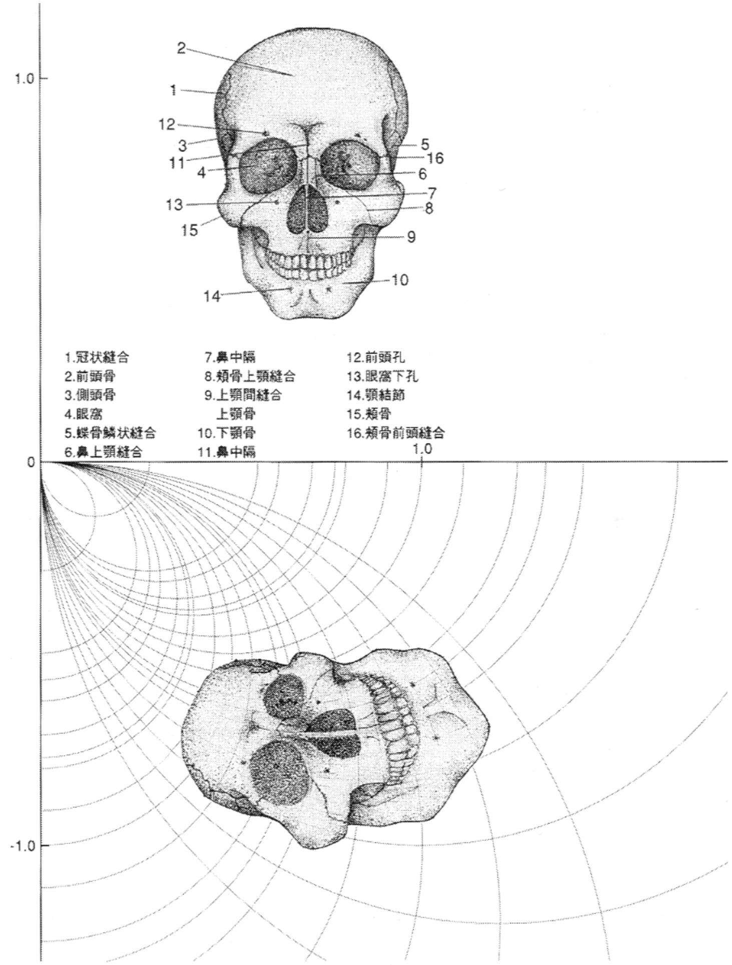

Fig. 6. “Skeleton” by M. Watanabe.

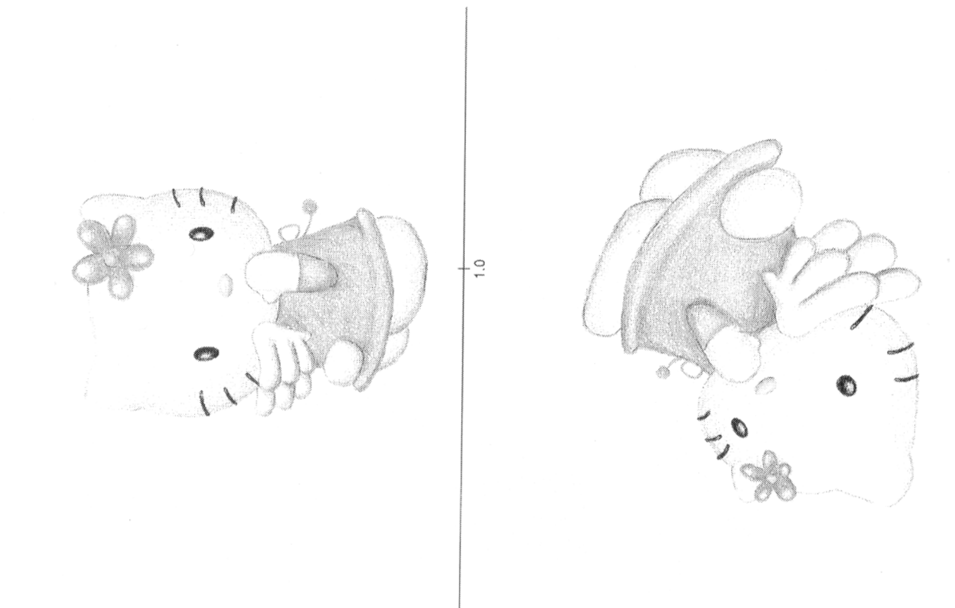

Fig. 7. “Hello kitty!” by Y. Moriya.

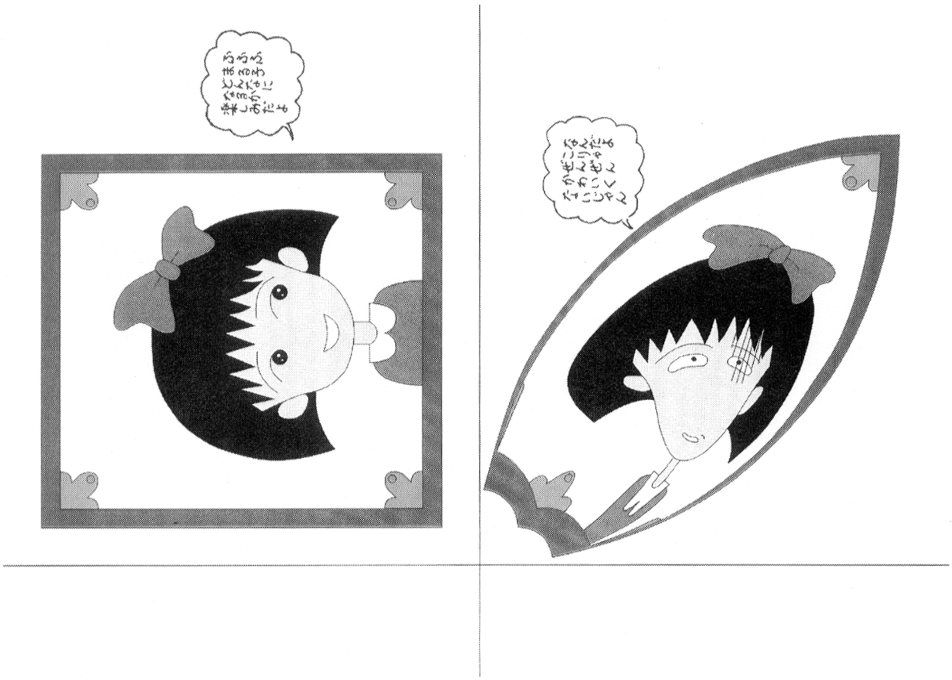

Fig. 8. “Hello chibi-marukochan!” by T. Moriue.

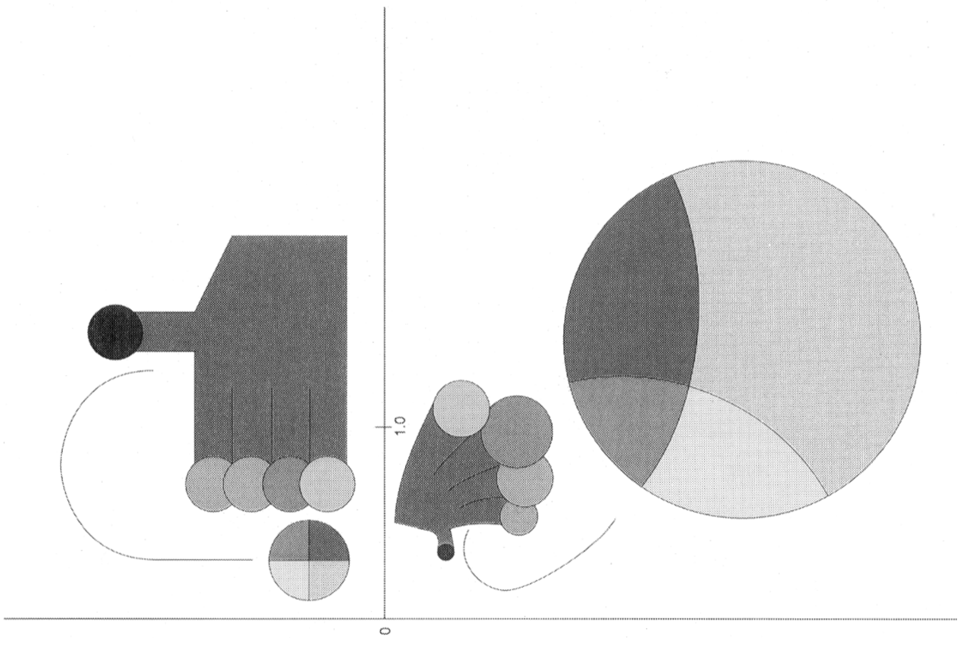

Fig. 9. “Finger play” by N. Kitagawa.

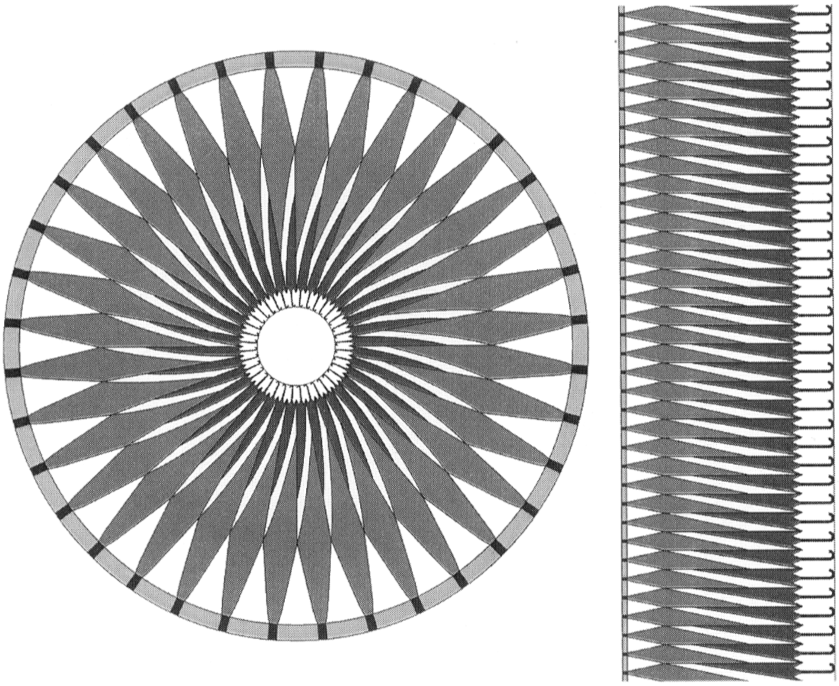

Fig. 10. “Open the umbrella” by Y. Nakano.

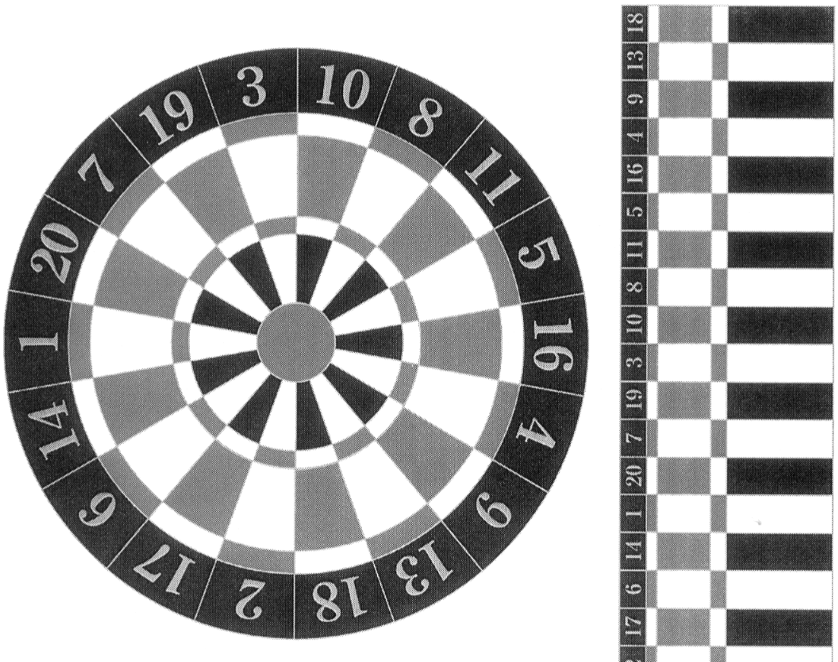

Fig. 11. “The keyboard changed the target” by R. Masuda.

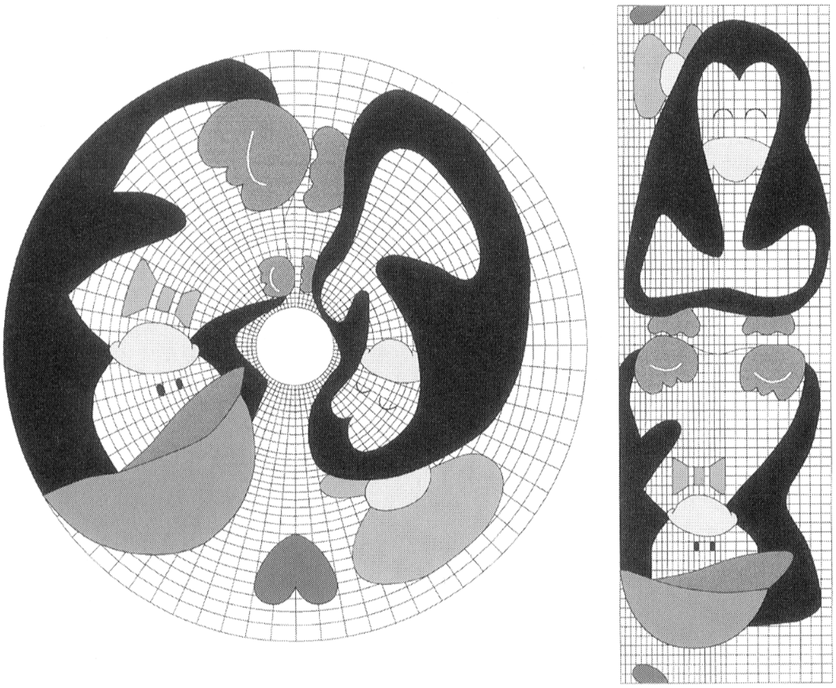

Fig. 12. “In fact, they are intimate friend” by T. Nishiyama.

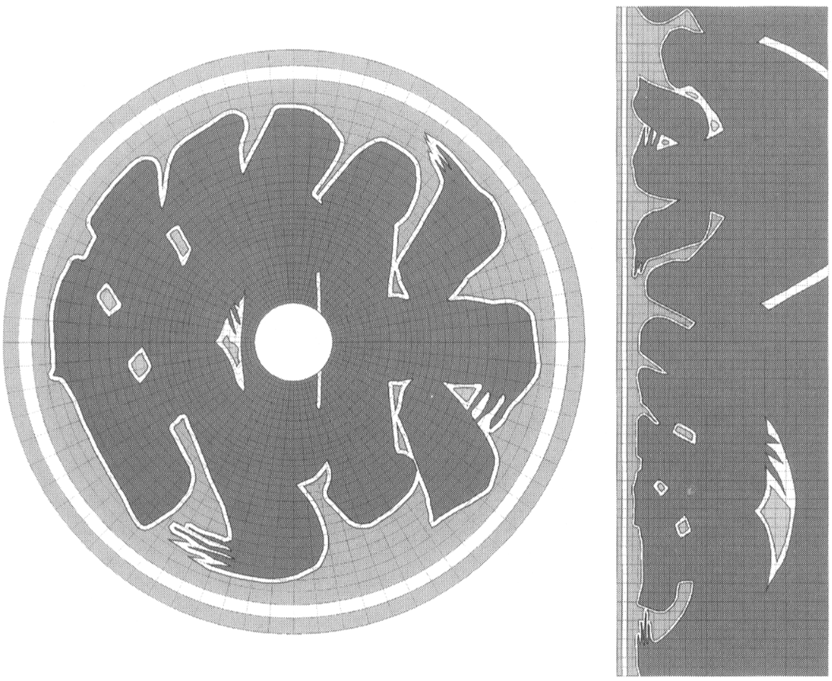

Fig. 13. “The back of festival dress” by T. Ishibashi.

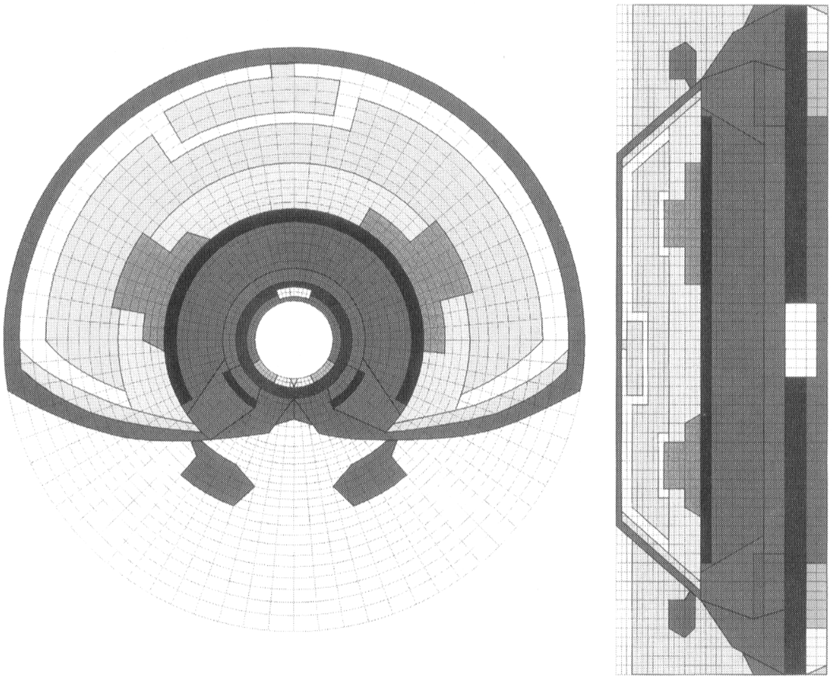

Fig. 14. “My car became to a crab” by K. Kawahara.

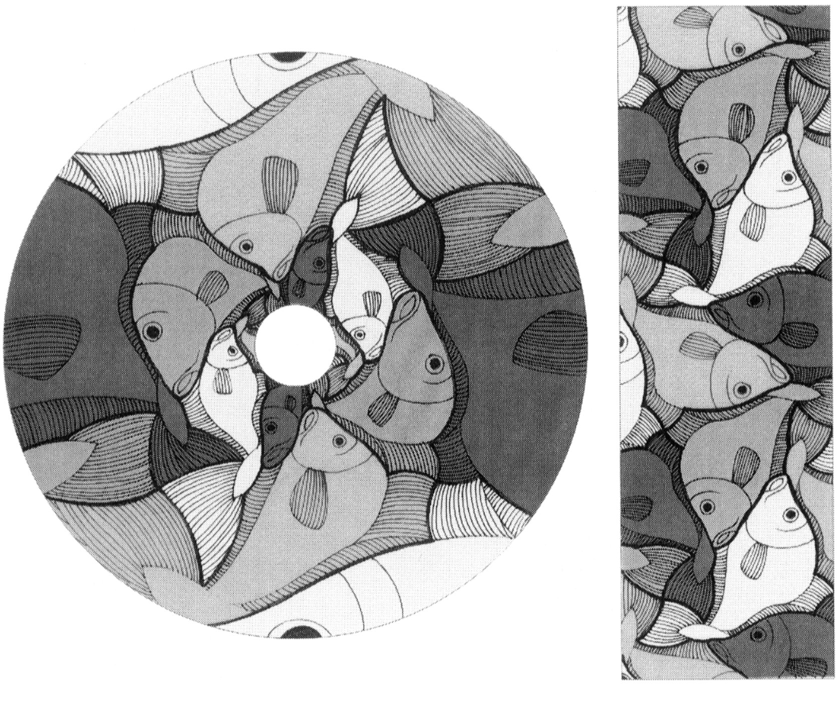

Fig. 15. "Tricked picture" by T. Yumiba.

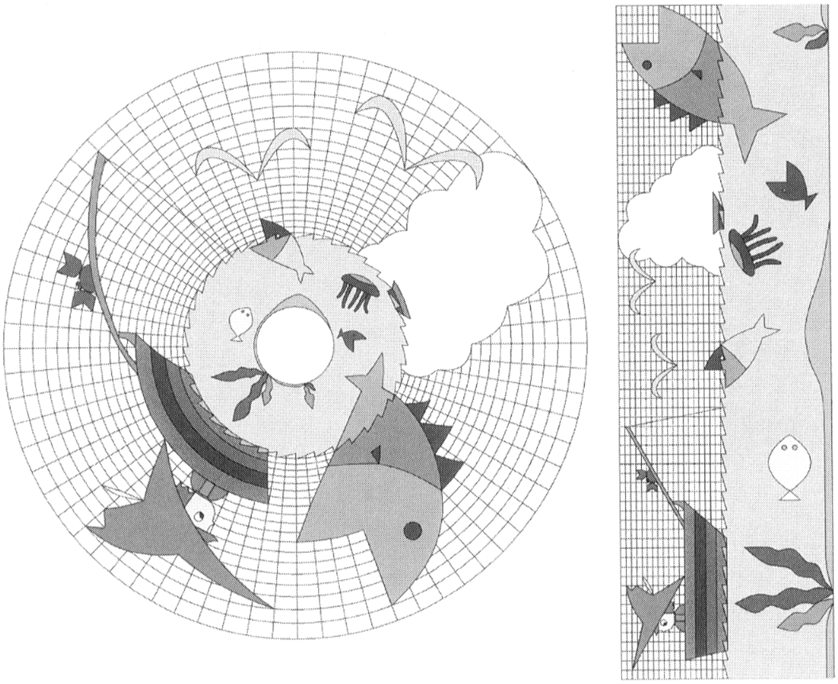

Fig. 16. "The movie's scene" by H. Imachi.

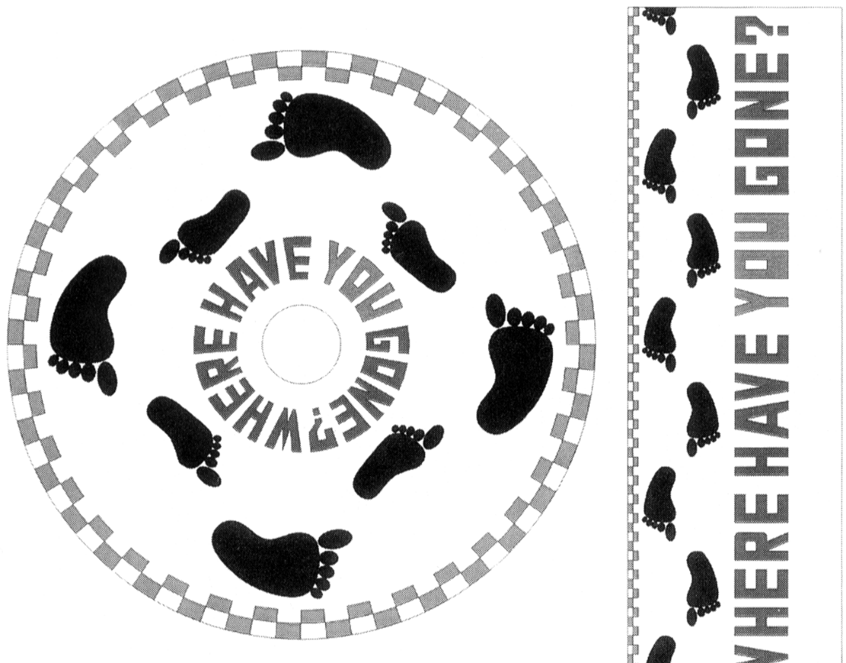

Fig. 17. "Where have you gone?" by M. Kuroda.

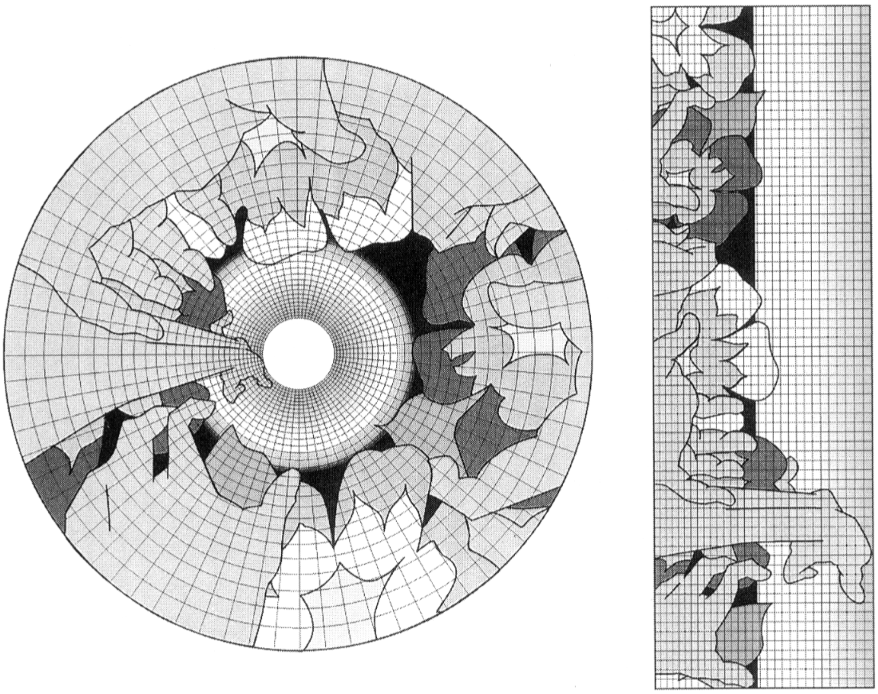

Fig. 18. "Arms" by W. Toriyama.

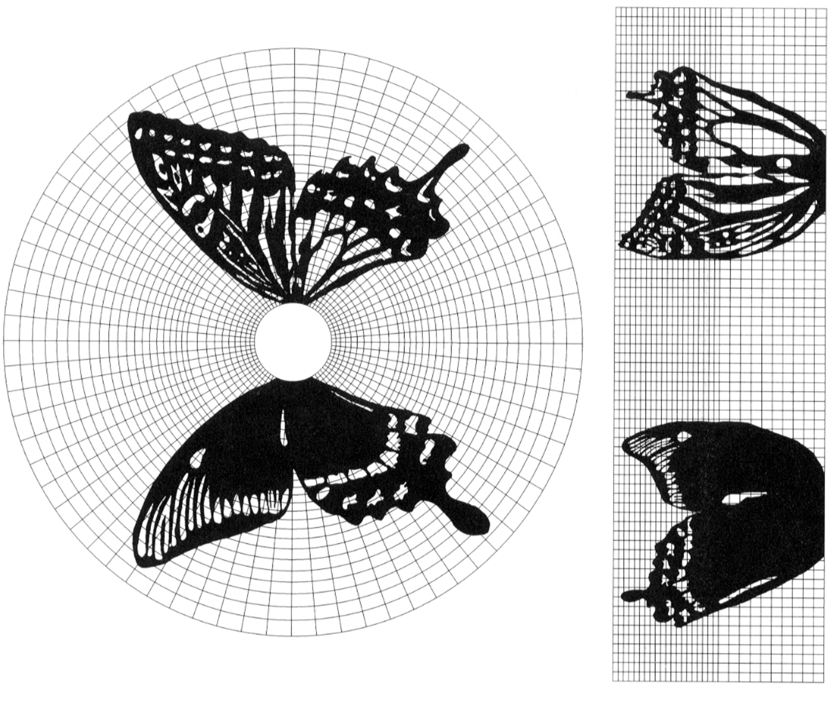

Fig. 19. “Changing to butterfly” by H. Itoh.

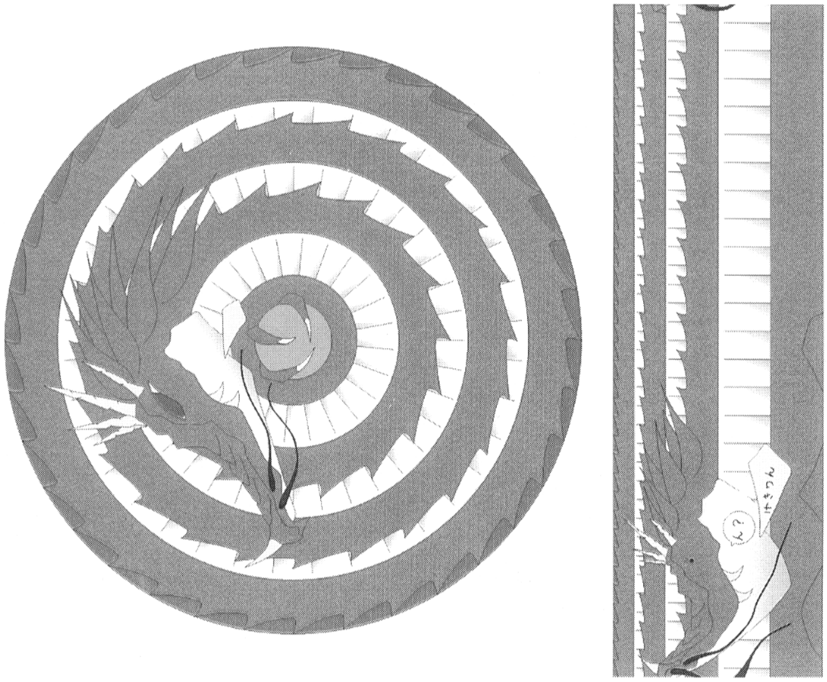

Fig. 20. “The dragon” by K. Sunabori.

4. Pictures

In this section, I want to present some pictures (Figs. 1–20) by some students of mine, which were handed in as part of their assignments with respect to two functions $f(z) = 1/z$ and $f(z) = \exp(-iz)$ in the last fourteen years. All the pictures are drawn by freehand touching: some of these pictures have been used as cover pictures of the printed matter of the University.

5. Conclusion

One very important thing is that anyone can easily understand the theory and properties of the regular functions from drawing a picture with the use of conformal mapping. Observing students' works, it give rise to more educational effectiveness than I can image. I would like to mention that I have been very much impressed with the drawing of pictures and the efforts a majority of students exert encourage me naturally to continue with the teaching of conformal mapping property of the regular function for the purpose of charming mathematics.

Before I close this paper, I would like to express my sincere gratitude to Dr. Toshihiko Numahara for his invitation to the Congress of FORMS.

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